Turk. J. Math. Comput. Sci.
14(1)(2022) 129–137

DOI : 10.47000/tjmcs.1014841

Some New Inequalities via Berezin Numbers

Mualla Birgül Huban[1,*], Hamdullah Başaran[2], Mehmet Gürdal[2]

[1]*Isparta University of Applied Sciences, Isparta, Turkey.*
[2]*Department of Mathematics, Suleyman Demirel University, 32260 Isparta, Turkey.*

Received: 26-10-2021 • Accepted: 10-03-2022

Abstract. The Berezin transform $\widetilde{T}$ and the Berezin radius of an operator T on the reproducing kernel Hilbert space $\mathcal{H}(Q)$ over some set Q with the reproducing kernel K_η are defined, respectively, by

$$\widetilde{T}(\eta) = \left\langle T\frac{K_\eta}{\|K_\eta\|}, \frac{K_\eta}{\|K_\eta\|}\right\rangle, \ \eta \in Q \text{ and } \mathrm{ber}(T) := \sup_{\eta\in Q}\left|\widetilde{T}(\eta)\right|.$$

We study several sharp inequalities by using this bounded function $\widetilde{T}$, involving powers of the Berezin radius and the Berezin norms of reproducing kernel Hilbert space operators. We also give some inequalities regarding the Berezin transforms of sum of two operators.

2010 AMS Classification: 47A30, 47A63

Keywords: Reproducing kernel Hilbert space, Berezin transform, Berezin radius, Jensen's inequality, mixed Schwarz inequality.

1. Introduction

In this paper, we present some inequalities for Berezin transforms of some operators on the reproducing kernel Hilbert space $\mathcal{H}(Q)$ over some set Q. By using Berezin transforms, we study several sharp inequalities involving powers of Berezin radius of some operators.

A reproducing kernel Hilbert space (shortly, RKHS) is the Hilbert space $\mathcal{H} = \mathcal{H}(Q)$ of complex-valued functions on some set Q such that:

(a) the evaluation functionals

$$\varphi_\eta(f) = f(\eta), \eta \in Q,$$

are continuous on $\mathcal{H}$;

(b) for every $\eta \in Q$ there exists a function $f_\eta \in \mathcal{H}$ such that $f_\eta(\eta) \neq 0$.

Then, via the classical Riesz representation theorem, we know if $\mathcal{H}$ is an RKHS on Q, there is a unique element $K_\eta \in \mathcal{H}$ such that $h(\eta) = \left\langle h, K_\eta\right\rangle_{\mathcal{H}}$ for every $\eta \in Q$ and all $h \in \mathcal{H}$. The element K_η is called the reproducing kernel at η. Further, we will denote the normalized reproducing kernel at η as $k_\eta := \frac{K_\eta}{\|K_\eta\|}$. Let $\mathcal{L}(\mathcal{H})$ denote the C^*-algebra of

---

*Corresponding Author
Email addresses: muallahuban@isparta.edu.tr (M.B. Huban), 07hamdullahbasaran@gmail.com (H. Başaran), gurdalmehmet@sdu.edu.tr (M. Gürdal)

all bounded linear operators on a complex Hilbert space $(\mathcal{H}, \langle ., . \rangle)$ with the identity operator $1_{\mathcal{H}}$ in $\mathcal{L}(\mathcal{H})$. The Berezin transform (symbol) of T is the complex-valued function on Q defined by

$$\widetilde{T}(\eta) := \left\langle T k_\eta, k_\eta \right\rangle,$$

for an operator $T \in \mathcal{L}(\mathcal{H})$.

The concept of the Berezin transform was initiated by F. Berezin in [4].

It is obvious that, the Berezin transform $\widetilde{T}$ is a bounded function on Q and $\sup_{\eta \in Q} \left|\widetilde{T}(\eta)\right|$, which is called the Berezin radius (number) of operator T [22,23], does not exceed $\|T\|$, i.e.,

$$\mathrm{ber}(T) := \sup_{\eta \in Q} \left|\widetilde{T}(\eta)\right| \leq \|T\|.$$

It is also clear from the definition of Berezin transform that, the range of the Berezin transform $\widetilde{T}$, which is said to be the Berezin set of operator T, lies in the numerical range $W(T)$ of operator T, i.e.,

$$\mathrm{Ber}(T) := \mathrm{Range}\left(\widetilde{T}\right) = \left\{\widetilde{T}(\eta) : \eta \in Q\right\} \subset W(T) := \{\langle Tx, x \rangle : x \in \mathcal{H} \text{ and } \|x\| = 1\}$$

which implies that $\mathrm{ber}(T) \leq w(T) := \sup_{\|x\|=1} |\langle Tx, x \rangle|$ (numerical radius of operator T).

Berezin set and Berezin radius of operators are new numerical characteristics of operators on the RKHS which are introduced by Karaev in [22]. For the basic properties and facts on these new concepts, see [1,3,24,32].

It is well-known that

$$\mathrm{ber}(T) \leq w(T) \leq \|T\|$$

and

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\|, \tag{1.1}$$

for any $T \in \mathcal{L}(\mathcal{H})$. For more information about the numerical radius, one can refer to [6–8,17,18,28,29,31].

