Filomat 31:15 (2017), 4989–4997
https://doi.org/10.2298/FIL1715989C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some New Inequalities of Simpson's Type for s-convex Functions via Fractional Integrals

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Abstract. In this paper, we establish some new inequalities of Simpson's type based on s-convexity via fractional integrals. Our results generalize the results obtained by Sarikaya et al. [1].

1. Introduction and Preliminaries

It is well known that the following inequality, named Simpson's inequality, is one of the best known results in the literature.

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right]\right| \leq \frac{1}{2880}\|f^{(4)}\|_\infty (b-a)^4,$$

where $f:[a,b]\to\mathfrak{R}$ be a four times continuously differentiable mapping on (a,b) and $\|f^{(4)}\|_\infty = \sup_{x\in(a,b)}|f^{(4)}(x)| < \infty$.

In [2], the class of functions which are s-convex in the second sense has been introduced by Breckner as the following.

Definition 1.1. *Let s be a real number $s\in(0,1]$. A function $f:[0,\infty)\to\mathfrak{R}$, is said to be s-convex (in the second sense), or f belongs to the class K_s^2, if*

$$f(\lambda x+(1-\lambda)y)\leq \lambda^s f(x)+(1-\lambda)^s f(y)$$

holds for all $x,y\in[0,\infty)$ and $\lambda\in[0,1]$.

It can be easily seen that for $s=1$, s-convexity reduces to ordinary convexity of functions defined on $[0,\infty)$.

In [3], Dragomir and Fitzpatrick proved a variant of Hermite-Hadamard's inequality which holds for s-convex functions in the second sense. In [13], the authors proved some new integral inequalities of these classes of functions via $(h-(\alpha,m))$-logarithmically convexity.

2010 *Mathematics Subject Classification*. 26D15, 26D10
Keywords. Simpson's type; s-convex functions; fractional integrals.
Received: 30 August 2016; Accepted: 09 December 2016
Communicated by Dragan S. Djordjević
This work supported by the National Natural Science Foundation of China (11461043 and 11661053) and supported partly by the Provincial Natural Science Foundation of Jiangxi, China (20161BAB201009) and the Science and Technology Project of Educational Commission of Jiangxi Province, China (150013) and Hunan Provincial Innovation Foundation for Postgraduates (No. CX2016B037).
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Theorem 1.2. *Suppose that $f : [0,\infty) \to [0,\infty)$ is an s-convex function in the second sense, where $s \in (0,1)$, and let $a, b \in [0,\infty)$, $a < b$. If $f \in L_1([a,b])$, then the following inequalities hold:*

$$2^{s-1} f(\frac{a+b}{2}) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{s+1}. \tag{1}$$

The following Lemma is proved by Sarikaya et al.(see [1]).

Lemma 1.3. *Let $f : I \to \Re$ be an absolutely continuous mapping on I^o such that $f' \in L_1([a,b])$, where $a, b \in I^o$ with $a < b$. Then the following equality holds:*

$$\begin{aligned}&\frac{1}{6}\Big[f(a)+4f(\frac{a+b}{2})+f(b)\Big]-\frac{1}{b-a}\int_a^b f(x)dx\\ &=\frac{b-a}{2}\int_0^1\Big[\Big(\frac{t}{2}-\frac{1}{3}\Big)f'\Big(\frac{1+t}{2}b+\frac{1-t}{2}a\Big)+\Big(\frac{1}{3}-\frac{t}{2}\Big)f'\Big(\frac{1+t}{2}a+\frac{1-t}{2}b\Big)\Big]dt.\end{aligned} \tag{2}$$

Using Lemma 1.3, Sarikaya et al. in [1] established the following results which hold for s-convex functions in the second sense.

Theorem 1.4. *Let $f : I \subset [0,\infty) \to \Re$ be a differentiable mapping on I^o such that $f' \in L_1([a,b])$, where $a, b \in I^o$ with $a < b$. If $|f'|$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$, then the following inequality holds:*

$$\begin{aligned}&\Big|\frac{1}{6}\Big[f(a)+4f(\frac{a+b}{2})+f(b)\Big]-\frac{1}{b-a}\int_a^b f(x)dx\Big|\\ &\le (b-a)\frac{(s-4)6^{s+1}+2\times 5^{s+2}-2\times 3^{s+2}+2}{6^{s+2}(s+1)(s+2)}\Big[|f'(a)|+|f'(b)|\Big].\end{aligned} \tag{3}$$

