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Some new Hermite–Hadamard type inequalities for *s*-convex functions and their applications

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Abstract

In this paper, we establish some new integral inequalities of Hermite–Hadamard type for s-convex functions by using the Hölder–İşcan integral inequality. We also compare our new results with the known results and show that the results which we obtained are better than the known results. Finally, we give some applications to trapezoidal formula and to special means.

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1 Introduction

The classical or the usual convexity is defined as follows:

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on interval I if

$$f\big(tx + (1-t)y\big) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0,1]$.

Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on I and $a, b \in I$ with $a < b$. The following inequality is known in the literature as the Hermite–Hadamard inequality for convex functions:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

In [8], Hudzik and Maligranda considered the class of s-convex functions in second sense. This is defined as follows:

A function $f : [0,\infty) \to \mathbb{R}$ is said to be s-convex in second sense if the inequality

$$f\big(tx + (1-t)y\big) \leq t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in [0,\infty)$, $t \in [0,1]$, and $s \in (0,1]$.

In [4], Dragomir and Fitzpatrick established a variant of Hermite–Hadamard inequality which holds for the s-convex functions.

Theorem 1.1 *Suppose that $f:[0,\infty)\to[0,\infty)$ is an s-convex function in second sense, where $s\in(0,1]$. Let $a,b\in[0,\infty)$ and $a<b$. If $f\in L[0,1]$, then the following inequality holds*:

$$2^{s-1}f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\,dx\le\frac{f(a)+f(b)}{s+1}.$$

For the generalizations and applications of the Hermite–Hadamard inequalities, see [1–3, 5–7, 9, 12–15, 17, 19–21].

A number of studies have shown that many of the results obtained about the theory of inequalities has a close relationship with the theory of convex functions.

The celebrated inequality of Hölder is well known for its fundamental role in many branches of pure and applied sciences. It has also important applications to the theory of convex functions as well as in many disciplines of applied mathematics. The power-mean integral inequality is also one of the most famous inequalities on applications to convex functions.

Theorem 1.2 (Hölder inequality for integrals [16]) *Let $p>1$ and $1/p+1/q=1$. If f and g are real functions defined on $[a,b]$ and if $|f|^p$ and $|g|^q$ are integrable on $[a,b]$, then*

$$\int_a^b\left|f(x)g(x)\right|dx\le\left(\int_a^b\left|f(x)\right|^p dx\right)^{\frac{1}{p}}\left(\int_a^b\left|g(x)\right|^q dx\right)^{\frac{1}{q}},$$

with equality holds if and only if $A|f(x)|^p=B|g(x)|^q$ almost everywhere, where A and B are constants.

A different version of Hölder integral inequality is given as follows.

Theorem 1.3 (Power-mean integral inequality) *Let $q\ge 1$. If f and g are real functions defined on $[a,b]$ and if $|f|$, $|f||g|^q$ are integrable on $[a,b]$, then*

$$\int_a^b\left|f(x)g(x)\right|dx\le\left(\int_a^b\left|f(x)\right|dx\right)^{1-\frac{1}{q}}\left(\int_a^b\left|f(x)\right|\left|g(x)\right|^q dx\right)^{\frac{1}{q}}.$$

In [10], İşcan obtained the following inequality for integrals which gives better results than the classical Hölder inequality.

Theorem 1.4 (Hölder–İşcan integral inequality) *Let $p>1$ and $1/p+1/q=1$. If f and g are real functions defined on $[a,b]$ and if $|f|^p$ and $|g|^q$ are integrable on $[a,b]$, then*

$$\begin{aligned}\int_a^b\left|f(x)g(x)\right|dx&\le\frac{1}{b-a}\left\{\left(\int_a^b(b-x)\left|f(x)\right|^p dx\right)^{\frac{1}{p}}\left(\int_a^b(b-x)\left|g(x)\right|^q dx\right)^{\frac{1}{q}}\right.\\&\quad\left.+\left(\int_a^b(x-a)\left|f(x)\right|^p dx\right)^{\frac{1}{p}}\left(\int_a^b(x-a)\left|g(x)\right|^q dx\right)^{\frac{1}{q}}\right\}\\&\le\left(\int_a^b\left|f(x)\right|^p dx\right)^{\frac{1}{p}}\left(\int_a^b\left|g(x)\right|^q dx\right)^{\frac{1}{q}}.\end{aligned}\tag{1}$$

In [11], a different representation of Hölder–İşcan inequality was given as follows.