In [20], Huban et al. gave the following inequality for $T \in \mathcal{L}(\mathcal{H})$ as :

$$\frac{1}{4}\|T^*T + TT^*\| \leq (\mathrm{ber}(T))^2 \leq \frac{1}{2}\|T^*T + TT^*\|. \tag{1.2}$$

Now, let $T = T_1 + iT_2$ be the Cartesian decomposition of T. Then T_1 and T_2 are self-adjoint, and $T^*T + TT^* = 2\left(T_1^2 + T_2^2\right)$. Thus, the inequalities in (1.2) can be written as

$$\frac{1}{4}\left\|T_1^2 + T_2^2\right\|_{\mathrm{ber}} \leq (\mathrm{ber}(T))^2 \leq \frac{1}{2}\left\|T_1^2 + T_2^2\right\|_{\mathrm{ber}}, \tag{1.3}$$

or equivalently, as

$$\frac{1}{4}\left\|(T_1 + T_2)^2 + (T_1 - T_2)^2\right\|_{\mathrm{ber}} \leq (\mathrm{ber}(T))^2 \leq \frac{1}{2}\left\|(T_1 + T_2)^2 + (T_1 - T_2)^2\right\|_{\mathrm{ber}}. \tag{1.4}$$

Also, Berezin radius inequalities were given by using the other inequalities in [10–12,14,15,34–36].

In the present paper, we investigate considerable generalizations of Berezin radius inequalities by using some classical convexity inequalities and some RKHS operator inequalities. The related results are obtained in [8].

2. Auxiliary Theorems

In this section, we present some required lemmas and related inequalities.

A simple consequence of the classical Jensen's inequality says that for $T, b \geq 0$, $0 < \zeta < 1$, $\varepsilon \neq 0$, $M_\varepsilon(T, b, \zeta) = (\zeta T^\varepsilon + (1 - \zeta) b^\varepsilon)^{\frac{1}{\varepsilon}}$ and $M_0(T, b, \zeta) = T^\zeta b^{1-\zeta}$, we have

$$M_\varepsilon(T, b, \zeta) \leq M_\delta(T, b, \zeta) \tag{2.1}$$

for $\varepsilon \leq \delta$ [19].

The following inequality is another application of Jensen's inequality : for $T, b \geq 0$, and $\varepsilon > 0$, we have

$$N_\delta(T, b) \leq N_\varepsilon(T, b) \text{ for } \delta \geq \varepsilon \geq 0, \tag{2.2}$$

where $N_\varepsilon(T, b) = (T^\varepsilon + b^\varepsilon)^{\frac{1}{\varepsilon}}$.

Lemma 2.1 ([25]). *If $T \in \mathcal{L}(\mathcal{H})$, $T \geq 0$, and $x \in \mathcal{H}$ is an any unit vector, then*

$$\langle Tx, x\rangle^{\varepsilon} \leq \langle T^{\varepsilon}x, x\rangle \text{ for } \varepsilon \geq 1, \tag{2.3}$$

$$\langle T^{\varepsilon}x, x\rangle \leq \langle Tx, x\rangle^{\varepsilon} \text{ for } 0 < \varepsilon \leq 1. \tag{2.4}$$

Now, we need several well-known lemmas which are respectively the simple consequences of the classical Jensen and Young inequalities [19]; spectral theorem for positive operators and Jensen's inequality [25,30]; and the generalized mixed Schwarz inequality [25].

Lemma 2.2. *If $T \in \mathcal{L}(\mathcal{H})$ is self-adjoint and $x \in \mathcal{H}$ is an any vector, then*

$$|\langle Tx, x\rangle| \leq \langle |T|\, x, x\rangle. \tag{2.5}$$

Lemma 2.3. *If $T \in \mathcal{L}(\mathcal{H})$ and $0 \leq \zeta \leq 1$, then we have*

$$|\langle Tx_1, x_2\rangle|^2 \leq \left\langle |T|^{2\zeta}\, x_1, x_1\right\rangle \left\langle |T^*|^{2(1-\zeta)}\, x_2, x_2\right\rangle, \tag{2.6}$$

for all $x_1, x_2 \in \mathcal{H}$.

Lemma 2.4 ([5]). *If $T_1, T_2 \in \mathcal{L}(\mathcal{H})$, and $T_1, T_2 \geq 0$, then we have*

$$\|(T_1 + T_2)^{\varepsilon}\| \leq \left\|T_1^{\varepsilon} + T_2^{\varepsilon}\right\| \text{ for } 0 < \varepsilon \leq 1. \tag{2.7}$$

Lemma 2.5 ([18]). *Let $T \in \mathcal{L}(\mathcal{H})$. Then,*

$$|\langle Tx_1, x_2\rangle|^2 \leq \langle |T|\, x_1, x_1\rangle^{1/2} \langle |T^*|\, x_2, x_2\rangle^{1/2}, \tag{2.8}$$

for all $x_1, x_2 \in \mathcal{H}$.