Theorem 1.5. *Let $f : I \subset [0,\infty) \to \Re$ be a differentiable mapping on I^o such that $f' \in L_1([a,b])$, where $a, b \in I^o$ with $a < b$. If $|f'|^q$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$ and $q > 1$, then the following inequality holds:*

$$\begin{aligned}&\Big|\frac{1}{6}\Big[f(a)+4f(\frac{a+b}{2})+f(b)\Big]-\frac{1}{b-a}\int_a^b f(x)dx\Big|\\ &\le \frac{(b-a)}{12}\Big(\frac{1+2^{p+1}}{3(p+1)}\Big)^{\frac{1}{p}}\Big\{\Big(\frac{|f'(b)|^q+|f'(\frac{a+b}{2})|^q}{s+1}\Big)^{\frac{1}{q}}+\Big(\frac{|f'(a)|^q+|f'(\frac{a+b}{2})|^q}{s+1}\Big)^{\frac{1}{q}}\Big\},\end{aligned} \tag{4}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 1.6. *Let $f : I \subset [0,\infty) \to \Re$ be a differentiable mapping on I^o such that $f' \in L_1([a,b])$, where $a, b \in I^o$ with $a < b$. If $|f'|^q$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$ and $q > 1$, then the following inequality holds:*

$$\begin{aligned}&\Big|\frac{1}{6}\Big[f(a)+4f(\frac{a+b}{2})+f(b)\Big]-\frac{1}{b-a}\int_a^b f(x)dx\Big|\\ &\le \frac{(b-a)}{12}\Big(\frac{1+2^{p+1}}{3(p+1)}\Big)^{\frac{1}{p}}\Big\{\Big(\frac{(2^{s+1}-1)|f'(b)|^q+|f'(a)|^q}{2^s(s+1)}\Big)^{\frac{1}{q}}+\Big(\frac{(2^{s+1}-1)|f'(a)|^q+|f'(b)|^q}{2^s(s+1)}\Big)^{\frac{1}{q}}\Big\},\end{aligned} \tag{5}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 1.7. *Let $f : I \subset [0,\infty) \to \Re$ be a differentiable mapping on I^o such that $f' \in L_1([a,b])$, where $a, b \in I^o$ with $a < b$. If $|f'|^q$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$ and $q > 1$, then the following inequality holds:*

$$\begin{aligned}&\Big|\frac{1}{6}\Big[f(a)+4f(\frac{a+b}{2})+f(b)\Big]-\frac{1}{b-a}\int_a^b f(x)dx\Big|\\ &\le \frac{(b-a)}{2}(\frac{5}{36})^{1-\frac{1}{q}}\Big\{\Big(\frac{2\times 5^{s+2}+(s-4)6^{s+1}-(2s+7)3^{s+1}}{3\times 6^{s+1}(s+1)(s+2)}|f'(b)|^q+\frac{(2s+1)3^{s+1}+2}{3\times 6^{s+1}(s+1)(s+2)}|f'(a)|^q\Big)^{\frac{1}{q}}\\ &\quad+\Big(\frac{(2s+1)3^{s+1}+2}{3\times 6^{s+1}(s+1)(s+2)}|f'(b)|^q+\frac{2\times 5^{s+2}+(s-4)6^{s+1}-(2s+7)3^{s+1}}{3\times 6^{s+1}(s+1)(s+2)}|f'(a)|^q\Big)^{\frac{1}{q}}\Big\},\end{aligned} \tag{6}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

In the following, we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. For more details, one can consult [4-5].

Definition 1.8. *Let* $f \in L_1([a,b])$. *The Riemann-Liouville integrals* $J^{\alpha}_{a+}f$ *and* $J^{\alpha}_{b-}f$ *of order* $\alpha > 0$ *with* $a \geq 0$ *are defined by*

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad x < b$$

respectively, where $\Gamma(\alpha) = \int_0^\infty e^{-t}u^{\alpha-1}du$ *is Euler gamma function. Here* $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. For some recent results connected with fractional integral inequalities, see [6-12]. For more fractional integral applications, please see [14-21]

The aim of this paper is to establish some new inequalities for s-convex functions in the second sense via Riemann-Liouville fractional integral. Our results generalize the results obtained by Sarikaya [1] and provide new estimates on these types of inequalities for fractional integrals.

2. Main Results

In this section, we introduce some inequalities via fractional integrals. First, a new identity is presented as follows:

Lemma 2.1. *Let* $f : I \to \Re$ *be an absolutely continuous mapping on* I^o *such that* $f' \in L_1([a,b])$, *where* $a, b \in I^o$ *with* $a < b$. *Then the following equality holds:*

$$\begin{aligned}&\frac{1}{6}\Big[f(a) + 4f(\frac{a+b}{2}) + f(b)\Big] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\Big[J^{\alpha}_{b-}f(\frac{a+b}{2}) + J^{\alpha}_{a+}f(\frac{a+b}{2})\Big]\\ &= \frac{b-a}{2}\int_0^1\Big[\Big(\frac{t^{\alpha}}{2} - \frac{1}{3}\Big)f'\Big(\frac{1+t}{2}b + \frac{1-t}{2}a\Big) + \Big(\frac{1}{3} - \frac{t^{\alpha}}{2}\Big)f'\Big(\frac{1+t}{2}a + \frac{1-t}{2}b\Big)\Big]dt.\end{aligned} \tag{7}$$

