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Some new inequalities for (k, s)-fractional integrals

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Abstract

In this paper, the (k, s)-fractional integral operator is used to generate new classes of integral inequalities using a family of n positive functions, $(n \in \mathbb{N})$. Two classes of integral inequalities involving the (k, s)-fractional integral operator are derived here and these results allow us in particular to generalize some classical inequalities. Certain interesting consequent results of the main theorems are also pointed out.

Keywords: Integral inequalities, fractional integral inequalities, (k, s)-fractional integrals.
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1. Introduction and preliminaries

Fractional calculus and its wide application have recently been paid to an ever increasing extent attentions. In mathematical analysis, the fractional calculus is a very helpful tool to perform differentiation and integration with the real number or complex number powers of the differential or integral operators. A detailed account of fractional calculus operators along with their properties and applications can be found in the research monographs by Miller and Ross [18] and Kiryakova [15]. It is fairly well-known that there are a number of different definitions of fractional integrals and their applications. Each definition has its own advantages and suitable for applications to different type of problems. Recently, Atangana and Baleanu

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[1] added one more dimension to this study by proposing a derivative that is based upon the generalized Mittag-Leffler function, since the Mittag-Leffler function is more suitable in expressing nature than power function. For the more recent development of fractional calculus, we refer the reader to the recent papers [2, 3, 10, 14, 22, 23, 30].

Integral inequalities are taken up to be important as these are useful in the study of different classes of differential and integral equations (see [19]). During the past several years, many researchers have obtained various fractional integral inequalities comprising the different fractional differential and integral operators. This subject has earned the attention of many researchers and mathematicians during the last few decades. There is a large number of the fractional integral operators discussed in the literature, but because of their applications in many fields of sciences, the Riemann-Liouville and Hadamard fractional integral operators have been studied extensively [7, 8, 11, 15, 16, 21, 29, 31]. Further, for inequalities involving generalized fractional operators one can see [4–6, 24–28].

Recently, fractional k-fractional integral operators have been investigated in the literature by some authors. For this, we begin with the following properties in the literature. The Pochhammer k-symbol $(x)_{n,k}$ and the k-gamma function Γ_k are defined as follows (see [9]):

$$(x)_{n,k} := x(x+k)(x+2k)\cdots(x+(n-1)k)\,, \quad (n\in\mathbb{N}, \quad k>0)\,, \tag{1.1}$$

and

$$\Gamma_k(x) := \lim_{n\to\infty}\frac{n!\,k^n\,(nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, \quad \left(k>0, \quad x\in\mathbb{C}\setminus k\mathbb{Z}_0^-\right), \tag{1.2}$$

where $k\mathbb{Z}_0^- := \left\{kn : n\in\mathbb{Z}_0^-\right\}$. It is noted that the case $k=1$ of (1.1) and (1.2) reduces to the familiar Pochhammer symbol $(x)_n$ and the gamma function Γ. The function Γ_k is given by the following integral:

$$\Gamma_k(x) = \int_0^\infty t^{x-1}\,e^{-\frac{t^k}{k}}\,dt, \quad (\Re(x)>0).$$

The function Γ_k defined on $\mathbb{R}^+$ is characterized by the following three properties:

(i) $\Gamma_k(x+k) = x\,\Gamma_k(x)$;

(ii) $\Gamma_k(k) = 1$;

(iii) $\Gamma_k(x)$ is logarithmically convex.

It is easy to see that

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\,\Gamma\left(\frac{x}{k}\right), \quad (\Re(x)>0, \quad k>0)\,.$$

We want to recall the preliminaries and notations of some well-known fractional integral operators that will be used to obtain some remarks and corollaries.

The (k,s)-Riemann-Liouville fractional integral operator ${}_k^s\mathcal{J}_a^\alpha$ of order $\alpha>0$ for a real-valued continuous function $f(t)$ is defined as (see [28, p. 79, Definition 2.1.]):

$${}_k^s\mathcal{J}_a^\alpha f(t) = \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^x (x^{s+1}-t^{s+1})^{\frac{\alpha}{k}-1}t^s f(t)dt, \tag{1.3}$$

where $k>0$, $\beta>0$ and $s\in\mathbb{R}\setminus\{-1\}$.