Lemma 2.6 ([9,26]). *If $T_1, T_2 \in \mathcal{L}(\mathcal{H})$, and $T_1, T_2 \geq 0$, then we have*

$$\left\|T_1^{1/2}T_2^{1/2}\right\| \leq \|T_1 T_2\|^{1/2}.$$

Lemma 2.7 ([27]). *If $T_1, T_2 \in \mathcal{L}(\mathcal{H})$, and $T_1, T_2 \geq 0$, then we have*

$$\|T_1 + T_2\| \leq \frac{1}{2}\left(\|T_1\| + \|T_2\| + \sqrt{(\|T_1\| - \|T_2\|)^2 + 4\left\|T_1^{1/2}T_2^{1/2}\right\|^2}\right).$$

3. Generalized Berezin Radius Inequalities

Our refined Berezin radius inequality could be presented like this:

Theorem 3.1. *If $T \in \mathcal{L}(\mathcal{H}(Q))$, then we have*

$$\operatorname{ber}(T) \leq \frac{1}{2}\, \||T| + |T^*|\|_{\text{ber}} \leq \frac{1}{2}\left(\|T\|_{\text{ber}} + \left\|T^2\right\|_{\text{ber}}^{1/2}\right). \tag{3.1}$$

Proof. By the inequality (2.8) and by the AM-GM inequality, we obtain

$$\begin{aligned}\left|\left\langle Tk_\eta, k_\eta\right\rangle\right| &\leq \left\langle |T|\, k_\eta, k_\eta\right\rangle^{1/2} \left\langle |T^*|\, k_\eta, k_\eta\right\rangle^{1/2} \\ &\leq \frac{1}{2}\left(\left\langle |T|\, k_\eta, k_\eta\right\rangle + \left\langle |T^*|\, k_\eta, k_\eta\right\rangle\right) \\ &= \frac{1}{2}\left(\left\langle (|T| + |T^*|)\, k_\eta, k_\eta\right\rangle\right).\end{aligned}$$

Thus,

$$\begin{aligned}\operatorname{ber}(T) = \sup_{\eta\in Q}\left|\widetilde{T}(\eta)\right| &\leq \frac{1}{2}\sup_{\eta\in Q}\left\langle (|T| + |T^*|)\, k_\eta, k_\eta\right\rangle \\ &= \frac{1}{2}\, \||T| + |T^*|\|_{\text{ber}}.\end{aligned}$$

Applying Lemmas 2.6 and 2.7 to the positive operators $|T|$ and $|T^*|$, and using the facts that $\||T|\| = \||T^*|\| = \|T\|$ and $\||T|\,|T^*|\| = \left\|T^2\right\|$, we have

$$\||T| + |T^*|\|_{\text{ber}} \leq \|T\|_{\text{ber}} + \left\|T^2\right\|_{\text{ber}}^{1/2}$$

as required. □

Generalization of the first inequality in (3.1) reveals our second result.

Theorem 3.2. *If $T \in \mathcal{L}(\mathcal{H}(Q))$, $0 < \zeta < 1$ and $\varepsilon \geq 1$, then we have*

$$(\operatorname{ber}(T))^{\varepsilon} \leq \frac{1}{2} \left\| |T|^{2\zeta\varepsilon} + |T^*|^{2(1-\zeta)\varepsilon} \right\|_{\text{ber}}. \tag{3.2}$$

Proof. Let $\eta \in \Omega$ be any number. Then, we get

$$\begin{aligned}
\left|\left\langle Tk_\eta, k_\eta \right\rangle\right| &\leq \left\langle |T|^{2\zeta} k_\eta, k_\eta \right\rangle^{1/2} \left\langle |T^*|^{2(1-\zeta)} k_\eta, k_\eta \right\rangle^{1/2} \\
&\quad \text{(by the inequality (2.6))} \\
&\leq \left(\frac{\left\langle |T|^{2\zeta} k_\eta, k_\eta \right\rangle^{\varepsilon} + \left\langle |T^*|^{2(1-\zeta)} k_\eta, k_\eta \right\rangle^{\varepsilon}}{2} \right)^{1/\varepsilon} \\
&\quad \text{(by the inequality (2.1))} \\
&\leq \left(\frac{\left\langle |T|^{2\zeta\varepsilon} k_\eta, k_\eta \right\rangle + \left\langle |T^*|^{2(1-\zeta)\varepsilon} k_\eta, k_\eta \right\rangle}{2} \right)^{1/\varepsilon} \\
&\quad \text{(by the inequality (2.3))}
\end{aligned}$$

and

$$\left|\left\langle Tk_\eta, k_\eta \right\rangle\right|^{\varepsilon} \leq \frac{1}{2} \left\langle \left(|T|^{2\zeta\varepsilon} + |T^*|^{2(1-\zeta)\varepsilon} \right) k_\eta, k_\eta \right\rangle.$$

So by taking supremum over $\eta \in Q$, we deduce

$$\sup_{\eta \in Q} \left|\left\langle Tk_\eta, k_\eta \right\rangle\right|^{\varepsilon} \leq \frac{1}{2} \sup_{\eta \in Q} \left\langle \left(|T|^{2\zeta\varepsilon} + |T^*|^{2(1-\zeta)\varepsilon} \right) k_\eta, k_\eta \right\rangle$$

which is equivalent to

$$(\operatorname{ber}(T))^{\varepsilon} \leq \frac{1}{2} \left\| |T|^{2\zeta\varepsilon} + |T^*|^{2(1-\zeta)\varepsilon} \right\|_{\text{ber}}.$$

This inequality gives the inequality (3.2). □

Generalization of the second inequality in (1.2) comes out our third result.