Proof. It suffices to note that

$$\begin{aligned}I &= \int_0^1\Big[\Big(\frac{t^{\alpha}}{2} - \frac{1}{3}\Big)f'\Big(\frac{1+t}{2}b + \frac{1-t}{2}a\Big) + \Big(\frac{1}{3} - \frac{t^{\alpha}}{2}\Big)f'\Big(\frac{1+t}{2}a + \frac{1-t}{2}b\Big)\Big]dt\\ &= \int_0^1\Big(\frac{t^{\alpha}}{2} - \frac{1}{3}\Big)f'\Big(\frac{1+t}{2}b + \frac{1-t}{2}a\Big)dt + \int_0^1(\frac{1}{3} - \frac{t^{\alpha}}{2}\Big)f'\Big(\frac{1+t}{2}a + \frac{1-t}{2}b\Big)dt\\ &= I_1 + I_2.\end{aligned} \tag{8}$$

Integrating by parts

$$
\begin{aligned}
I_1 &= \int_0^1 \Big(\frac{t^\alpha}{2}-\frac{1}{3}\Big) f'\Big(\frac{1+t}{2}b+\frac{1-t}{2}a\Big)dt\\
&= \frac{2}{b-a}\int_0^1 \Big(\frac{t^\alpha}{2}-\frac{1}{3}\Big) d\Big(f(\frac{1+t}{2}b+\frac{1-t}{2}a)\Big)\\
&= \frac{2}{b-a}\Big\{\Big[(\frac{t^\alpha}{2}-\frac{1}{3})f(\frac{1+t}{2}b+\frac{1-t}{2}a)\Big]\Big|_0^1-\int_0^1 f(\frac{1+t}{2}b+\frac{1-t}{2}a)d(\frac{t^\alpha}{2}-\frac{1}{3})\Big\}\\
&= \frac{2}{b-a}\Big[\frac{1}{6}f(b)+\frac{1}{3}f(\frac{a+b}{2})-\int_0^1 f(\frac{1+t}{2}b+\frac{1-t}{2}a)d(\frac{t^\alpha}{2}-\frac{1}{3})\Big]\\
&= \frac{2}{b-a}\Big[\frac{1}{6}f(b)+\frac{1}{3}f(\frac{a+b}{2})-\frac{\alpha}{2}\int_0^1 t^{\alpha-1}f(\frac{1+t}{2}b+\frac{1-t}{2}a)dt\Big]\\
&= \frac{2}{b-a}\Big[\frac{1}{6}f(b)+\frac{1}{3}f(\frac{a+b}{2})-\frac{\alpha}{2}\int_{\frac{a+b}{2}}^b (\frac{x-\frac{a+b}{2}}{\frac{b-a}{2}})^{\alpha-1}f(x)\frac{1}{\frac{b-a}{2}}dx\Big]\\
&= \frac{2}{b-a}\Big[\frac{1}{6}f(b)+\frac{1}{3}f(\frac{a+b}{2})-\frac{\alpha}{2}(\frac{2}{b-a})^\alpha\int_{\frac{a+b}{2}}^b (x-\frac{a+b}{2})^{\alpha-1}f(x)dx\Big]\\
&= \frac{2}{b-a}\Big[\frac{1}{6}f(b)+\frac{1}{3}f(\frac{a+b}{2})-\frac{\Gamma(\alpha+1)}{2}(\frac{2}{b-a})^\alpha\frac{1}{\Gamma(\alpha)}\int_{\frac{a+b}{2}}^b (x-\frac{a+b}{2})^{\alpha-1}f(x)dx\Big]\\
&= \frac{2}{b-a}\Big[\frac{1}{6}f(b)+\frac{1}{3}f(\frac{a+b}{2})-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}J^\alpha_{b-}f(\frac{a+b}{2})\Big],
\end{aligned}
\tag{9}
$$

and similarly we have,

$$
\begin{aligned}
I_2 &= \int_0^1 \Big(\frac{1}{3}-\frac{t^\alpha}{2}\Big) f'\Big(\frac{1+t}{2}a+\frac{1-t}{2}b\Big)dt\\
&= \frac{2}{b-a}\Big[\frac{1}{6}f(a)+\frac{1}{3}f(\frac{a+b}{2})-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}J^\alpha_{a+}f(\frac{a+b}{2})\Big].
\end{aligned}
\tag{10}
$$

From (8), (9) and (10), we have conclusion (7). This completes the proof. □

Remark 2.2. *In Lemma 2.1 , if $\alpha = 1$, then we obtain Lemma 1.3.*

Using this lemma, we can obtain the following fractional integral inequalities which a new result of Simpson's inequality for s-convex functions.