Throughout this paper, we will obtain our main results by assuming $s\in\mathbb{R}^+\setminus\{-1\}$ instead of $s\in\mathbb{R}\setminus\{-1\}$.

2. Main results

In this section, we prove two classes of integral inequalities involving (k,s)-fractional integral operator. These results allow us in particular to generalize some classical inequalities.

Theorem 2.1. *Let $(f_i)_{i=1,\dots,n}$ be n positive continuous and decreasing functions on $[a,b]$ and $a < t \leq b$, $\alpha > 0$, $k > 0$, $s \in \mathbb{R}^+ \setminus \{-1\}$, $\delta > 0$, $\zeta \geq \gamma_p > 0$. Then, the following inequality holds true*

$$\frac{{}_k^s\mathcal{J}_a^\alpha\left[\prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\zeta}(t)\right]}{{}_k^s\mathcal{J}_a^\alpha\left[\prod_{i=1}^n f_i^{\gamma_i}(t)\right]} \geq \frac{{}_k^s\mathcal{J}_a^\alpha\left[(t-a)^\delta\prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\zeta}(t)\right]}{{}_k^s\mathcal{J}_a^\alpha\left[(t-a)^\delta\prod_{i=1}^n f_i^{\gamma}(t)\right]}\ , \tag{2.1}$$

where p is any fixed integer in $\{1,2,...,n\}$.

Proof. Since $(f_i)_{i=1,\dots,n}$ are n positive continuous and decreasing functions on $[a\ ,b]$, then we have

$$\left((\rho-a)^\delta-(\tau-a)^\delta\right)\left(f_p^{\zeta-\gamma_p}(\tau)-f_p^{\zeta-\gamma_p}(\rho)\right)\geq 0$$

for any fixed $p\in\{1,\dots,n\}$ and $\zeta \geq \gamma_p > 0$, $\delta > 0$, $\tau,\ \rho \in [a,t]\,;\ a < t \leq b$.

Let us consider the following functional,

$$\begin{aligned}\mathcal{F}_s^\alpha(t,\tau) &= \frac{(s+1)^{1-\frac{\alpha}{k}}\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\tau^s}{k\Gamma_k(\alpha)}\\ &\quad\times\prod_{i=1}^n f_i^{\gamma_i}(\tau)\left((\rho-a)^\delta-(\tau-a)^\delta\right)\left(f_p^{\zeta-\gamma_p}(\tau)-f_p^{\zeta-\gamma_p}(\rho)\right),\end{aligned}$$

where $\alpha, k > 0,\ t > a$ and $s \in \mathbb{R}^+ \setminus \{-1\}$

We observe that each factor of the above functional is positive in view of the conditions stated with Theorem 2.1, and hence, the function $\mathcal{F}_s^\alpha(t,\tau)$ remains positive for all $\tau \in (a,\, t)$ $(t > a)$.

Since $\mathcal{F}_s^\alpha(t,\tau) \geq 0$ and by using the definition of (k,s)-fractional integral, then we have

$$\begin{aligned}0 \leq \int_a^t \mathcal{F}_s^\alpha(t,\tau)d\tau &= (\rho-a)^\delta\,{}_k^s\mathcal{J}_a^\alpha\left[\prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\zeta}(t)\right] + f_p^{\zeta-\gamma_p}(\rho)\,{}_k^s\mathcal{J}_a^\alpha\left[(t-a)^\delta\prod_{i=1}^n f_i^{\gamma_i}(t)\right]\\ &\quad-(\rho-a)^\delta f_p^{\zeta-\gamma_p}(\rho)\,{}_k^s\mathcal{J}_a^\alpha\left[\prod_{i=1}^n f_i^{\gamma_i}(t)\right]-{}_k^s\mathcal{J}_a^\alpha\left[(t-a)^\delta\prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\zeta}(t)\right].\end{aligned} \tag{2.2}$$