Theorem 3.3. *If $T \in \mathcal{L}(\mathcal{H}(Q))$, $0 < \zeta < 1$ and $\varepsilon \geq 1$, then we have*

$$(\operatorname{ber}(T))^{2\varepsilon} \leq \left\| \zeta |T|^{2\varepsilon} + (1-\zeta) |T^*|^{2\varepsilon} \right\|_{\text{ber}}.$$

Proof. Let $\eta \in \Omega$ be any number. Then we get

$$\begin{aligned}
\left|\left\langle Tk_\eta, k_\eta \right\rangle\right|^2 &\leq \left\langle |T|^{2\zeta} k_\eta, k_\eta \right\rangle \left\langle |T^*|^{2(1-\zeta)} k_\eta, k_\eta \right\rangle \\
&\quad \text{(by the inequality (2.6))} \\
&\leq \left\langle |T|^{2} k_\eta, k_\eta \right\rangle^{\zeta} \left\langle |T^*|^{2} k_\eta, k_\eta \right\rangle^{(1-\zeta)} \\
&\quad \text{(by the inequality (2.4))} \\
&\leq \left(\zeta \left\langle |T|^{2} k_\eta, k_\eta \right\rangle^{\varepsilon} + (1-\zeta) \left\langle |T^*|^{2} k_\eta, k_\eta \right\rangle^{\varepsilon} \right)^{1/\varepsilon} \\
&\quad \text{(by the inequality (2.1))} \\
&\leq \left(\zeta \left\langle |T|^{2\varepsilon} k_\eta, k_\eta \right\rangle + (1-\zeta) \left\langle |T^*|^{2\varepsilon} k_\eta, k_\eta \right\rangle \right)^{1/\varepsilon} \\
&\quad \text{(by the inequality (2.3))} \\
&\leq \left\langle \left(\zeta |T|^{2\varepsilon} + (1-\zeta) |T^*|^{2\varepsilon} \right) k_\eta, k_\eta \right\rangle^{1/\varepsilon}
\end{aligned}$$

and so

$$\left|\left\langle Tk_\eta, k_\eta \right\rangle\right|^{2\varepsilon} \leq \left\langle \left(\zeta |T|^{2\varepsilon} + (1-\zeta) |T^*|^{2\varepsilon} \right) k_\eta, k_\eta \right\rangle.$$

By taking supremum over $\eta \in Q$ above inequality, we reach that

$$\sup_{\eta\in Q}\left|\left\langle Tk_\eta,k_\eta\right\rangle\right|^{2\varepsilon}\leq\sup_{\eta\in Q}\left\langle\left(\zeta\,|T|^{2\varepsilon}+(1-\zeta)\,|T^*|^{2\varepsilon}\right)k_\eta,k_\eta\right\rangle$$

which clearly implies that

$$(\mathrm{ber}\,(T))^{2\varepsilon}\leq\left\|\zeta\,|T|^{2\varepsilon}+(1-\zeta)\,|T^*|^{2\varepsilon}\right\|_{\mathrm{ber}}.$$

Then, the desired result has been obtained. □

Our next results are generalizations of the second inequality in (1.3).

Theorem 3.4. *If $T\in\mathcal{L}(\mathcal{H}(Q))$ with the Cartesian decomposition $T=T_1+iT_2$ and $0<\varepsilon\leq 2$, then*

$$(\mathrm{ber}\,(T))^{\varepsilon}\leq\left\||T_1|^{\varepsilon}+|T_2|^{\varepsilon}\right\|_{\mathrm{ber}}.\tag{3.3}$$

Proof. First we prove an inequality stronger than (3.3) for the special case where $1\leq\varepsilon\leq 2$. Let k_η be a normalized reproducing kernel, and for $1\leq\varepsilon\leq 2$, we get

$$\begin{aligned}\left|\left\langle Tk_\eta,k_\eta\right\rangle\right|&\leq\left(\left\langle T_1k_\eta,k_\eta\right\rangle^2+\left\langle T_2k_\eta,k_\eta\right\rangle^2\right)^{1/2}\\&\leq\left(\left|\left\langle T_1k_\eta,k_\eta\right\rangle\right|^{\varepsilon}+\left|\left\langle T_2k_\eta,k_\eta\right\rangle\right|^{\varepsilon}\right)^{1/\varepsilon}\ \text{(by the inequality (2.2))}\\&\leq\left(\left\langle |T_1|\,k_\eta,k_\eta\right\rangle^{\varepsilon}+\left\langle |T_2|\,k_\eta,k_\eta\right\rangle^{\varepsilon}\right)^{1/\varepsilon}\ \text{(by the inequality (2.5))}\\&\leq\left(\left\langle |T_1|^{\varepsilon}\,k_\eta,k_\eta\right\rangle+\left\langle |T_2|^{\varepsilon}\,k_\eta,k_\eta\right\rangle\right)^{1/\varepsilon}\ \text{(by the inequality (2.3))}\\&=\left(\left\langle (|T_1|^{\varepsilon}+|T_2|^{\varepsilon})\,k_\eta,k_\eta\right\rangle\right)^{1/\varepsilon}.\end{aligned}$$

So, we will get a stronger inequality

$$\left|\left\langle Tk_\eta,k_\eta\right\rangle\right|^{\varepsilon}\leq\left\langle (|T_1|^{\varepsilon}+|T_2|^{\varepsilon})\,k_\eta,k_\eta\right\rangle,$$

and so by taking supremum over $\eta\in Q$ above inequality, we deduce

$$\sup_{\eta\in Q}\left|\left\langle Tk_\eta,k_\eta\right\rangle\right|^{\varepsilon}\leq\sup_{\eta\in Q}\left\langle (|T_1|^{\varepsilon}+|T_2|^{\varepsilon})\,k_\eta,k_\eta\right\rangle$$

which is equivalent to

$$\mathrm{ber}^{\varepsilon}(T)\leq\left\||T_1|^{\varepsilon}+|T_2|^{\varepsilon}\right\|_{\mathrm{ber}}.$$