Theorem 1.5 (Improved power-mean integral inequality) *Let $q \geq 1$. If f and g are real functions defined on $[a,b]$ and if $|f|$, $|f||g|^q$ are integrable functions on $[a,b]$, then*

$$\begin{aligned}\int_a^b \left|f(x)g(x)\right| dx &\leq \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\left|f(x)\right| dx\right)^{1-\frac{1}{q}}\left(\int_a^b (b-x)\left|f(x)\right|\left|g(x)\right|^q dx\right)^{\frac{1}{q}}\right.\\ &\quad \left.+\left(\int_a^b (x-a)\left|f(x)\right| dx\right)^{1-\frac{1}{q}}\left(\int_a^b (x-a)\left|f(x)\right|\left|g(x)\right|^q dx\right)^{\frac{1}{q}}\right\}\\ &\leq \left(\int_a^b \left|f(x)\right| dx\right)^{1-\frac{1}{q}}\left(\int_a^b \left|f(x)\right|\left|g(x)\right|^q dx\right)^{\frac{1}{q}}. \end{aligned} \tag{2}$$

2 Main results

In this section we obtain some new results about Hermite–Hadamard inequality for s-convex functions by using Hölder–İşcan integral inequality and improved power-mean integral inequality which provide better approach than the classical Hölder and power-mean integral inequalities, respectively.

In [5], Dragomir and Pearce obtained the following equality for differentiable functions.

Lemma 2.1 *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function on I° where $a, b \in I$ with $a < b$. If $f' \in L[a,b]$, then the following equality holds*:

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{b-a}{2}\int_0^1 (1-2t)f'\big(ta+(1-t)b\big)\,dt.$$

In [17], Muddassar et al. obtained the following inequality for s-convex functions by using Lemma 2.1 and Hölder integral inequality.

Theorem 2.1 *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$ and $f' \in L[a,b]$. If $|f'|^q$ is s-convex on $[a,b]$, $p > 1$ such that $q = \frac{p}{p-1}$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left(\frac{|f'(a)|^q + |f'(b)|^q}{s+1}\right)^{\frac{1}{q}}. \tag{3}$$

In the following theorem, we will obtain a new upper bound for the right-hand side of Hermite–Hadamard inequality for s-convex functions, which is better than inequality (3).

Theorem 2.2 *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$ and $f' \in L[a,b]$. If $|f'|^q$ is s-convex on $[a,b]$, where $p > 1$ such that $q = \frac{p}{p-1}$, then*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ &\quad\leq \frac{b-a}{2^{\frac{p+1}{p}}(p+1)^{\frac{1}{p}}}\cdot\frac{1}{(s+1)^{\frac{1}{q}}}\\ &\qquad\times\left\{\left[\frac{(s+1)|f'(a)|^q + |f'(b)|^q}{s+2}\right]^{\frac{1}{q}} + \left[\frac{|f'(a)|^q + (s+1)|f'(b)|^q}{s+2}\right]^{\frac{1}{q}}\right\}. \end{aligned} \tag{4}$$

Proof From Lemma 2.1 and using Hölder–İşcan integral inequality (1), we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\quad\le \frac{b-a}{2}\int_0^1 |1-2t|\left|f'\bigl(ta+(1-t)b\bigr)\right|dt \\
&\quad\le \frac{b-a}{2}\left\{\left(\int_0^1 (1-t)|1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)\left|f'\bigl(ta+(1-t)b\bigr)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\qquad\left.+\left(\int_0^1 t|1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left|f'\bigl(ta+(1-t)b\bigr)\right|^q dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Since $|f'|^q$ is s-convex on $[a,b]$, then

