Some new generalizations of Domination using restrictions on degrees of vertices

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Abstract

A set D of vertices in a graph G = (V, E) is a degree restricted dom*inating set* for G if each vertex v_i in D is dominating at most $g(d_i)$ vertices of V - D, where g is a function restricting the degree value d_i with respect to the given function value k_i for a natural valued function f from the vertex set of the graph. We define three different types of Degree Restricted Domination by varying the way how the restricted function $g(v_i)$ is defined. If $g(d_i) = \begin{bmatrix} d_i \\ k_i \end{bmatrix}$, the corresponding domination is called the *ceil degree restricted domination*, in short, CDRD, and the dominating set obtained in this manner is the CDRD-set. If $g(d_i) = \lfloor \frac{d_i}{k_i} \rfloor$ or $g(d_i) = d_i - k_i + 1$, then the corresponding dominations are respectively called the *floor degree restricted domination*, in short FDRD, or the translate degree restricted domination, TDRD. The dominating sets obtained in this manner are the FDRD-set and the TDRD-set respectively. In this paper, we introduce these new generalizations of the domination number in line with the different DRD-sets and study these types of domination for some classes of graphs like complete graphs, caterpillar graphs etc. Degree restricted domination has a vital role in retaining the efficiency of nodes in a network and has many interesting applications.

Keywords: Graph Domination, Degree Restricted Domination, Ceil Degree Restricted Domination, Floor Degree Restricted Domination, Translate Degree Restricted Domination.

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1 Introduction

Let G = (V, E) be a graph with order n and size m, where $V = \{v_1, v_2, \ldots, v_n\}$. The *degree* of a vertex $v_i \in V$ is the number of edges incident with it and is

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denoted by $d_G(v_i)$ or d_i . A caterpillar graph is a tree which can be obtained from a path by adding pendant edges with its vertices. The initial path sans the pendant vertices is called the *spine* of the caterpillar.

Two vertices v_i and v_j dominate each other in a graph G = (V, E) if v_i and v_j are adjacent in G, i.e., $v_i v_j \in E$. A set $D \subseteq V$ in a graph G is called a dominating set if every vertex in V - D is dominated by atleast one vertex in D. Property of domination is superhereditary and so, the minimal dominating sets are of much importance. The minimum cardinality of a minimal dominating set is called the domination number, denoted as $\gamma(G)$.

In networks, to retain the efficiency of those nodes which are in contact with more number of nodes, we may have to restrict the transfer of data only through a certain pairs of nodes. This restriction can be done in various forms. If the number of such data transformation is restricted equally at every node, that is, if every vertex v_i can dominate atmost $\left\lceil \frac{d_i}{k} \right\rceil$ vertices adjacent to it, then such a domination is called *k*-part degree restricted domination [3, 4]. But practically such restriction need not be uniform. It can vary depending on the situation on the type of the network and its applications. In this paper, we model some such restrictions through graphs.

The reader is referred to [5] for the notations and terminologies and [1, 2] for the domination concepts.

2 Main Results

2.1 Degree Restricted Domination

Definition 2.1. Let G = (V, E) be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and let the degree sequence be (d_1, d_2, \ldots, d_n) where $d_i = d(v_i)$. Suppose $f : V \to \mathbb{N}$ is a function defined as $f(v_i) = k_i$, where $1 \le k_i \le d_i$ and $f(v_i) = 1$ if v_i is an isolated vertex. A dominating set $D \subseteq V$ is a degree restricted dominating set for the graph G if each vertex v_i in D is dominating atmost $g(d_i)$ vertices of V - D, where g is a function restricting the degree value d_i with respect to the given function f.

By varying the way the function g is defined we get different generalizations for the dominating sets. We define here three types of degree restricted domination.

2.1.1 Ceil Degree Restricted Domination(CDRD)

If $g(d_i) = \left\lceil \frac{d_i}{k_i} \right\rceil$, the corresponding domination is called the *ceil degree re*stricted domination, in short *CDRD*, and a dominating set obtained in this manner is a CDRD-set. The minimum cardinality of a CDRD-set, the CDRD number of G is denoted as $\gamma_{\overline{f}}(G)$ or $\gamma_{\overline{f}}$. A CDRD-set with minimum cardinality is a $\gamma_{\overline{f}}$ -set.