In general, where $0<\varepsilon\leq 2$, we have

$$\begin{aligned}\mathrm{ber}^{\varepsilon}(T)&\leq\left\|T_1^2+T_2^2\right\|^{\frac{\varepsilon}{2}}\ \text{(by the second inequality (1.3))}\\&\leq\left\|\left(T_1^2+T_2^2\right)^{\frac{\varepsilon}{2}}\right\|\\&\leq\left\||T_1|^{\varepsilon}+|T_2|^{\varepsilon}\right\|\ \text{(by the inequality (2.7))},\end{aligned}$$

as required. □

Theorem 3.5. *If $T\in\mathcal{L}(\mathcal{H}(Q))$ with the Cartesian decomposition $T=T_1+iT_2$ and $\varepsilon\geq 2$, then we have*

$$(\mathrm{ber}\,(T))^{\varepsilon}\leq 2^{\frac{\varepsilon}{2}-1}\left\||T_1|^{\varepsilon}+|T_2|^{\varepsilon}\right\|_{\mathrm{ber}}.$$

Proof. Let $\eta \in \Omega$ be any number. Then we have

$$\begin{aligned}\frac{\left|\left\langle Tk_\eta, k_\eta\right\rangle\right|}{\sqrt{2}} &= \left(\frac{\left\langle T_1k_\eta, k_\eta\right\rangle^2 + \left\langle T_2k_\eta, k_\eta\right\rangle^2}{2}\right)^{\frac{1}{2}} \leq \left(\frac{\left|\left\langle T_1k_\eta, k_\eta\right\rangle\right|^\varepsilon + \left|\left\langle T_2k_\eta, k_\eta\right\rangle\right|^\varepsilon}{2}\right)^{\frac{1}{\varepsilon}} \\ &\text{(by the inequality (2.1))} \\ &\leq 2^{-\frac{1}{\varepsilon}}\left(\left\langle |T_1|\, k_\eta, k_\eta\right\rangle^\varepsilon + \left\langle |T_2|^\varepsilon\, k_\eta, k_\eta\right\rangle\right)^{\frac{1}{\varepsilon}} \\ &\text{(by the inequality (2.5))} \\ &\leq 2^{-\frac{1}{\varepsilon}}\left(\left\langle |T_1|^\varepsilon\, k_\eta, k_\eta\right\rangle + \left\langle |T_2|^\varepsilon\, k_\eta, k_\eta\right\rangle\right)^{\frac{1}{\varepsilon}} \\ &\text{(by the inequality (2.3))} \\ &= 2^{-\frac{1}{\varepsilon}}\left\langle \left(|T_1|^\varepsilon + |T_2|^\varepsilon\right) k_\eta, k_\eta\right\rangle^{\frac{1}{\varepsilon}}.\end{aligned}$$

Thus,

$$\left|\left\langle Tk_\eta, k_\eta\right\rangle\right|^\varepsilon \leq 2^{\frac{\varepsilon}{2}-1}\left\langle \left(|T_1|^\varepsilon + |T_2|^\varepsilon\right) k_\eta, k_\eta\right\rangle$$

and so by taking supremum over $\eta \in Q$

$$\mathrm{ber}^\varepsilon(T) \leq 2^{\frac{\varepsilon}{2}-1}\, \||T_1|^\varepsilon + |T_2|^\varepsilon\|_{\mathrm{ber}}$$

as required. □

The following is a generalization of the inequalities in (1.4).

Theorem 3.6. *If $T \in \mathcal{L}(\mathcal{H}(Q))$ with the Cartesian decomposition $T = T_1 + iT_2$, and $\varepsilon \geq 2$, then we have*

$$2^{-\frac{\varepsilon}{2}-1}\, \|(T_1+T_2)^\varepsilon + (T_1-T_2)^\varepsilon\|_{\mathrm{ber}} \leq (\mathrm{ber}(T))^\varepsilon \leq \frac{1}{2}\, \|(T_1+T_2)^\varepsilon + (T_1-T_2)^\varepsilon\|_{\mathrm{ber}}. \tag{3.4}$$

Proof. The proof essentially depends on some general arguments of Huban et al.'s paper (see Theorem 3.1 in [20]). Therefore, we have

$$\mathrm{ber}^2(T) \geq \frac{1}{2}\left\|(T_1 \pm T_2)^2\right\|_{\mathrm{ber}}.$$

Hence,

$$\mathrm{ber}^\varepsilon(T) \geq 2^{-\frac{\varepsilon}{2}}\left\|(T_1 \pm T_2)^2\right\|_{\mathrm{ber}}^{\varepsilon/2} = 2^{-\frac{\varepsilon}{2}}\, \||T_1 \pm T_2|^\varepsilon\|,$$

and so

$$\begin{aligned}2\mathrm{ber}^\varepsilon(T) &\geq 2^{-\frac{\varepsilon}{2}}\left(\||T_1+T_2|^\varepsilon\|_{\mathrm{ber}} + \||T_1-T_2|^\varepsilon\|_{\mathrm{ber}}\right) \\ &\geq 2^{-\frac{\varepsilon}{2}}\, \||T_1+T_2|^\varepsilon + |T_1-T_2|^\varepsilon\|_{\mathrm{ber}} \text{ (by the triangle inequality).}\end{aligned}$$