Theorem 2.3. *Let $f : I \subset [0,\infty) \to \Re$ be a differentiable mapping on I^o such that $f' \in L_1([a,b])$, where $a, b \in I^o$ with $a < b$. If $|f'|$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$, then the following inequality holds:*

$$
\begin{aligned}
&\Big|\frac{1}{6}\Big[f(a)+4f(\frac{a+b}{2})+f(b)\Big]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\Big[J^\alpha_{b-}f(\frac{a+b}{2})+J^\alpha_{a+}f(\frac{a+b}{2})\Big]\Big|\\
&\le \frac{b-a}{2^{s+1}}\Big[|f'(a)|+|f'(b)|\Big]\{I_3(\alpha,s)+I_4(\alpha,s)\},
\end{aligned}
\tag{11}
$$

where

$$
\begin{aligned}
I_3(\alpha,s) &= \int_0^{(\frac{2}{3})^{\frac{1}{\alpha}}} (\frac{1}{3}-\frac{t^\alpha}{2})[(1+t)^s+(1-t)^s]dt,\\
I_4(\alpha,s) &= \int_{(\frac{2}{3})^{\frac{1}{\alpha}}}^1 (\frac{t^\alpha}{2}-\frac{1}{3})[(1+t)^s+(1-t)^s]dt.
\end{aligned}
\tag{12}
$$

Proof. From Lemma 2.1 and since $|f'|$ is s-convex on $[a,b]$, we get

$$
\begin{aligned}
&\Big|\frac{1}{6}\Big[f(a)+4f(\frac{a+b}{2})+f(b)\Big]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\Big[J^{\alpha}_{b-}f(\frac{a+b}{2})+J^{\alpha}_{a+}f(\frac{a+b}{2})\Big]\Big|\\
&\le \frac{b-a}{2}\int_0^1\Big[\Big|\frac{t^{\alpha}}{2}-\frac{1}{3}\Big|\Big|f'(\frac{1+t}{2}b+\frac{1-t}{2}a)\Big|+\Big|\frac{1}{3}-\frac{t^{\alpha}}{2}\Big|\Big|f'(\frac{1+t}{2}a+\frac{1-t}{2}b)\Big|\Big]dt\\
&\le \frac{b-a}{2}\int_0^1\Big[\Big|\frac{t^{\alpha}}{2}-\frac{1}{3}\Big|\Big|f'(\frac{1+t}{2}b+(1-\frac{1+t}{2})a)\Big|+\Big|\frac{1}{3}-\frac{t^{\alpha}}{2}\Big|\Big|f'(\frac{1+t}{2}a+(1-\frac{1+t}{2}b))\Big|\Big]dt\\
&= \frac{b-a}{2}\int_0^1\Big[\Big|\frac{t^{\alpha}}{2}-\frac{1}{3}\Big|\Big|f'(\frac{1+t}{2}b+(1-\frac{1+t}{2})a)\Big|\Big]dt+\int_0^1\Big[\Big|\frac{1}{3}-\frac{t^{\alpha}}{2}\Big|\Big|f'(\frac{1+t}{2}a+(1-\frac{1+t}{2}b))\Big|\Big]dt\\
&\le \frac{b-a}{2}\int_0^1\Big[\Big|\frac{t^{\alpha}}{2}-\frac{1}{3}\Big|\Big|(\frac{1+t}{2})^{s}|f'(b)|+(1-\frac{1+t}{2})^{s}|f'(a)|\Big|\Big]dt\\
&\quad+\int_0^1\Big[\Big|\frac{1}{3}-\frac{t^{\alpha}}{2}\Big|\Big|(\frac{1+t}{2})^{s}|f'(a)|+(1-\frac{1+t}{2})^{s}|f'(b)|\Big|\Big]dt\\
&\le \frac{b-a}{2^{s+1}}\Big[|f'(a)|+|f'(b)|\Big]\Big\{\int_0^1\Big|\frac{t^{\alpha}}{2}-\frac{1}{3}\Big|[(1+t)^{s}+(1-t)^{s}]dt\Big\}\\
&\le \frac{b-a}{2^{s+1}}\Big[|f'(a)|+|f'(b)|\Big]\Big\{\int_0^{(\frac{2}{3})^{\frac{1}{\alpha}}}(\frac{1}{3}-\frac{t^{\alpha}}{2})[(1+t)^{s}+(1-t)^{s}]dt+\int_{(\frac{2}{3})^{\frac{1}{\alpha}}}^1(\frac{t^{\alpha}}{2}-\frac{1}{3})[(1+t)^{s}+(1-t)^{s}]dt\Big\}\\
&\le \frac{b-a}{2^{s+1}}\Big[|f'(a)|+|f'(b)|\Big]\{I_3(\alpha,s)+I_4(\alpha,s)\},
\end{aligned}
$$

where $I_3(\alpha,s)$ and $I_4(\alpha,s)$ are defined in (12). This completes the proof. □

Corollary 2.4. *Let $f: I\subset[0,\infty)\to\Re$ be a differentiable mapping on I^{o} such that $f'\in L_1([a,b])$, where $a,b\in I^{o}$ with $a<b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$
\begin{aligned}
&\Big|\frac{1}{6}\Big[f(a)+4f(\frac{a+b}{2})+f(b)\Big]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\Big[J^{\alpha}_{b-}f(\frac{a+b}{2})+J^{\alpha}_{a+}f(\frac{a+b}{2})\Big]\Big|\\
&\le \frac{b-a}{4}\Big[|f'(a)|+|f'(b)|\Big]\{I_3(\alpha,1)+I_4(\alpha,1)\}.
\end{aligned}
\tag{13}
$$

Proof. Setting $s=1$ in (11), we get the required result. □

Remark 2.5. *In Theorem 2.3 , if $\alpha=1$, then we obtain Theorem 1.4.*

In the following theorem, we shall propose a new upper bound for the right-hand side of Simpsons inequality for s-convex mapping with fractional integral type.