By multiplying both sides of (2.2) by $\frac{(s+1)^{1-\frac{\alpha}{k}}(t^{s+1}-\rho^{s+1})^{\frac{\alpha}{k}-1}\rho^s}{k\Gamma_k(\alpha)}$ and integrating with respect to ρ from a to t, and by using the definition of (k,s)-fractional integral, we get

$${}_k^s\mathcal{J}_a^\alpha\left[\prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\zeta}(t)\right]{}_k^s\mathcal{J}_a^\alpha\left[(t-a)^\delta\prod_{i=1}^n f_i^{\gamma_i}(t)\right]\geq {}_k^s\mathcal{J}_a^\alpha\left[(t-a)^\delta\prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\zeta}(t)\right]{}_k^s\mathcal{J}_a^\alpha\left[\prod_{i=1}^n f_i^{\gamma_i}(t)\right],$$

which arrives the result (2.1). □

Remark 2.2. The inequality (2.1) is reversed, if the functions $(f_i)_{i=1,\dots,n}$ are increasing on $[a,b]$. Further, on setting $s=0$, $\alpha=k=n=1$ and $t=b$, Theorem 2.1 leads to the known Theorem 3 due to [17, p. 205].

Theorem 2.3. *Let $(f_i)_{i=1,\dots,n}$ be n positive continuous and decreasing functions on $[a,b]$ and $a < t \leq b$, $\alpha, \beta > 0$, $k > 0$, $s \in \mathbb{R}^+ \setminus \{-1\}$, $\zeta \geq \gamma_p > 0$, where p is a fixed integer in $\{1,2,\dots,n\}$. Then, the following inequality holds true*

$$\frac{{}_k^s\mathcal{J}_a^\alpha\left[\prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\zeta}(t)\right]{}_k^s\mathcal{J}_a^\beta\left[(t-a)^\delta\prod_{i=1}^n f_i^{\gamma_i}(t)\right]+{}_k^s\mathcal{J}_a^\beta\left[\prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\zeta}(t)\right]{}_k^s\mathcal{J}_a^\alpha\left[(t-a)^\delta\prod_{i=1}^n f_i^{\gamma_i}(t)\right]}{{}_k^s\mathcal{J}_a^\alpha\left[(t-a)^\delta\prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\zeta}(t)\right]{}_k^s\mathcal{J}_a^\beta\left[\prod_{i=1}^n f_i^{\gamma_i}(t)\right]+{}_k^s\mathcal{J}_a^\beta\left[(t-a)^\delta\prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\zeta}(t)\right]{}_k^s\mathcal{J}_a^\alpha\left[\prod_{i=1}^n f_i^{\gamma_i}(t)\right]}\geq 1.$$

Proof. By multiplying both side of (2.2) by $\frac{(s+1)^{1-\frac{\beta}{k}}(t^{s+1}-\rho^{s+1})^{\frac{\alpha}{k}-1}\rho^s}{k\Gamma_k(\beta)}\prod_{i=1}^n f_i^{\gamma_i}(\rho)$, then integrating the resulting inequality with respect to ρ over (a,t), $a<t\leq b$ and by using Fubini's theorem, we get