Hence,

$$\mathrm{ber}^\varepsilon(T) \geq 2^{-\frac{\varepsilon}{2}-1}\, \||T_1+T_2|^\varepsilon + |T_1-T_2|^\varepsilon\|_{\mathrm{ber}},$$

which could be shown as a proof of the first inequality in (3.4). In order to proof of the second inequality in (3.4), let k_η be a normalized reproducing kernel. Hence, by the convexity of the function $f(x) = x^{\frac{\varepsilon}{2}}$ on $[0, \infty)$,

$$\begin{aligned}
\left|\left\langle T k_\eta, k_\eta\right\rangle\right|^\varepsilon &= \left(\left\langle T_1 k_\eta, k_\eta\right\rangle^2 + \left\langle T_2 k_\eta, k_\eta\right\rangle^2\right)^{\frac{\varepsilon}{2}} \\
&= 2^{-\frac{\varepsilon}{2}}\left(\left\langle (T_1+T_2) k_\eta, k_\eta\right\rangle^2\right)^{\frac{\varepsilon}{2}} + 2^{-\frac{\varepsilon}{2}}\left(\left\langle (T_1-T_2) k_\eta, k_\eta\right\rangle^2\right)^{\frac{\varepsilon}{2}} \\
&\leq 2^{-\frac{\varepsilon}{2}} 2^{\frac{\varepsilon}{2}-1}\left(\left|\left\langle (T_1+T_2) k_\eta, k_\eta\right\rangle\right|^\varepsilon + 2^{-\frac{\varepsilon}{2}}\left|\left\langle (T_1-T_2) k_\eta, k_\eta\right\rangle\right|^\varepsilon\right) \\
&\leq \frac{1}{2}\left(\left\langle |T_1+T_2| k_\eta, k_\eta\right\rangle^\varepsilon + \left\langle (|T_1-T_2|) k_\eta, k_\eta\right\rangle^\varepsilon\right) \\
&\text{(by the inequality (2.5))} \\
&\leq \frac{1}{2}\left(\left\langle |T_1+T_2|^\varepsilon k_\eta, k_\eta\right\rangle + \left\langle (|T_1-T_2|^\varepsilon) k_\eta, k_\eta\right\rangle\right) \\
&\text{(by the inequality (2.3))} \\
&= \frac{1}{2}\left(\left\langle (|T_1+T_2|^\varepsilon + |T_1-T_2|^\varepsilon) k_\eta, k_\eta\right\rangle\right).
\end{aligned}$$

Now, by taking supremum over $\eta \in Q$, we have

$$\mathrm{ber}^\varepsilon(T) \leq \frac{1}{2}\left\|(T_1+T_2)^\varepsilon + (T_1-T_2)^\varepsilon\right\|_{\mathrm{ber}}$$

which proves the second inequality in (3.4). Then, the desired result has been obtained. □

Now, we present Berezin norm inequalities and a related Berezin radius inequality.

Theorem 3.7. *If $T_1, T_2 \in \mathcal{L}(\mathcal{H}(Q))$, $0 < \zeta < 1$, and $\varepsilon \geq 1$, then we have*

$$\mathrm{ber}^\varepsilon(T_1+T_2) \leq 2^{\varepsilon-2}\left(\left\||T_1|^{2\zeta\varepsilon} + |T_2|^{2\zeta\varepsilon}\right\|_{\mathrm{ber}} + \left\|\left|T_1^*\right|^{2(1-\zeta)\varepsilon} + \left|T_2^*\right|^{2(1-\zeta)\varepsilon}\right\|_{\mathrm{ber}}\right). \tag{3.5}$$

Proof. For any $\eta, \tau \in Q$, we have

$$\begin{aligned}
\left|\left\langle (T_1+T_2) k_\eta, k_\tau\right\rangle\right| &\leq \left|\left\langle T_1 k_\eta, k_\tau\right\rangle\right| + \left|\left\langle T_2 k_\eta, k_\tau\right\rangle\right| \\
&\leq \left(\left\langle |T_1|^{2\zeta} k_\eta, k_\eta\right\rangle^{1/2}\left\langle \left|T_1^*\right|^{2(1-\zeta)} k_\tau, k_\tau\right\rangle^{1/2}\right. \\
&\quad \left. + \left\langle |T_2|^{2\zeta} k_\eta, k_\eta\right\rangle^{1/2}\left\langle \left|T_2^*\right|^{2(1-\zeta)} k_\tau, k_\tau\right\rangle^{1/2}\right) \\
&\quad \text{(by the inequality (2.6))}
\end{aligned}$$

$$\leq \left(\frac{\left\langle |T_1|^{2\zeta} k_\eta, k_\eta \right\rangle^{\varepsilon} + \left\langle \left|T_1^*\right|^{2(1-\zeta)} k_\tau, k_\tau \right\rangle^{\varepsilon}}{2} \right)^{1/\varepsilon}$$

$$+ \left(\frac{\left\langle |T_2|^{2\zeta} k_\eta, k_\eta \right\rangle^{\varepsilon} + \left\langle \left|T_2^*\right|^{2(1-\zeta)} k_\tau, k_\tau \right\rangle^{\varepsilon}}{2} \right)^{1/\varepsilon}$$