Theorem 2.6. *Let $f: I\subset[0,\infty)\to\Re$ be a differentiable mapping on I^{o} such that $f'\in L_1([a,b])$, where $a,b\in I^{o}$ with $a<b$. If $|f'|^{q}$ is s-convex on $[a,b]$, for some fixed $s\in(0,1]$ and $q>1$, then the following inequality holds:*

$$
\begin{aligned}
&\Big|\frac{1}{6}\Big[f(a)+4f(\frac{a+b}{2})+f(b)\Big]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\Big[J^{\alpha}_{b-}f(\frac{a+b}{2})+J^{\alpha}_{a+}f(\frac{a+b}{2})\Big]\Big|\\
&\le \frac{(b-a)}{2}\Big(\int_0^1\Big|\frac{t^{\alpha}}{2}-\frac{1}{3}\Big|^{p}dt\Big)^{\frac{1}{p}}\Big\{\Big(\frac{|f'(b)|^{q}+|f'(\frac{a+b}{2})|^{q}}{s+1}\Big)^{\frac{1}{q}}+\Big(\frac{|f'(a)|^{q}+|f'(\frac{a+b}{2})|^{q}}{s+1}\Big)^{\frac{1}{q}}\Big\},
\end{aligned}
\tag{14}
$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.1 and Hölder's inequality, we get

$$\begin{aligned}&\Big|\frac{1}{6}\Big[f(a)+4f(\frac{a+b}{2})+f(b)\Big]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\Big[J^{\alpha}_{b-}f(\frac{a+b}{2})+J^{\alpha}_{a+}f(\frac{a+b}{2})\Big]\Big|\\&=\frac{b-a}{2}\int_0^1\Big[\Big|\frac{t^{\alpha}}{2}-\frac{1}{3}\Big|\Big|f'(\frac{1+t}{2}b+\frac{1-t}{2}a)\Big|+\Big|\frac{1}{3}-\frac{t^{\alpha}}{2}\Big|\Big|f'(\frac{1+t}{2}a+\frac{1-t}{2}b)\Big|\Big]dt\\&\le\frac{b-a}{2}\Big\{(\int_0^1|\frac{t^{\alpha}}{2}-\frac{1}{3}|^p dt)^{\frac{1}{p}}\Big(\int_0^1\Big|f'(\frac{1+t}{2}b+\frac{1-t}{2}a)\Big|^q dt\Big)^{\frac{1}{q}}\\&\quad+(\int_0^1|\frac{t^{\alpha}}{2}-\frac{1}{3}|^p dt)^{\frac{1}{p}}\Big(\int_0^1\Big|f'(\frac{1+t}{2}a+\frac{1-t}{2}b)\Big|^q dt\Big)^{\frac{1}{q}}\Big\}.\end{aligned}$$

Since $|f'|^q$ is s-convex on $[a,b]$, by using in (1), we get

$$\int_0^1\Big|f'(\frac{1+t}{2}b+\frac{1-t}{2}a)\Big|^q dt\le\frac{|f'(b)|^q+|f'(\frac{a+b}{2})|^q}{s+1}$$

and

$$\int_0^1\Big|f'(\frac{1+t}{2}a+\frac{1-t}{2}b)\Big|^q dt\le\frac{|f'(a)|^q+|f'(\frac{a+b}{2})|^q}{s+1}.$$

Hence

$$\begin{aligned}&\Big|\frac{1}{6}\Big[f(a)+4f(\frac{a+b}{2})+f(b)\Big]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\Big[J^{\alpha}_{b-}f(\frac{a+b}{2})+J^{\alpha}_{a+}f(\frac{a+b}{2})\Big]\Big|\\&\le\frac{(b-a)}{2}\Big(\int_0^1|\frac{t^{\alpha}}{2}-\frac{1}{3}|^p dt\Big)^{\frac{1}{p}}\Big\{\Big(\frac{|f'(b)|^q+|f'(\frac{a+b}{2})|^q}{s+1}\Big)^{\frac{1}{q}}+\Big(\frac{|f'(a)|^q+|f'(\frac{a+b}{2})|^q}{s+1}\Big)^{\frac{1}{q}}\Big\}.\end{aligned}$$