$$\begin{aligned}
0 \leq &\int_a^t\int_a^t \frac{(s+1)^{1-\frac{\beta}{k}}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\beta}{k}-1}\rho^s}{k\Gamma_k(\beta)}\prod_{i=1}^n f_i^{\gamma_i}(\rho)\mathcal{F}_s^{\alpha}(t,\tau)d\tau d\rho\\
=&{}_k^s\mathcal{J}_a^{\alpha}\left[\prod_{i\neq p}^n f_i^{\gamma_i}f_p^{\zeta}(t)\right]{}_k^s\mathcal{J}_a^{\beta}\left[(t-a)^{\delta}\prod_{i=1}^n f_i^{\gamma_i}(t)\right]\\
&+{}_k^s\mathcal{J}_a^{\beta}\left[\prod_{i\neq p}^n f_i^{\gamma_i}f_p^{\zeta}(t)\right]{}_k^s\mathcal{J}_a^{\alpha}\left[(t-a)^{\delta}\prod_{i=1}^n f_i^{\gamma_i}(t)\right]\\
&-{}_k^s\mathcal{J}_a^{\alpha}\left[(t-a)^{\delta}\prod_{i\neq p}^n f_i^{\gamma_i}f_p^{\zeta}(t)\right]{}_k^s\mathcal{J}_a^{\beta}\left[\prod_{i=1}^n f_i^{\gamma_i}(t)\right]\\
&-{}_k^s\mathcal{J}_a^{\beta}\left[(t-a)^{\delta}\prod_{i\neq p}^n f_i^{\gamma_i}f_p^{\zeta}(t)\right]{}_k^s\mathcal{J}_a^{\alpha}\left[\prod_{i=1}^n f_i^{\gamma_i}(t)\right].
\end{aligned}$$

This completes the proof of Theorem 2.3. □

Remark 2.4. It may be noted that for $\alpha=\beta$, Theorem 2.3 immediately reduces to Theorem 2.1. Again, for $s=0$, $\alpha=\beta=k=n=1$ and $t=b$, Theorem 2.3 reduces to Theorem 3 of [17, p. 205].

Now, we consider another class of fractional integral inequalities which generalizes the above theorems, in the following manner.

Theorem 2.5. *Let $(f_i)_{i=1,\dots,n}$ and g be continuous functions on $[a,b]$ such that g is increasing and $(f_i)_{i=1,\dots,n}$ are decreasing on $[a,b]$ and $\alpha>0$, $k>0$, $s\in\mathbb{R}^+\setminus\{-1\}$, $a<t\leq b$, $\zeta\geq\gamma_p>0$ where p is a fixed integer in $\{1,2,\dots,n\}$. Then, the following inequality holds true*

$$\frac{{}_k^s\mathcal{J}_a^{\alpha}\left[\prod_{i\neq p}^n f_i^{\gamma_i}f_p^{\zeta}(t)\right]{}_k^s\mathcal{J}_a^{\alpha}\left[g^{\delta}(t)\prod_{i=1}^n f_i^{\gamma_i}(t)\right]}{{}_k^s\mathcal{J}_a^{\alpha}\left[g^{\delta}(t)\prod_{i\neq p}^n f_i^{\gamma_i}f_p^{\zeta}(t)\right]{}_k^s\mathcal{J}_a^{\alpha}\left[\prod_{i=1}^n f_i^{\gamma_i}(t)\right]}\geq 1.$$

Proof. Under the valid condition of Theorem 2.5, we can write

$$\left(g^{\delta}(\rho)-g^{\delta}(\tau)\right)\left(f_p^{\zeta-\gamma_p}(\tau)-f_p^{\zeta-\gamma_p}(\rho)\right)\geq 0$$

for all $p=1,\dots,n$, $a<t\leq b$, $\alpha>0$, $k>0$, $s\in\mathbb{R}^+\setminus\{-1\}$, $\delta>0$, $\zeta\geq\gamma_p>0$, $\tau,\rho\in[a\ ,b]$.