(by the inequality (2.1))

$$\leq \left(\frac{\left\langle |T_1|^{2\zeta\varepsilon} k_\eta, k_\eta \right\rangle + \left\langle \left|T_1^*\right|^{2(1-\zeta)\varepsilon} k_\tau, k_\tau \right\rangle}{2} \right)^{1/\varepsilon}$$

$$+ \left(\frac{\left\langle |T_2|^{2\zeta\varepsilon} k_\eta, k_\eta \right\rangle + \left\langle \left|T_2^*\right|^{2(1-\zeta)\varepsilon} k_\tau, k_\tau \right\rangle}{2} \right)^{1/\varepsilon}$$

(by the inequality (2.3))

$$\leq 2^{1-\frac{1}{\varepsilon}} \left(\left\langle |T_1|^{2\zeta\varepsilon} k_\eta, k_\eta \right\rangle + \left\langle \left|T_1^*\right|^{2(1-\zeta)\varepsilon} k_\tau, k_\tau \right\rangle \right.$$

$$\left. + \left\langle |T_2|^{2\zeta\varepsilon} k_\eta, k_\eta \right\rangle + \left\langle \left|T_2^*\right|^{2(1-\zeta)\varepsilon} k_\tau, k_\tau \right\rangle \right)^{1/\varepsilon}$$

from the concavity of the function $f(x) = x^{1/\varepsilon}$ on $[0, \infty)$. Thus,

$$\left| \left\langle (T_1 + T_2) k_\eta, k_\tau \right\rangle \right|^{\varepsilon} \leq 2^{\varepsilon-2} \left(\left\langle |T_1|^{2\zeta\varepsilon} k_\eta, k_\eta \right\rangle + \left\langle \left|T_1^*\right|^{2(1-\zeta)\varepsilon} k_\tau, k_\tau \right\rangle \right.$$

$$\left. + \left\langle |T_2|^{2\zeta\varepsilon} k_\eta, k_\eta \right\rangle + \left\langle \left|T_2^*\right|^{2(1-\zeta)\varepsilon} k_\tau, k_\tau \right\rangle \right).$$

Now, by taking supremum over $\eta \in Q$ with $\eta = \tau$, and we have

$$\mathrm{ber}^{\varepsilon}(T_1 + T_2) \leq 2^{\varepsilon-2} \left(\left\| |T_1|^{2\zeta\varepsilon} + |T_2|^{2\zeta\varepsilon} \right\|_{\mathrm{ber}} + \left\| \left|T_1^*\right|^{2(1-\zeta)\varepsilon} + \left|T_2^*\right|^{2(1-\zeta)\varepsilon} \right\|_{\mathrm{ber}} \right).$$

□

Putting $\eta = \tau$ in the proof of Theorem 3.7, we get

$$\mathrm{ber}^{\varepsilon}(T_1 + T_2) \leq 2^{\varepsilon-2} \left\| |T_1|^{2\zeta\varepsilon} + |T_2|^{2\zeta\varepsilon} + \left|T_1^*\right|^{2(1-\zeta)\varepsilon} + \left|T_2^*\right|^{2(1-\zeta)\varepsilon} \right\|_{\mathrm{ber}}. \tag{3.6}$$

If $T_1 = T_2$, then the inequality (3.6) reduces in particular to the inequality (3.2).

The following is an important case of the inequality (3.5).

Corollary 3.8. *If $T_1, T_2 \in \mathcal{L}(\mathcal{H}(Q))$, T_1, T_2 are normal, $\zeta = 1/2$, and $\varepsilon \geq 1$, then*

$$\mathrm{ber}^{\varepsilon}(T_1 + T_2) \leq 2^{\varepsilon-1} \left\| |T_1|^{\varepsilon} + |T_2|^{\varepsilon} \right\|_{\mathrm{ber}}.$$

For more recent results concerning Berezin radius inequalities for operators and other related results, we suggest [2, 13, 16, 21, 32, 33].

Acknowledgement

The abstract of this work was presented in the "3rd International Conference on Artificial Intelligence and Applied Mathematics in Engineering (ICAIAME 2021)" which was held between 1-3 October 2021 in Belek/Turkey.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

All authors have contributed sufficiently to the planning, execution, or analysis of this study to be included as authors. All authors have read and agreed to the published version of the manuscript.

References

[1] Bakherad, M., Garayev, M.T., *Berezin number inequalities for operators*, Concr. Oper., **6**(1)(2019), 33–43.

[2] Başaran, H. Gürdal, M., *Berezin number inequalities via Young inequality*, Honam Mathematical J., **43**(3)(2021), 523–537.

[3] Başaran, H., Gürdal, M., Güncan, A.N., *Some operator inequalities associated with Kantorovich and Hölder-McCarthy inequalities and their applications*, Turkish J. Math., **43**(1)(2019), 523–532.

[4] Berezin, F.A., *Covariant and contravariant symbols for operators*, Izv. Akad. Nauk SSSR Ser. Mat., **36**(1972), 1134–1167.

[5] Bhatia, R., Kittaneh, F., *Norm inequalities for positive operators*, Left. Math. Phys., **43**(1998), 225–231.

[6] Dragomir, S.S., *Some inequalities for the Euclidean operator radius of two operators in Hilbert spaces*, Linear Algebra Appl., **419**(1)(2006), 256–264.