This completes the proof. □

Corollary 2.7. *Let $f: I\subset[0,\infty)\to\Re$ be a differentiable mapping on I^o such that $f'\in L_1([a,b])$, where $a,b\in I^o$ with $a<b$. If $|f'|^q$ is convex on $[a,b]$ and $q>1$, then the following inequality holds:*

$$\begin{aligned}&\Big|\frac{1}{6}\Big[f(a)+4f(\frac{a+b}{2})+f(b)\Big]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\Big[J^{\alpha}_{b-}f(\frac{a+b}{2})+J^{\alpha}_{a+}f(\frac{a+b}{2})\Big]\Big|\\&\le\frac{(b-a)}{2}\Big(\int_0^1|\frac{t^{\alpha}}{2}-\frac{1}{3}|^p dt\Big)^{\frac{1}{p}}\Big\{\Big(\frac{|f'(b)|^q+|f'(\frac{a+b}{2})|^q}{2}\Big)^{\frac{1}{q}}+\Big(\frac{|f'(a)|^q+|f'(\frac{a+b}{2})|^q}{2}\Big)^{\frac{1}{q}}\Big\},\end{aligned}\tag{15}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Setting $s=1$ in (14), we get the required result. □

Remark 2.8. *In Theorem 2.6, if $\alpha=1$, then we obtain Theorem 1.5.*

Next, we shall give another versions of Simpson's type inequality for s-convex functions with fractional integral type as follows:

Theorem 2.9. *Let $f: I\subset[0,\infty)\to\Re$ be a differentiable mapping on I^o such that $f'\in L_1([a,b])$, where $a,b\in I^o$ with $a<b$. If $|f'|^q$ is s-convex on $[a,b]$, for some fixed $s\in(0,1]$ and $q>1$, then the following inequality holds:*

$$\begin{aligned}&\Big|\frac{1}{6}\Big[f(a)+4f(\frac{a+b}{2})+f(b)\Big]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\Big[J^{\alpha}_{b-}f(\frac{a+b}{2})+J^{\alpha}_{a+}f(\frac{a+b}{2})\Big]\Big|\\&\le\frac{(b-a)}{2}\Big(\int_0^1|\frac{t^{\alpha}}{2}-\frac{1}{3}|^p dt\Big)^{\frac{1}{p}}\Big\{\Big(\frac{(2^{s+1}-1)|f'(b)|^q+|f'(a)|^q}{2^s(s+1)}\Big)^{\frac{1}{q}}+\Big(\frac{(2^{s+1}-1)|f'(a)|^q+|f'(b)|^q}{2^s(s+1)}\Big)^{\frac{1}{q}}\Big\},\end{aligned}\tag{16}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.1 and Hölder's inequality, we get

$$\begin{aligned}
&\Big|\frac{1}{6}\Big[f(a)+4f(\frac{a+b}{2})+f(b)\Big]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\Big[J^{\alpha}_{b-}f(\frac{a+b}{2})+J^{\alpha}_{a+}f(\frac{a+b}{2})\Big]\Big|\\
&=\frac{b-a}{2}\int_0^1\Big[\Big|\frac{t^{\alpha}}{2}-\frac{1}{3}\Big|\Big|f'(\frac{1+t}{2}b+\frac{1-t}{2}a)\Big|+\Big|\frac{1}{3}-\frac{t^{\alpha}}{2}\Big|\Big|f'(\frac{1+t}{2}a+\frac{1-t}{2}b)\Big|\Big]dt\\
&\leq\frac{b-a}{2}\Big\{\Big(\int_0^1|\frac{t^{\alpha}}{2}-\frac{1}{3}|^p dt\Big)^{\frac{1}{p}}\Big(\int_0^1\Big|f'(\frac{1+t}{2}b+\frac{1-t}{2}a)\Big|^q dt\Big)^{\frac{1}{q}}\\
&\quad+\Big(\int_0^1|\frac{t^{\alpha}}{2}-\frac{1}{3}|^p dt\Big)^{\frac{1}{p}}\Big(\int_0^1\Big|f'(\frac{1+t}{2}a+\frac{1-t}{2}b)\Big|^q dt\Big)^{\frac{1}{q}}\Big\}.
\end{aligned}\tag{17}$$

Since $|f'|^q$ is s-convex on $[a,b]$, we know that for $t\in[0,1]$ and $s\in(0,1]$

$$\Big|f'(\frac{1+t}{2}b+\frac{1-t}{2}a)\Big|^q\leq(\frac{t+1}{2})^s|f'(b)|^q+(\frac{1-t}{2})^s|f'(a)|^q.\tag{18}$$