$$\begin{aligned}
\int_0^1 t\left|f'\bigl(ta+(1-t)b\bigr)\right|^q dt &\le \int_0^1 t\left[t^s\left|f'(a)\right|^q+(1-t)^s\left|f'(b)\right|^q\right]dt \\
&= \frac{(s+1)|f'(a)|^q+|f'(b)|^q}{(s+1)(s+2)}
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
\int_0^1 (1-t)\left|f'\bigl(ta+(1-t)b\bigr)\right|^q dt &= \int_0^1 t\left|f'\bigl(tb+(1-t)a\bigr)\right|^q dt \\
&= \frac{|f'(a)|^q+(s+1)|f'(b)|^q}{(s+1)(s+2)},
\end{aligned} \tag{6}$$

and also

$$\begin{aligned}
\int_0^1 t|1-2t|^p\,dt &= \int_0^1 (1-t)|1-2t|^p\,dt \\
&= \frac{1}{2(p+1)}.
\end{aligned} \tag{7}$$

By inequalities (5), (6), and (7), we get inequality (4). □

Corollary 2.1 *Let $f: I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I^\circ$ with $a<b$ and $f' \in L[a,b]$. If $|f'|^q$ is s-convex on $[a,b]$, where $p>1$ such that $q=\frac{p}{p-1}$, then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\quad\le \frac{b-a}{2^{\frac{p+1}{p}}(p+1)^{\frac{1}{p}}}\frac{1}{(s+2)^{\frac{1}{q}}}\left(1+\frac{1}{(s+1)^{\frac{1}{q}}}\right)\left(\left|f'(a)\right|+\left|f'(b)\right|\right).
\end{aligned} \tag{8}$$

Proof Using the fact $\sum_{i=1}^n (a_i+b_i)^k \le \sum (a_i)^k + \sum (b_i)^k$ for $k \in (0,1)$ with $p>1$ such that $q=\frac{p}{p-1}$ completes the proof. □

Remark 2.1 Inequality (4) is better than inequality (3). Indeed, since the function $g : [0,\infty) \to \mathbb{R}$, $g(x) = x^r$, $r \in (0,1]$ is concave, we can write

$$\frac{\alpha^r + \beta^r}{2} = \frac{g(\alpha) + g(\beta)}{2} \le g\left(\frac{\alpha+\beta}{2}\right) = \left(\frac{\alpha+\beta}{2}\right)^r \tag{9}$$

for all $\alpha, \beta \ge 0$. In inequality (9), if we choose

$$\alpha = \frac{(s+1)|f'(a)|^q + |f'(b)|^q}{s+2}, \qquad \beta = \frac{|f'(a)|^q + (s+1)|f'(b)|^q}{s+2}$$

and $r = \frac{1}{q}$, then we have

$$\begin{aligned} &\frac{1}{2}\left[\frac{(s+1)|f'(a)|^q + |f'(b)|^q}{s+2}\right]^{\frac{1}{q}} + \frac{1}{2}\left[\frac{|f'(a)|^q + (s+1)|f'(b)|^q}{s+2}\right]^{\frac{1}{q}} \\ &\quad \le \left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{\frac{1}{q}}. \end{aligned}$$

Thus, we obtain the following inequality:

$$\begin{aligned} &\frac{b-a}{2^{\frac{p+1}{p}}(p+1)^{\frac{1}{p}}} \cdot \frac{1}{(s+1)^{\frac{1}{q}}} \\ &\qquad \times \left\{\left[\frac{(s+1)|f'(a)|^q + |f'(b)|^q}{s+2}\right]^{\frac{1}{q}} + \left[\frac{|f'(a)|^q + (s+1)|f'(b)|^q}{s+2}\right]^{\frac{1}{q}}\right\} \\ &\quad \le \frac{b-a}{2^{\frac{1}{p}}(p+1)^{\frac{1}{p}}} \cdot \frac{1}{(s+1)^{\frac{1}{q}}}\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{\frac{1}{q}} \\ &\quad = \frac{b-a}{2(p+1)^{\frac{1}{p}}} \cdot \frac{1}{(s+1)^{\frac{1}{q}}}\left(\left|f'(a)\right|^q + \left|f'(b)\right|^q\right)^{\frac{1}{q}}. \end{aligned}$$

If we take $s = 1$ in Remark 2.1, we have the following result which was obtained by İşcan in [10].