Observation 2.2. 1. If $k_i = 1$, for all $v_i \in V$, then the CDRD is same as the fundamental domination. Thus $\gamma_{\overline{f}}(G) = \gamma(G)$ in this case.

2. If $k_i = k$, where $k \leq \delta(G)$ for each $v_i \in V$, then the corresponding domination is the k-part degree restricted domination defined in [4].

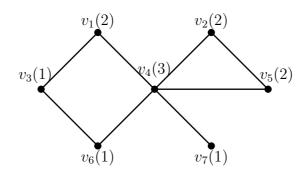


Figure 2.1.1: Graph G

For the graph G in Fig.2.1.1 with the given function $f(v_1) = 2$, $f(v_2) = 2$, $f(v_3) = 1$, $f(v_4) = 3$, $f(v_5) = 2$, $f(v_6) = 1$ and $f(v_7) = 1$, indicated in the parantheses, v_1, v_2, v_5, v_7 can dominate atmost one vertex and v_3, v_4, v_6 can dominate atmost two vertices in accordance with *CDRD*. Thus $\{v_2, v_3, v_4\}$ forms a minimal *CDRD* set and which is also a minimum *CDRD*-set. Thus $\gamma_{\overline{f}}(G) = 3$.

2.1.2 Floor Degree Restricted Domination(FDRD)

If $g(d_i) = \lfloor \frac{d_i}{k_i} \rfloor$, the corresponding domination is the floor degree restricted domination, in short FDRD, and a dominating set obtained in this manner is a FDRD-set. The minimum cardinality of a FDRD-set, FDRD number is denoted as $\gamma_{\underline{f}}(G)$ or $\gamma_{\underline{f}}$. A FDRD-set with minimum cardinality is a γ_f -set.

Observation 2.3. When each d_i is divisible by the corresponding k_i , for i = 1, 2, ..., n then $\left\lfloor \frac{d_i}{k_i} \right\rfloor = \left\lfloor \frac{d_i}{k_i} \right\rfloor$ and hence the CDRD-set and FDRD-set will be same.

In Fig.2.1.1 v_1, v_2, v_4, v_5, v_7 can dominate atmost one vertex and v_3, v_6 can dominate atmost two vertices in accordance with *FDRD*. Thus $\{v_2, v_3, v_4\}$ forms a minimal *FDRD* set and which is also a minimum *FDRD*-set. Thus $\gamma_f(G) = 3$.

2.1.3 Translate Degree Restricted Domination(TDRD)

If $g(d_i) = d_i - k_i + 1$, then such a domination is the translate degree restricted domination, in short TDRD, and such dominating set is called a TDRDset. The minimum cardinality of a TDRD-set, TDRD number is denoted as $\gamma_{f_t}(G)$ or γ_{f_t} . A TDRD-set with minimum cardinality is a γ_{f_t} -set. In Fig.2.1.1 v_1, v_2, v_5, v_7 can dominate atmost one vertex, v_3, v_6 can dominate atmost two vertices and v_4 can dominate atmost 3 vertices in accordance with TDRD. Thus $\{v_2, v_3, v_7\}$ forms a minimal TDRD set; but is not a minimum TDRD-set. Here $\{v_3, v_4\}$ forms a minimum TDRD-set and thus $\gamma_{f_t}(G) = 2$.

Observation 2.4. The newly defined domination varies for the same graph with different function values. Consider the graph G in Fig.2.1.1 with different function value as $f(v_1) = 2$, $f(v_2) = 2$, $f(v_3) = 1$, $f(v_4) = 2$, $f(v_5) = 2$, $f(v_6) = 1$ and $f(v_7) = 1$, then v_4 can dominate atmost three vertices with respect to CDRD and thus $\gamma_{\overline{f}}(G) = 2$ with the $\gamma_{\overline{f}}$ -set $\{v_3, v_4\}$.