From (17) and (18), we have

$$\begin{aligned}
&\Big|\frac{1}{6}\Big[f(a)+4f(\frac{a+b}{2})+f(b)\Big]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\Big[J^{\alpha}_{b-}f(\frac{a+b}{2})+J^{\alpha}_{a+}f(\frac{a+b}{2})\Big]\Big|\\
&\leq\frac{b-a}{2}\Big\{\Big(\int_0^1|\frac{t^{\alpha}}{2}-\frac{1}{3}|^p dt\Big)^{\frac{1}{p}}\Big(\int_0^1\Big|f'(\frac{1+t}{2}b+\frac{1-t}{2}a)\Big|^q dt\Big)^{\frac{1}{q}}\\
&\quad+\Big(\int_0^1|\frac{t^{\alpha}}{2}-\frac{1}{3}|^p dt\Big)^{\frac{1}{p}}\Big(\int_0^1\Big|f'(\frac{1+t}{2}a+\frac{1-t}{2}b)\Big|^q dt\Big)^{\frac{1}{q}}\Big\}.\\
&\leq\frac{b-a}{2}\Big\{\Big(\int_0^1|\frac{t^{\alpha}}{2}-\frac{1}{3}|^p dt\Big)^{\frac{1}{p}}\Big(\int_0^1\Big|(\frac{t+1}{2})^s|f'(b)|^q+(\frac{1-t}{2})^s|f'(a)|^q dt\Big)^{\frac{1}{q}}\\
&\quad+\Big(\int_0^1|\frac{t^{\alpha}}{2}-\frac{1}{3}|^p dt\Big)^{\frac{1}{p}}\Big(\int_0^1(\frac{t+1}{2})^s|f'(a)|^q+(\frac{1-t}{2})^s|f'(b)|^q dt\Big)^{\frac{1}{q}}\Big\}\\
&=\frac{b-a}{2}\Big(\int_0^1|\frac{t^{\alpha}}{2}-\frac{1}{3}|^p dt\Big)^{\frac{1}{p}}\Big\{\Big(\int_0^1\Big(\frac{t+1}{2})^s|f'(b)|^q+(\frac{1-t}{2})^s|f'(a)|^q dt\Big)^{\frac{1}{q}}\\
&\quad+\Big(\int_0^1(\frac{t+1}{2})^s|f'(a)|^q+(\frac{1-t}{2})^s|f'(b)|^q dt\Big)^{\frac{1}{q}}\Big\}\\
&\leq\frac{(b-a)}{2}\Big(\int_0^1|\frac{t^{\alpha}}{2}-\frac{1}{3}|^p dt\Big)^{\frac{1}{p}}\Big\{\Big(\frac{(2^{s+1}-1)|f'(b)|^q+|f'(a)|^q}{2^s(s+1)}\Big)^{\frac{1}{q}}+\Big(\frac{(2^{s+1}-1)|f'(a)|^q+|f'(b)|^q}{2^s(s+1)}\Big)^{\frac{1}{q}}\Big\}.
\end{aligned}$$

This completes the proof. □

Corollary 2.10. *Let $f:I\subset[0,\infty)\to\Re$ be a differentiable mapping on I^o such that $f'\in L_1([a,b])$, where $a,b\in I^o$ with $a<b$. If $|f'|^q$ is convex on $[a,b]$ and $q>1$, then the following inequality holds:*

$$\begin{aligned}
&\Big|\frac{1}{6}\Big[f(a)+4f(\frac{a+b}{2})+f(b)\Big]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\Big[J^{\alpha}_{b-}f(\frac{a+b}{2})+J^{\alpha}_{a+}f(\frac{a+b}{2})\Big]\Big|\\
&\leq\frac{(b-a)}{2}\Big(\int_0^1|\frac{t^{\alpha}}{2}-\frac{1}{3}|^p dt\Big)^{\frac{1}{p}}\Big\{\Big(\frac{3|f'(b)|^q+|f'(a)|^q}{4}\Big)^{\frac{1}{q}}+\Big(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\Big)^{\frac{1}{q}}\Big\},
\end{aligned}\tag{19}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Setting $s=1$ in (16), we get the required result. □

Remark 2.11. *In Theorem 2.9, if $\alpha=1$, then we obtain Theorem 1.6.*

Theorem 2.12. *Let $f : I \subset [0,\infty) \to \Re$ be a differentiable mapping on I^o such that $f' \in L_1([a,b])$, where $a,b \in I^o$ with $a < b$. If $|f'|^q$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$ and $q > 1$, then the following inequality holds:*

$$\begin{aligned}&\Big|\frac{1}{6}\Big[f(a)+4f(\frac{a+b}{2})+f(b)\Big]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\Big[J^\alpha_{b-}f(\frac{a+b}{2})+J^\alpha_{a+}f(\frac{a+b}{2})\Big]\Big|\\&\le \frac{b-a}{2}I_5(\alpha,s)\times\{I_6(\alpha,s)^{\frac{1}{q}}+I_7(\alpha,s)^{\frac{1}{q}}\},\end{aligned}\tag{20}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and

$$\begin{aligned}I_5(\alpha,s)&=(\int_0^1|\frac{t^\alpha}{2}-\frac{1}{3}|dt)^{1-\frac{1}{q}}=(\int_0^1|\frac{1}{3}-\frac{t^\alpha}{2}|dt)^{1-\frac{1}{q}},\\I_6(\alpha,s)&=\int_0^1|\frac{t^\alpha}{2}-\frac{1}{3}|\Big((\frac{1+t}{2})^s|f'(b)|^q+(\frac{1-t}{2})^s|f'(a)|^q\Big)dt,\\I_7(\alpha,s)&=\int_0^1|\frac{1}{3}-\frac{t^\alpha}{2}|\Big((\frac{1+t}{2})^s|f'(a)|^q+(\frac{1-t}{2})^s|f'(b)|^q\Big)dt.\end{aligned}\tag{21}$$