Now, let us consider the quantity

$$\begin{aligned}
\mathcal{L}_{k,s}(t;\tau,\rho)=&\frac{(s+1)^{1-\frac{\alpha}{k}}\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\tau^s}{k\Gamma_k(\alpha)}\\
&\times\prod_{i=1}^n f_i^{\gamma_i}(\tau)\left(\left(g^{\delta}(\rho)-g^{\delta}(\tau)\right)\right)\left(f_p^{\zeta-\gamma_p}(\tau)-f_p^{\zeta-\gamma_p}(\rho)\right).
\end{aligned}$$

It is clear that

$$L_{k,s}(t;\tau,\rho)\geq 0,$$

therefore,

$$0 \leq \int_0^t \mathcal{L}_{k,s}(t;\tau,\rho)\,d\tau = g^{\delta}(\rho)\, {}_k^s\mathcal{J}_a^{\alpha}\left[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] + f_p^{\zeta-\gamma_p}(\rho)\, {}_k^s\mathcal{J}_a^{\alpha}\left[g^{\delta}(t)\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right] \tag{2.3}$$
$$- {}_k^s\mathcal{J}_a^{\alpha}\left[g^{\delta}(t)\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] - g^{\delta}(\rho) f_p^{\zeta-\gamma_p}(\rho)\, {}_k^s\mathcal{J}_a^{\alpha}\left[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right].$$

Consequently,

$${}_k^s\mathcal{J}_a^{\alpha}\left[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] {}_k^s\mathcal{J}_a^{\alpha}\left[g^{\delta}(t)\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right] \geq {}_k^s\mathcal{J}_a^{\alpha}\left[g^{\delta}(t)\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] {}_k^s\mathcal{J}_a^{\alpha}\left[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right].$$

Theorem 2.5 is thus proved. □

Another generalization of Theorem 2.5 is as below.

Theorem 2.6. *Let $(f_i)_{i=1,\dots,n}$ and g be positive continuous functions on $[a,b]$, such that g is increasing and $(f_i)_{i=1,\dots,n}$ are decreasing on $[a,b]$. Then for any fixed $p \in \{1,2,\dots,n\}$ and for all $a < t \leq b$, $\alpha,\beta > 0$, $k > 0$, $s \in \mathbb{R}^+ \setminus \{-1\}$, $\delta > 0$, $\zeta \geq \gamma_p > 0$ we have*

$$\frac{{}_k^s\mathcal{J}_a^{\alpha}\left[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] {}_k^s\mathcal{J}_a^{\beta}\left[g^{\delta}(t)\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right] + {}_k^s\mathcal{J}_a^{\beta}\left[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] {}_k^s\mathcal{J}_a^{\alpha}\left[g^{\delta}(t)\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]}{{}_k^s\mathcal{J}_a^{\alpha}\left[g(t)\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] {}_k^s\mathcal{J}_a^{\beta}\left[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right] + {}_k^s\mathcal{J}_a^{\beta}\left[g(t)\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] {}_k^s\mathcal{J}_a^{\alpha}\left[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]} \geq 1.$$

Proof. By using (2.3) we can write

$$0 \leq \int_a^t \int_a^t \frac{(s+1)^{1-\frac{\beta}{k}} \left(t^{s+1}-\rho^{s+1}\right)^{\frac{\beta}{k}-1} \rho^s}{k\Gamma_k(\beta)} \prod_{i=1}^{n} f_i^{\gamma_i}(\rho) \mathcal{L}_{k,s}(t;\tau,\rho)\,d\tau d\rho$$
$$= {}_k^s\mathcal{J}_a^{\alpha}\left[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] {}_k^s\mathcal{J}_a^{\beta}\left[g^{\delta}(t)\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]$$
$$+ {}_k^s\mathcal{J}_a^{\beta}\left[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] {}_k^s\mathcal{J}_a^{\alpha}\left[g^{\delta}(t)\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]$$
$$- {}_k^s\mathcal{J}_a^{\alpha}\left[g(t)\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] {}_k^s\mathcal{J}_a^{\beta}\left[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]$$
$$- {}_k^s\mathcal{J}_a^{\beta}\left[g(t)\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] {}_k^s\mathcal{J}_a^{\alpha}\left[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right].$$

On some simplification, this completes the proof of Theorem 2.6. □

Remark 2.7. For $\alpha = \beta$, Theorem 2.6 immediately reduces to Theorem 2.5. Further, by applying Theorem 2.5 for $s = 0$, $\alpha = \beta = k = n = 1$ and $t = b$, we obtain Theorem 4 of [17, p. 206].