2.2 DRD Number

In a graph G with the vertex set $\{v_1, v_2, \ldots, v_n\}$, a vertex can dominate maximum number of vertices if $k_i = 1$ for every *i*. Then as observed above $\gamma_{\overline{f}}(G) = \gamma(G)$. A vertex v_i can dominate only one of its neighbours when $k_i = d_i$, and thus the *CDRD* number will be maximum if all the vertices have $k_i = d_i$. In a star graph $K_{1,n-1}$, if $k_i = d_i$ for each vertex, then the central vertex will dominate one of its neighbours and all other vertices must be in the *CDRD* set. Thus $\gamma_{\overline{f}}(K_{1,n-1}) = n - 1$. Hence $\gamma \leq \gamma_{\overline{f}} \leq n - 1$.

Theorem 2.5. For any graph G, $\gamma(G) \leq \gamma_{\overline{f}}(G) \leq \gamma_f(G)$.

Proof. Any CDRD-set or FDRD-set is also a dominating set for any graph G. Thus $\gamma(G) \leq \gamma_{\overline{f}}(G)$ and $\gamma(G) \leq \gamma_{\underline{f}}(G)$. Also, for any vertex v_i (with the function value k_i and the degree d_i), $\left\lfloor \frac{d_i}{k_i} \right\rfloor \leq \left\lceil \frac{d_i}{k_i} \right\rceil$ and hence $\gamma_{\overline{f}}(G) \leq \gamma_{\underline{f}}(G)$. Thus $\gamma(G) \leq \gamma_{\overline{f}}(G) \leq \gamma_{\underline{f}}(G)$.

Corollary 2.5.1. If G is a graph for which degree d_i of each vertex is divisible by the corresponding function value k_i , then $\gamma_{\overline{f}}(G) = \gamma_f(G)$.

Proof. Since every k_i divides d_i , we have $\left\lceil \frac{d_i}{k_i} \right\rceil = \left\lfloor \frac{d_i}{k_i} \right\rfloor$. So, every CDRD-set must be a FDRD-set and hence the result.

Corollary 2.5.2. If G is a graph for which degree d_i of each vertex in a $\gamma_{\overline{f}}$ or γ_f -set is divisible by the corresponding function value k_i , then $\gamma_{\overline{f}}(G) = \gamma_f(G)$.

Corollary 2.5.3. If G is a graph for which each vertex has the function value $k_i = 1$, then $\gamma(G) = \gamma_{\overline{f}}(G) = \gamma_f(G)$.

The corollaries to Theorem 2.5 are only the sufficient conditions. Figure 2.2.2 is a counter example for the converse of above stated corollaries where the function values for the vertices are indicated in the parantheses.

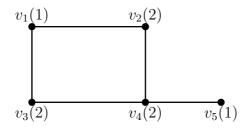


Figure 2.2.2: Counter example for the converse of Corollaries to Theorem 2.5

Theorem 2.6. For any graph G, $\left\lceil \frac{n}{1+\left\lceil \frac{\Delta}{k} \right\rceil} \right\rceil \leq \gamma_{\overline{f}} \leq n - \max_{i \in [n]} \left\lceil \frac{d_i}{k_i} \right\rceil$ where $k = \min_{i \in [n]} k_i$.

Proof. For any vertex v_i in a $\gamma_{\overline{f}}$ -set, it can dominate at most $1 + \left\lceil \frac{d_i}{k_i} \right\rceil$ vertices. Maximum possible value for $\left\lceil \frac{d_i}{k_i} \right\rceil$ is when $d_i = \Delta$ and $k_i = k$, where $k = \min_{i \in [n]} k_i$. Thus $\left\lceil \frac{n}{1 + \left\lceil \frac{\Delta}{k} \right\rceil} \right\rceil \leq \gamma_{\overline{f}}$.