Remark 2.2 Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$ and $f' \in L[a,b]$. If $|f'|^q$ is convex on $[a,b]$, then

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\quad \le \frac{b-a}{4(p+1)^{\frac{1}{p}}}\left\{\left[\frac{2|f'(a)|^q + |f'(b)|^q}{3}\right]^{\frac{1}{q}} + \left[\frac{|f'(a)|^q + 2|f'(b)|^q}{3}\right]^{\frac{1}{q}}\right\} \\ &\quad \le \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{\frac{1}{q}}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

In [17], Muddassar et al. obtained the following inequality by using Lemma 2.1 and power-mean integral inequality.

Theorem 2.3 *Let $f : I^\circ \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function on I° where $a, b \in I$ with $a < b$ and $f' \in L[a,b]$. If $|f'|^q$ is s-convex on $[a,b]$ for some $p > 1$ such that $q = \frac{p}{p-1}$, then the following inequality holds*:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{b-a}{2^{\frac{p+1}{p}}}\left[\left(\frac{s+2^{-s}}{(s+1)(s+2)}\right)\left(\left|f'(a)\right|^q + \left|f'(b)\right|^q\right)\right]^{\frac{1}{q}}. \tag{10}$$

If Theorem 2.3 is proved again by using improved power-mean integral inequality, then we get the following result.

Theorem 2.4 *Let $f : I^\circ \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function on I° where $a, b \in I$ with $a < b$ and $f' \in L[a,b]$. If $|f'|^q$ is s-convex on $[a,b]$ for some $p > 1$ such that $q = \frac{p}{p-1}$, then the following inequality holds*:

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\quad\le \frac{b-a}{2.4^{\frac{1}{p}}} \cdot \frac{1}{(s+1)^{\frac{1}{q}}} \\
&\qquad\times \left\{\left[\frac{(s+1)(s+1+2^{-s-1})|f'(a)|^q + (s-1+2^{-s-1}(s+5))|f'(b)|^q}{(s+2)(s+3)}\right]^{\frac{1}{q}}\right. \\
&\qquad\left. + \left[\frac{(s-1+2^{-s-1}(s+5))|f'(a)|^q + (s+1)(s+1+2^{-s-1})|f'(b)|^q}{(s+2)(s+3)}\right]^{\frac{1}{q}}\right\}.
\end{aligned} \tag{11}$$

Proof From Lemma 2.1 and applying improved power-mean integral inequality (2) for $q > 1$, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\quad\le \frac{b-a}{2}\left\{\left(\int_0^1 (1-t)|1-2t|\,dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)|1-2t|\left|f'\big(ta+(1-t)b\big)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\qquad\left. + \left(\int_0^1 t|1-2t|\,dt\right)^{\frac{1}{p}}\left(\int_0^1 t|1-2t|\left|f'\big(ta+(1-t)b\big)\right|^q dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

By s-convexity of $|f'|^q$ on $[a,b]$, we have

$$\begin{aligned}
&\int_0^1 t|1-2t|\left|f'\big(ta+(1-t)b\big)\right|^q dt \\
&\quad\le \int_0^1 t|1-2t|\left[t^s\left|f'(a)\right|^q + (1-t)^s\left|f'(b)\right|^q\right] dt \\
&\quad= \frac{(s+1)(s+1+2^{-s-1})|f'(a)|^q + (s-1+2^{-s-1}(s+5))|f'(b)|^q}{(s+1)(s+2)(s+3)}
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
&\int_0^1 (1-t)|1-2t|\left|f'\left(ta+(1-t)b\right)\right|^q dt \\
&\quad = \int_0^1 t|1-2t|\left|f'\left(tb+(1-t)a\right)\right|^q dt \\
&\quad \le \frac{(s-1+2^{-s-1}(s+5))|f'(a)|^q + (s+1)(s+1+2^{-s-1})|f'(b)|^q}{(s+1)(s+2)(s+3)}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\int_0^1 t|1-2t|\,dt &= \int_0^1 (1-t)|1-2t|\,dt \\
&= \frac{1}{4}. \qquad (13)
\end{aligned}$$