Theorem 2.8. *Let $(f_i)_{i=1,\dots,n}$ and g be continuous functions on $[a,b]$. Suppose that for any fixed $p \in \{1,2,...,n\}$, $\left(f_p^{\delta}(\tau) g^{\delta}(\rho) - f_p^{\delta}(\rho) g^{\delta}(\tau)\right)\left(f_p^{\zeta-\gamma_p}(\tau) - f_p^{\zeta-\gamma_p}(\rho)\right) \geq 0$, $\delta > 0$, $\alpha > 0$, $k > 0$, $s \in \mathbb{R}^+\{-1\} \setminus \{-1\}$, $\zeta \geq \gamma_p > 0$, $\tau,\rho \in [a,t]$, $t \in (a,b]$, then we have*

$$\frac{{}_k^s\mathcal{J}_a^{\alpha}\left[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta+\delta}(t)\right]\ {}_k^s\mathcal{J}_a^{\alpha}\left[g^{\delta}(t)\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]}{{}_k^s\mathcal{J}_a^{\alpha}\left[g^{\delta}(t)\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] {}_k^s\mathcal{J}_a^{\alpha}\left[f_p^{\delta}\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]} \geq 1.$$

Proof. The proof is quite similar to Theorem 2.5, provided if we replace the quantity

$$\left(g^{\delta}\left(\rho\right)-g^{\delta}\left(\tau\right)\right) \text{ by } \left(f_p^{\delta}\left(\tau\right)g^{\delta}\left(\rho\right)-f_p^{\delta}(\rho)g^{\delta}\left(\tau\right)\right).$$

□

Theorem 2.9. *Let $(f_i)_{i=1,\ldots,n}$ and g be positive continuous functions on $[a,b]$. Suppose that for any fixed $p\in\{1,2,\ldots,n\}$, $\left(f_p^{\delta}\left(\tau\right)g^{\delta}\left(\rho\right)-f_p^{\delta}(\rho)g^{\delta}\left(\tau\right)\right)\left(f_p^{\zeta-\gamma_p}\left(\tau\right)-f_p^{\zeta-\gamma_p}(\rho)\right)\geq 0$, $\delta>0$, $\alpha>0$, $\zeta\geq\gamma_p>0$, $\tau,\rho,\in[a,t]$, $t\in(a,b]$, then the inequality*

$$\frac{{}_k^s\mathcal{J}_a^{\alpha}\left[\prod_{i\neq p}^n f_i^{\gamma_i}f_p^{\zeta+\delta}(t)\right]{}_k^s\mathcal{J}_a^{\beta}\left[g^{\delta}\left(t\right)\prod_{i=1}^n f_i^{\gamma_i}(t)\right]+{}_k^s\mathcal{J}_a^{\beta}\left[\prod_{i\neq p}^n f_i^{\gamma_i}f_p^{\zeta+\delta}(t)\right]{}_k^s\mathcal{J}_a^{\alpha}\left[g^{\delta}\left(t\right)\prod_{i=1}^n f_i^{\gamma_i}(t)\right]}{{}_k^s\mathcal{J}_a^{\alpha}\left[g^{\delta}\left(t\right)\prod_{i\neq p}^n f_i^{\gamma_i}f_p^{\zeta+\delta}(t)\right]{}_k^s\mathcal{J}_a^{\beta}\left[f_p^{\delta}\prod_{i=1}^n f_i^{\gamma_i}(t)\right]+{}_k^s\mathcal{J}_a^{\beta}\left[g^{\delta}\left(t\right)\prod_{i\neq p}^n f_i^{\gamma_i}f_p^{\zeta}(t)\right]{}_k^s\mathcal{J}_a^{\alpha}\left[\prod_{i=1}^n f_i^{\gamma_i}(t)\right]}\geq 1$$

holds true.