[7] Dragomir, S. S., Inequalities for the Numerical Radius of Linear Operators in Hilbert Spaces, Springer Briefs in Mathematics, Springer, Cham, Switzerland, 2013.

[8] El-Haddad, M., Kittaneh, F., *Numerical radius inequalities for Hilbert space operators (II)*, Studia Math., **182**(2)(2007), 133–140.

[9] Furuta, T., *Norm inequalities equivalent to Löwner–Heinz theorem*, Rev. Math. Phys., **1**(1989), 135–137

[10] Garayev, M., Bouzeffour, F., Gürdal, M., Yangöz, C.M., *Refinements of Kantorovich type, Schwarz and Berezin number inequalities*, Extracta Math., **35**(2020), 1–20.

[11] Garayev, M.T., Gürdal, M., Okudan, A., *Hardy-Hilbert's inequality and a power inequality for Berezin numbers for operators*, Math. Inequal. Appl., **19**(3)(2016), 883–891.

[12] Garayev, M.T., Gürdal, M., Saltan, S., *Hardy type inequaltiy for reproducing kernel Hilbert space operators and related problems*, Positivity, **21**(4)(2017), 1615–1623.

[13] Garayev, M.T., Gürdal, M., Yamanci, U., Alsahli, G.M., *Boundary behavior of Berezin symbols and related results*, Filomat, **33**(14)(2019), 4433–4439

[14] Garayev, M.T., Guedri, H., Gürdal, M., Alsahli, G.M., *On some problems for operators on the reproducing kernel Hilbert space*, Linear Multilinear Algebra, **69**(11)(2021), 2059–2077.

[15] Garayev, M., Saltan, S., Bouzeffour, F., Aktan, B., *Some inequalities involving Berezin symbols of operator means and related questions*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM, **114**(85)(2020), 1–17.

[16] Gürdal, M., Duman, N., *Berezin number inequalities involving superquadratic functions*, Casp. J. Math. Sci., **10**(1)(2021), 18–27

[17] Gustafson, K.E., Rao, D.K.M., Numerical Range, Springer-Verlag, New York, 1997.

[18] Halmos, P. R., A Hilbert Space Problem Book , 2nd ed., Springer, New York, 1982.

[19] Hardy, G.H., Littlewood, J.E., Polya, G., Inequalities. 2nd ed., Cambridge Univ. Press, Cambridge, 1988.

[20] Huban, M.B., Başaran, H., Gürdal, M., *New upper bounds related to the Berezin number inequalities*, J. Inequal. Spec. Funct., **12**(3)(2021), 1–12.

[21] Huban, M.B., Gürdal, M., Tilki, H., *Some classical inequalities and their applications*, Filomat, **35**(7)(2021), 2165–2173.

[22] Karaev M.T., *Berezin set and Berezin number of operators and their applications*, The 8th Workshop on Numerical Ranges and Numerical Radii (WONRA -06),University of Bremen, July 15-17, (2006), p.14.

[23] Karaev, M.T., *Berezin symbol and invertibility of operators on the functional Hilbert spaces*, J. Funct. Anal., **238**(2006), 181–192.

[24] Karaev, M.T., *Reproducing kernels and Berezin symbols techniques in various questions of operator theory*, Complex Anal. Oper. Theory, **7**(2013), 983–1018.

[25] Kittaneh, F., *Notes on some inequalities for Hilbert space operators*, Publ. Res. Ins. Math. Sci., **24**(1988), 283–293.

[26] Kittaneh, F., *Singular values of companion matrices and bounds on zeros of polynomials*, SIAM J. Matrix Anal. Appl., **16**(1995), 333–340.

[27] Kittaneh, F., *Norm inequalities for sums of positive operators*, J. Operator Theory, **48**(2002), 95–103.

[28] Kittaneh, F., *A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix*, Studia Math., **158**(1)(2003), 11–17.

[29] Kittaneh, F., *Numerical radius inequalities for Hilbert space operators*, Studia Math., **168**(1)(2005), 73–80.

[30] Kittaneh, F., Manasrah, Y., *Improved Young and Heinz inequalities for matrices*, J. Math. Anal. Appl., **361**(2010), 262–269.

[31] Kittaneh, F., Moslehian, M.S., Yamazaki, T., *Cartesian decomposition and numerical radius inequalities*, Lin. Algeb. Appl., **471**(2015), 46–53.

[32] Tapdigoglu, R., *New Berezin symbol inequalities for operators on the reproducing kernel Hilbert space*, Oper. Matrices, **15**(3)(2021), 1031–1043.

[33] Tapdigoglu, R., Gürdal, M., Altwaijry, N., Sarı, N., *Davis-Wielandt-Berezin radius inequalities via Dragomir inequalities*, Oper. Matrices, **15**(4)(2021), 1445–1460.

[34] Yamancı, U., Gürdal, M., *On numerical radius and Berezin number inequalities for reproducing kernel Hilbert space*, New York J. Math., **23**(2017), 1531–1537.

[35] Yamancı, U., Gürdal, M., Garayev, M.T., *Berezin number inequality for convex function in reproducing kernel Hilbert space*, Filomat, **31**(2017), 5711–5717.

[36] Yamancı, U., Tunç, R., Gürdal, M., *Berezin numbers, Grüss type inequalities and their applications*, Bull. Malays. Math. Sci. Soc., **43**(3)(2020), 2287–2296.