By using inequalities (12) and (13), we get inequality (11). □

Remark 2.3 Inequality (11) is better than inequality (10). Indeed, using inequality (9) in Remark 2.1, we have

$$\begin{aligned}
&\frac{1}{2}\left[\frac{(s+1)(s+1+2^{-s-1})|f'(a)|^q + (s-1+2^{-s-1}(s+5))|f'(b)|^q}{(s+2)(s+3)}\right]^{\frac{1}{q}} \\
&\qquad + \frac{1}{2}\left[\frac{(s-1+2^{-s-1}(s+5))|f'(a)|^q + (s+1)(s+1+2^{-s-1})|f'(b)|^q}{(s+2)(s+3)}\right]^{\frac{1}{q}} \\
&\quad \le \left[\left(\frac{s+2^{-s}}{s+2}\right)\left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)\right]^{\frac{1}{q}}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\frac{b-a}{2.4^{\frac{1}{p}}} \cdot \frac{1}{(s+1)^{\frac{1}{q}}} \\
&\qquad \times \left\{\left[\frac{(s+1)(s+1+2^{-s-1})|f'(a)|^q + (s-1+2^{-s-1}(s+5))|f'(b)|^q}{(s+2)(s+3)}\right]^{\frac{1}{q}}\right. \\
&\qquad + \left.\left[\frac{(s-1+2^{-s-1}(s+5))|f'(a)|^q + (s+1)(s+1+2^{-s-1})|f'(b)|^q}{(s+2)(s+3)}\right]^{\frac{1}{q}}\right\} \\
&\quad \le \frac{b-a}{4^{\frac{1}{p}}} \cdot \frac{1}{(s+1)^{\frac{1}{q}}}\left[\left(\frac{s+2^{-s}}{s+2}\right)\left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)\right]^{\frac{1}{q}} \\
&\quad = \frac{b-a}{2^{\frac{p+1}{p}}}\left[\left(\frac{s+2^{-s}}{(s+1)(s+2)}\right)\left(\left|f'(a)\right|^q + \left|f'(b)\right|^q\right)\right]^{\frac{1}{q}}.
\end{aligned}$$

If we take $s = 1$ in Remark 2.3, we have the following result for convex functions which is better than the inequality given in [18, Theorem 1].

Remark 2.4 Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$ and $f' \in L[a,b]$. If $|f'|^q$ is convex on $[a,b]$, then

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\quad\le \frac{b-a}{4.2^{\frac{1}{p}}}\left\{\left[\frac{3|f'(a)|^q + |f'(b)|^q}{4}\right]^{\frac{1}{q}} + \left[\frac{|f'(a)|^q + 3|f'(b)|^q}{4}\right]^{\frac{1}{q}}\right\} \\
&\quad\le \frac{b-a}{4}\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{\frac{1}{q}},
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

3 An application to trapezoidal formula

Let D be a division of the interval $[a,b]$, i.e., $D : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ and consider the quadrature formula

$$I = \int_a^b f(x)\,dx = T(f,D) + R(f,D),$$

where

$$T(f,D) = \sum_{k=0}^{n-1} \frac{f(x_k) + f(x_{k+1})}{2}(x_{k+1} - x_k)$$

is the trapezoidal formula and $R(f,D)$ denotes the associated approximation error of the integral I.