Proof. The proof is quite similar to Theorem 2.6, provided if we replace the quantity

$$\left(g^{\delta}\left(\rho\right)-g^{\delta}\left(\tau\right)\right) \text{ by } \left(f_p^{\delta}\left(\tau\right)g^{\delta}\left(\rho\right)-f_p^{\delta}(\rho)g^{\delta}\left(\tau\right)\right).$$

□

Remark 2.10. Again it is interesting to observe that, for $\alpha=\beta$ Theorem 2.9 immediately reduces to Theorem 2.5. Also, by applying Theorem 2.9 for $s=0,\alpha=\beta=k=n=1$ and $t=b$, we obtain the known results [17, Theorem 5, p. 207].

3. Special cases

The most important feature of (k,s)-fractional integrals is that they generalize some types of fractional integrals (Riemann-Liouville fractional integral, k-Riemann-Liouville fractional integral, generalized fractional integral and Hadamard fractional integral). These important special cases of the integral operator ${}_k^s\mathcal{J}_a^{\alpha}$ are mentioned below:

1. For $k=1$, the operator in (1.3) yields the following generalized fractional integrals defined by Katugompola in [12]:
$$ {}_a^r\mathcal{I}_t^{\alpha}f(t)=\frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)}\int_a^x(x^{r+1}-t^{r+1})^{\alpha-1}t^rf(t)dt.$$
2. Firstly by taking $k=1$, after that by taking limit $r\to-1^+$ and by using L'hospital's rule, the operator in (1.3) leads to Hadamard fractional integral operator [11]. That is,
$$\begin{aligned}\lim_{r\to-1^+}{}_a^r\mathcal{I}_t^{\alpha}f(t)&=\lim_{r\to-1^+}\frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)}\int_a^x\frac{f(t)t^r}{(x^{r+1}-t^{r+1})^{1-\alpha}}dt\\&=\frac{1}{\Gamma(\alpha)}\int_a^x\lim_{r\to-1^+}f(t)t^r\left(\frac{r+1}{x^{r+1}-t^{r+1}}\right)^{1-\alpha}dt\\&=\frac{1}{\Gamma(\alpha)}\int_a^x f(t)\lim_{r\to-1^+}\left(\frac{r+1}{x^{r+1}-t^{r+1}}\right)^{1-\alpha}\frac{dt}{t}\\&=\frac{1}{\Gamma(\alpha)}\int_a^x f(t)\left(\lim_{r\to-1^+}\frac{r+1}{x^{r+1}-t^{r+1}}\right)^{1-\alpha}\frac{dt}{t}\\&=\frac{1}{\Gamma(\alpha)}\int_a^x\left(\log\frac{x}{t}\right)f(t)\frac{dt}{t}=\ {}_H\mathcal{J}^{\alpha}[f(t)],\end{aligned}$$
(see [13, p. 569, eqn. (3.13)]).

3. If we take $s = 0$ in (1.3), the operator (1.3), reduces to the k-Riemann-Liouville fractional integral operator, which firstly defined by Mubeen and Habibullah in [20], this relation is as follows:

$$\mathcal{I}_{a,k}^{\alpha} f(t) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt.$$

4. Again, by taking $s = 0$ and $k = 1$, the operator (1.3) gives us the Riemann-Liouville fractional integration operator:

$$\mathcal{R}^{\alpha}(f(t)) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt.$$

Now, by suitably choosing the values of parameters k and s the results presented in this paper may generate some more known and possibly new inequalities involving the various type of integral operator.

4. Conclusion

In this paper, we introduced new classes of integral inequalities involving the (k, s)-fractional integral operators. It is interesting to mention here that, whenever the (k, s)-fractional integral operators reduces to the other-related operators (by suitably choosing the values of parameters k and s), the results become relatively more important from the application viewpoint. We conclude this paper with the remark that the fractional integral inequalities derived in Section 2 can fruitfully be used in establishing uniqueness of solutions in fractional boundary value problems.

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