Let v_i be the vertex where $\left\lceil \frac{d_i}{k_i} \right\rceil$ is maximum. Then all but those $\left\lceil \frac{d_i}{k_i} \right\rceil$ vertices dominated by v_i will form a *CDRD*-set for *G*. Hence $\gamma_{\overline{f}} \leq n - \max_{i \in [n]} \left\lceil \frac{d_i}{k_i} \right\rceil$.

The upperbound is attained for the star graph and the lower bound is attained for the complete graph K_{3p} if $k_i \ge \lceil \frac{3p-1}{2} \rceil$ for all v_i or the cycle C_{3p} if $k_i = 1$ for all $i \equiv 0 \pmod{3}$.

Theorem 2.7. For any isolate free graph G, $\gamma_{\overline{f}} \leq \beta'$, the edge covering number of G.

Proof. Let $\{e_1, e_2, \ldots, e_{\beta'}\}$ be a maximum edge covering for the graph G. If e_i has the end vertices v_{i_1} and v_{i_2} , then the collection of all v_{i_1} 's forms a CDRD-set for the graph G. Thus $\gamma_{\overline{f}} \leq \beta'$.

If all the vertices in the graph has the function value $k_i = d_i$, then the bound mentioned above will be attained by the *CDRD* number $\gamma_{\overline{f}}$.

Corollary 2.7.1. For any isolate free bipartite graph G, $\gamma_{\overline{f}} \leq \alpha$, the independence number of G.

2.2.1 Complete graph K_n

CDRD number for a complete graph with some particular functions can be determined.

Theorem 2.8. For a complete graph K_n if there are atmost $\lfloor \frac{2n}{3} \rfloor$ vertices with $k_i = d_i$, then $\gamma_{\overline{f}}(K_n) \leq \lfloor \frac{n}{3} \rfloor$.

Proof. Since K_n is an n-1 regular graph of order n, the function value k_i can vary from 1 to n-1. Unless $k_i = d_i$, v_i can dominate at least two vertices. Thus a set of at most $\lceil \frac{n}{3} \rceil$ vertices with $k_i < d_i$ will form a *CDRD*-set for K_n . Hence $\gamma_{\overline{f}}(K_n) \leq \lceil \frac{n}{3} \rceil$.

This is only a necessary condition for the *CDRD* number to be bounded for K_n , but not sufficient since the presence of a vertex v_i with $k_i = 1$ and all other vertices with $k_i = d_i$ will give $\gamma_{\overline{f}}(K_n) \leq \left\lceil \frac{n}{3} \right\rceil$.

Similar result can be obtained for the FDRD number as below.

Theorem 2.9. For a complete graph K_n if there are atmost $\lfloor \frac{2n}{3} \rfloor$ vertices with $k_i > \lfloor \frac{n-1}{2} \rfloor$, then $\gamma_{\underline{f}}(K_n) \leq \lfloor \frac{n}{3} \rfloor$.

Here also it is not a sufficient condition since a vertex with $k_i = 1$ will give $\gamma_{\underline{f}}(K_n) = 1$, without considering the k_j values for other vertices. Similar result can be obtained for the *TDRD* number as below.

Theorem 2.10. For a complete graph K_n if there are atmost $\lfloor \frac{2n}{3} \rfloor$ vertices with $k_i = d_i$, then $\gamma_{f_t}(K_n) \leq \lfloor \frac{n}{3} \rfloor$.

Here also it is not a sufficient condition since a vertex with $k_i = 1$ will give $\gamma_{f_t}(K_n) = 1$, without considering the k_i values for other vertices.

2.2.2 Caterpillar graph

If G is a caterpillar graph whose spine is the path $P_n = v_1 v_2 \dots v_n$, where each vertex v_i in the spine has degree d_i and is attached to l_i leaves, then $d_i = \begin{cases} l_i + 1 & \text{if } i = 1 \text{ or } n \\ l_i + 2 & \text{if } i = 2, 3, \dots, n-1 \end{cases}$, assuming that $l_i \ge 1$, for all $i = 1, 2, \dots, n$. Let $f: V \to \mathbb{N}$ is defined as $f(v_i) = k_i$ and $f(v_{i_j}) = 1$, where v_{i_j} is the leaf attached to the vertex v_i , $1 \le j \le l_i$, and $1 \le i \le n$.