Proof. Using Lemma 2.1 , the power mean inequality and Hölder's inequality, we have

$$\begin{aligned}&\Big|\frac{1}{6}\Big[f(a)+4f(\frac{a+b}{2})+f(b)\Big]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\Big[J^\alpha_{b-}f(\frac{a+b}{2})+J^\alpha_{a+}f(\frac{a+b}{2})\Big]\Big|\\&=\frac{b-a}{2}\int_0^1\Big[\Big|\frac{t^\alpha}{2}-\frac{1}{3}\Big|\Big|f'(\frac{1+t}{2}b+\frac{1-t}{2}a)\Big|+\Big|\frac{1}{3}-\frac{t^\alpha}{2}\Big|\Big|f'(\frac{1+t}{2}a+\frac{1-t}{2}b)\Big|\Big]dt\\&\le\frac{b-a}{2}\Big\{(\int_0^1|\frac{t^\alpha}{2}-\frac{1}{3}|dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1|\frac{t^\alpha}{2}-\frac{1}{3}||f'(\frac{1+t}{2}b+\frac{1-t}{2}a)|^qdt\Big)^{\frac{1}{q}}\\&+(\int_0^1|\frac{t^\alpha}{2}-\frac{1}{3}|dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1|\frac{t^\alpha}{2}-\frac{1}{3}||f'(\frac{1+t}{2}b+\frac{1-t}{2}a)|^qdt\Big)^{\frac{1}{q}}\Big\}\end{aligned}\tag{22}$$

Since $|f'|^q$ is s-convex on $[a,b]$, we know that for $t \in [0,1]$ and $s \in (0,1]$

$$\Big|f'(\frac{1+t}{2}b+\frac{1-t}{2}a)\Big|^q\le(\frac{t+1}{2})^s|f'(b)|^q+(\frac{1-t}{2})^s|f'(a)|^q.\tag{23}$$

From (22) and (23), we have

$$\begin{aligned}&\Big|\frac{1}{6}\Big[f(a)+4f(\frac{a+b}{2})+f(b)\Big]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\Big[J^\alpha_{b-}f(\frac{a+b}{2})+J^\alpha_{a+}f(\frac{a+b}{2})\Big]\Big|\\&\le\frac{b-a}{2}\Big\{(\int_0^1|\frac{t^\alpha}{2}-\frac{1}{3}|dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1|\frac{t^\alpha}{2}-\frac{1}{3}|\Big[(\frac{1+t}{2})^s|f'(b)|^q+(\frac{1-t}{2})^s|f'(a)|^q\Big]dt\Big)^{\frac{1}{q}}\\&+(\int_0^1|\frac{t^\alpha}{2}-\frac{1}{3}|dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1|\frac{t^\alpha}{2}-\frac{1}{3}|\Big[(\frac{1+t}{2})^s|f'(a)|^q+(\frac{1-t}{2})^s|f'(b)|^q\Big]dt\Big)^{\frac{1}{q}}\Big\}\\&=\frac{b-a}{2}I_5(\alpha,s)\times\{I_6(\alpha,s)^{\frac{1}{q}}+I_7(\alpha,s)^{\frac{1}{q}}\}\end{aligned}$$

where $I_5(\alpha,s)$, $I_6(\alpha,s)$ and $I_7(\alpha,s)$ are defined in (21). This completes the proof. □

Corollary 2.13. *Let $f : I \subset [0,\infty) \to \Re$ be a differentiable mapping on I^o such that $f' \in L_1([a,b])$, where $a,b \in I^o$ with $a < b$. If $|f'|^q$ is convex on $[a,b]$ and $q > 1$, then the following inequality holds:*

$$\begin{aligned}&\Big|\frac{1}{6}\Big[f(a)+4f(\frac{a+b}{2})+f(b)\Big]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\Big[J^\alpha_{b-}f(\frac{a+b}{2})+J^\alpha_{a+}f(\frac{a+b}{2})\Big]\Big|\\&\le\frac{b-a}{2}I_5(\alpha,1)\times\{I_6(\alpha,1)^{\frac{1}{q}}+I_7(\alpha,1)^{\frac{1}{q}}\},\end{aligned}\tag{24}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Setting $s = 1$ in (20), we get the required result. □

Remark 2.14. *In Theorem 2.12, if $\alpha = 1$, then we obtain Theorem 1.7.*

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