Proposition 3.1 *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$ with $a < b$ and $f' \in L[a,b]$. If $|f'|^q$ is s-convex on $[a,b]$ for $p, q > 1$, for every division D of $[a,b]$, the following trapezoidal error estimate holds*:

$$\begin{aligned}
\left|R(f,D)\right| &\le \frac{1}{2^{\frac{p+1}{p}}(p+1)^{\frac{1}{p}}} \cdot \frac{1}{(s+2)^{\frac{1}{q}}}\left(1 + \frac{1}{(s+1)^{\frac{1}{q}}}\right) \\
&\quad\times \sum_{k=0}^{n-1}(x_{k+1} - x_k)^2\left(\left|f'(x_k)\right| + \left|f'(x_{k+1})\right|\right).
\end{aligned}$$

Proof Applying Corollary 2.1 on the subintervals $[x_k, x_{k+1}]$, $(k = 0, 1, 2, \ldots, n-1)$ of the division D, we have

$$\begin{aligned}
&\left|\frac{f(x_k) + f(x_{k+1})}{2} - \frac{1}{x_{k+1} - x_k}\int_{x_k}^{x_{k+1}} f(x)\,dx\right| \\
&\quad\le \frac{x_{k+1} - x_k}{2^{\frac{p+1}{p}}(p+1)^{\frac{1}{p}}} \cdot \frac{1}{(s+2)^{\frac{1}{q}}}\left(1 + \frac{1}{(s+1)^{\frac{1}{q}}}\right)\left(\left|f'(x_k)\right| + \left|f'(x_{k+1})\right|\right).
\end{aligned}$$

Summing over k from 0 to $n-1$ and using the generalized triangle inequality, we get

$$\begin{aligned} \left|R(f,D)\right| &= \left|T(f,D) - \int_a^b f(x)\,dx\right| \\ &= \left|\sum_{k=0}^{n-1}\left((x_{k+1}-x_k)\frac{f(x_k)+f(x_{k+1})}{2} - \int_{x_k}^{x_{k+1}} f(x)\,dx\right)\right| \\ &\leq \sum_{k=0}^{n-1}\left|(x_{k+1}-x_k)\frac{f(x_k)+f(x_{k+1})}{2} - \int_{x_k}^{x_{k+1}} f(x)\,dx\right|. \end{aligned}$$

So, we have

$$\left|R(f,D)\right| \leq \sum_{k=0}^{n-1}(x_{k+1}-x_k)\left|\frac{f(x_k)+f(x_{k+1})}{2} - \frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}} f(x)\,dx\right|,$$

which is the required result. □

4 An application to special means

In [6], Hudzik and Maligranda gave the following example.

Let $a,b,c \in \mathbb{R}$ and $s \in (0,1)$. For $t \in [0,\infty)$, define the function $f : [0,\infty) \to \mathbb{R}$ as

$$f(t) = \begin{cases} a, & t = 0, \\ bt^s + c, & t > 0. \end{cases}$$

If $b \geq 0$ and $0 \leq c \leq a$, then $f \in K_s^2$. So, we have $f : [0,1] \to [0,1]$, $f(t) = t^s$, $f \in K_s^2$ for $a = c = 0$ and $b = 1$.

Let us recall the following special means for arbitrary real numbers a and b ($a \neq b$).

(1) The arithmetic mean:

$$A := A(a,b) = \frac{a+b}{2}.$$

(2) The p-logarithmic mean:

$$L_p := L_p(a,b) = \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, \quad p \in \mathbb{R}\backslash\{-1,0\}.$$

Proposition 4.1 *Let $p > 1$, $a < b$, $s \in (0,1)$, and $q = \frac{p}{p-1}$. Then we have*

$$\begin{aligned} &\left|A^s(a,b) - L_s^s(a,b)\right| \\ &\quad\leq \frac{b-a}{2^{\frac{p+1}{p}}(p+1)^{\frac{1}{p}}} \cdot \frac{s}{(s+2)^{\frac{1}{q}}}\left(1 + \frac{1}{(s+1)^{\frac{1}{q}}}\right)\left(|a|^{s-1} + |b|^{s-1}\right). \end{aligned}$$

Proof By Corollary 2.1 applied for the s-convex function $f : [0,1] \to [0,1]$, $f(x) = x^s$ for $s \in (0,1)$, the proof is completed. □

5 Conclusion

In this paper, some new results of Hermite–Hadamard type for s-convex functions are established. It is shown that the results obtained here are better than the known results. Some applications of these results to trapezoidal formula and to special means have also been presented. The results of this paper may stimulate further research for the researchers working in this field.

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