Theorem 2.11. If G is the caterpillar graph defined as above then $\gamma_{\overline{f}}(G) =$ $\sum_{i=1,n} (l_i - \lceil \frac{l_i+1}{k_i} \rceil) + \sum_{i=2}^{n-1} (l_i - \lceil \frac{l_i+2}{k_i} \rceil) + n, \text{ provided } l_i > 0, k_i > 1 \text{ and if } l_i = 1$ then $k_i = 3$.

Proof. Since $l_i > 0$ for all the vertices in the spine $v_1 v_2 \dots v_n$, each v_i must be a member in every CDRD-set. Thus all the n vertices in the spine are necessary in the $\gamma_{\overline{f}}$ -set. In the spine, all but v_1 and v_n can dominate $\left\lceil \frac{l_i+2}{k_i} \right\rceil$ vertices and at the same time v_1 and v_n can dominate $\left\lceil \frac{l_i+1}{k_i} \right\rceil$ vertices. Let the number of pendant vertices adjacent to v_i but not dominated by it, be r_i . Then $r_i = \begin{cases} l_i - \left\lceil \frac{l_i+1}{k_i} \right\rceil$ if i = 1 or $n \\ l_i - \left\lceil \frac{l_i+2}{k_i} \right\rceil$ if $i = 2, 3, \dots, n-1$. Any of the r_i leaves adjacent to v_i must be in every *CDRD*-set. So, $\gamma_{\overline{f}}(G) =$

$$\sum_{i=1,n} \left(l_i - \left\lceil \frac{l_i+1}{k_i} \right\rceil\right) + \sum_{i=2} \left(l_i - \left\lceil \frac{l_i+2}{k_i} \right\rceil\right) + n.$$

Theorem 2.12. For the star graph $K_{1,n}$, $\gamma_{\overline{f}} \geq \lfloor \frac{n}{2} \rfloor + 1$ unless the central vertex has the function value 1.

Proof. Let v_1 be the central vertex in the star graph $K_{1,n}$. If $f(v_1) = 1$, then $\gamma_{\overline{f}} = \gamma = 1.$

If $f(v_1) = k_1$, where $1 < k_1 \le n$. Since $\left\lceil \frac{n}{k_1} \right\rceil \le \left\lceil \frac{n}{2} \right\rceil$, v_1 can dominate at most $\left\lceil \frac{n}{2} \right\rceil$ remaining vertices of the graph. Thus any *CDRD*-set will contain at least $n = \lfloor \frac{n}{2} \rfloor$ vertices of the graph other than v_1 . Hence $\gamma_{\overline{f}} \geq \lfloor \frac{n}{2} \rfloor + 1$ for any function $f: V \to \mathbb{N}$.

Let \mathfrak{C}_2 denotes the collection of caterpillars whose leaves are attached only with the vertices v_i , where $i \equiv 2 \pmod{3}$ on the spine $P_n = v_1 v_2 \dots v_n$. Fig. 2.2.3 is an example for a caterpillar in the class \mathfrak{C}_2 .

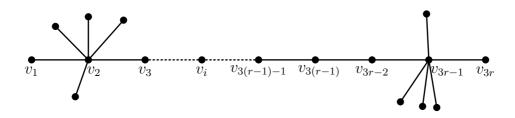


Figure 2.2.3: Graph $G \in \mathfrak{C}_2$

Theorem 2.13. If $G \in \mathfrak{C}_2$ with $d(v_i) = n_i$ and $f(v_i) = k_i$, then $\gamma_{\overline{f}}(G) \in \{k :$ $k = \sum_{i=1}^{n} n_i - \left\lceil \frac{n_i}{k_i} \right\rceil + 1 \}.$ $i \equiv 2 \pmod{3}$

Proof. Each v_i , where $i \equiv 2 \pmod{3}$ together with the leaves attached to it forms a star graph K_{1,n_i} . So in Theorem 2.12, $n_i - \left\lceil \frac{n_i}{k_i} \right\rceil + 1$ vertices are required to dominate that star. Hence to dominate the entire graph we have to consider *CDRD*-set of each star centered at v_i , where $i \equiv 2 \pmod{3}$.

2.2.3 Path and Cycle

Theorem 2.14. If P_n is a path of order n, $\left\lceil \frac{n}{3} \right\rceil \leq \gamma_{\overline{f}}(P_n) \leq \left\lceil \frac{n}{2} \right\rceil$.

Proof. In a path $P_n = v_1 v_2 \dots v_n$, degree of the vertex v_i be d_i and $d_i = \begin{cases} 1 & \text{if } i = 1 \text{ or } n \\ 2 & \text{if } i = 2, 3, \dots, n-1 \end{cases}$ Thus $k_i = \begin{cases} 1 & \text{if } i = 1 \text{ or } n \\ 1 \text{ or } 2 & \text{if } i = 2, 3, \dots, n-1 \end{cases}$. Hence v_1 and v_n can dominate exactly one vertex and all other vertices will dominate 1 or 2 vertices with respect to the k_i values 2 or 1 respectively. If each vertex v_i has $k_i = d_i$, then each vertex can dominate exactly one other vertex and in such case, those vertices with odd index will form a CDRD-set. So, $\gamma_{\overline{f}}(P_n) \leq \left\lceil \frac{n}{2} \right\rceil$. Also $\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil$ and $\gamma(G) \leq \gamma_{\overline{f}}(G)$ for any graph G. Thus $\left\lceil \frac{n}{3} \right\rceil \leq \gamma_{\overline{f}}(P_n) \leq \left\lceil \frac{n}{2} \right\rceil$.

Theorem 2.15. If C_n is a cycle of order n, $\left\lceil \frac{n}{3} \right\rceil \leq \gamma_{\overline{f}}(C_n) \leq \left\lceil \frac{n}{2} \right\rceil$.

Definition 2.16 (Restricted radius). Let G = (V, E) be any graph and $S \subseteq V$. We can define the radius of G restricted to S as $rad_S(G) = \min_{u,v\in S} \{d_G(u,v)\}$.

Clearly, if $S_1 \subseteq S_2$, then $\operatorname{rad}_{S_1}(G) \ge \operatorname{rad}_{S_2}(G)$.

Theorem 2.17. For a cycle C_n of order n with the given function f, if $rad_{S_1}(G) \geq 3$ where S_1 is the collection of vertices mapping to 1 by the function f, then $\gamma_{\overline{f}}(C_n) \leq n-2|S_1|$.

Proof. For any $v_i \in V(C_n)$, $d_i = 2$ and so $k_i = 1$ or 2. Let $S_1 = \{v_i \in V(C_n) : f(v_i) = 1\}$. If $\operatorname{rad}_{S_1}(G) \geq 3$, there exist vertices in V with both the function values 1 and 2. All the vertices in S_1 can dominate both of its neighbours and thus, the remaining vertices which are not being dominated are only those with $k_i = 2$, that is, $n - 3|S_1|$ in number. All these vertices together with those in S_1 will form a CDRD-set. Hence $\gamma_{\overline{f}}(C_n) \leq n - 2|S_1|$. If $\operatorname{rad}_{S_1}(G) < 3$, then can form a new set S from S_1 by deleting necessary vertices so that $\operatorname{rad}_S(G) \geq 3$. Then $\gamma_{\overline{f}}(C_n) \leq n - 2|S|$.

3 Conclusion

In this paper, some new generalized forms of domination were introduced by restricting the number of vertices a vertex can dominate. The newly defined variations of dominations are Ceil Degree Restricted Domination, Floor Degree Restricted Domination and Translate Degree Restricted Domination. As the name indicates, in each type of domination a vertex dominates a particular number of vertices as the given function indicates. Some bounds for the Degree Restricted Domination number have been discussed in this paper and also Degree Restricted Domination is studied for some particular classes of graphs like the complete graph K_n , paths, cycles and the caterpillars.

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