



SOME NEW INTEGRAL INEQUALITIES FOR N -TIMES DIFFERENTIABLE R -CONVEX AND R -CONCAVE FUNCTIONS

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Abstract. In this work, by using an integral identity together with both the Hölder and the Power-Mean integral inequality we establish several new inequalities for n -time differentiable r -convex and concave functions.

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1. INTRODUCTION

A function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

is valid for all $x, y \in I$ and $t \in [0, 1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$. This definition is well known in the literature. Convexity theory has appeared as a powerful technique to study a wide class of unrelated problems in pure and applied sciences. Many articles have been written by a number of mathematicians on convex functions and inequalities for their different classes, using, for example, the last articles [3, 8–15] and the references in these papers.

$f : [a, b] \rightarrow \mathbb{R}$ be a convex function, then the inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}$$

is known as the Hermite-Hadamard inequality (see [6] for more information). Since then, some refinements of the Hermite-Hadamard inequality on convex functions have been extensively investigated by a number of authors (e.g., [3, 4, 15]). In [20], the first author obtained a new refinement of the Hermite-Hadamard inequality for convex functions. The Hermite-Hadamard inequality was generalized in [17] to an r -convex positive function which is defined on an interval $[a, b]$.

Definition 1. A positive function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is called r -convex function on $[a, b]$, if for each the $x, y \in [a, b]$ and $t \in [0, 1]$,

$$f(tx + (1-t)y) \leq \begin{cases} [tf^r(x) + (1-t)f^r(y)]^{\frac{1}{r}}, & r \neq 0, \\ [f(x)]^t[f(y)]^{1-t}, & r = 0. \end{cases}$$

If the equality is reversed, then the function f is said to be r -concave.

It is obvious 0-convex functions are simply log-convex functions, 1-convex functions are ordinary convex functions and -1 -convex functions are arithmetically harmonically convex. One should note that if f is r -convex on $[a, b]$, then the function f^r is a convex function for $r > 0$ and f^r is a concave function for $r < 0$. We note that if f and g are convex and g is increasing, then gof is convex; moreover, since $f = \exp(\log f)$, it follows that a log-convex function is convex.

The definition of r -convexity naturally complements the concept of r -concavity, in which the inequality is reversed [18] and which plays an important role in statistics.

It is easily seen that if f is r -convex on $[a, b]$,

$$f^r\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f^r(x) dx \leq \frac{f^r(a) + f^r(b)}{2}, \quad r > 0 \quad (1.1)$$

$$f^r\left(\frac{a+b}{2}\right) \geq \frac{1}{b-a} \int_a^b f^r(x) dx \geq \frac{f^r(a) + f^r(b)}{2}, \quad r < 0 \quad (1.2)$$

Some refinements of the Hadamard inequality for r -convex functions could be found in [2, 7, 16, 19, 21]. In [1], Bessenyei studied Hermite-Hadamard-type inequalities for generalized 3-convex functions. In [16], the authors showed that if f is r -convex in $[a, b]$ and $0 < r \leq 1$, then

$$\frac{1}{b-a} \int_a^b f(x) dx \leq \frac{r}{r+1} [f^r(a) + f^r(b)]^{\frac{1}{r}}. \quad (1.3)$$

Theorem 1 ([5]). Suppose that f is a positive r -convex function on $[a, b]$. Then

$$\frac{1}{b-a} \int_a^b f(t) dt \leq L_r(f(a), f(b)).$$

If f is a positive r -concave function, then the inequality is reversed, where

$$L_r(f(a), f(b)) = \begin{cases} \frac{r}{r+1} \frac{f^{r+1}(a) - f^{r+1}(b)}{f^r(a) - f^r(b)}, & r \neq 0, -1, \quad f(a) \neq f(b) \\ \frac{f(a) - f(b)}{\ln f(a) - \ln f(b)}, & r = 0, \quad f(a) \neq f(b) \\ f(a) f(b) \frac{\ln f(a) - \ln f(b)}{f(a) - f(b)}, & r = -1, \quad f(a) \neq f(b) \\ f(a), & f(a) = f(b). \end{cases}$$

Theorem 2. Let $f : [a, b] \rightarrow (0, \infty)$ be r -convex function and $r \geq 1$. Then the following inequality holds:

$$\frac{1}{b-a} \int_a^b f(x) dx \leq \left[\frac{f^r(a) + f^r(b)}{2} \right]^{\frac{1}{r}}.$$

Lemma 1. Let $a \geq 0, b \geq 0$. Then $(a+b)^\lambda \leq a^\lambda + b^\lambda$, $0 < \lambda \leq 1$.

Let $0 < a < b$, throughout this paper we will use

$$A(a, b) = \frac{a+b}{2}, \quad G(a, b) = \sqrt{ab}$$

$$L_p(a, b) = \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right)^{\frac{1}{p}}, \quad a \neq b, \quad p \in \mathbb{R}, \quad p \neq -1, 0$$

for the arithmetic, geometric, generalized logarithmic mean, respectively. Also for shortness we will use the following notation:

$$I(a, b, n, f) = \sum_{k=0}^{n-1} (-1)^k \left(\frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!} \right) - \int_a^b f(x) dx$$

where an empty sum is understood to be nil.

2. MAIN RESULTS

We will use the following Lemma for obtain our main results.

Lemma 2 ([14]). Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be n -times differentiable mapping on I° for $n \in \mathbb{N}$ and $f^{(n)} \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$, we have the identity

$$I(a, b, n, f) = \frac{(-1)^{n+1}}{n!} \int_a^b x^n f^{(n)}(x) dx.$$

where an empty sum is understood to be nil.

Theorem 3. For $n \in \mathbb{N}$; let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be n -times differentiable function on I° , $r > 0$ and $a, b \in I^\circ$ with $a < b$. If $f^{(n)} \in L[a, b]$ and $|f^{(n)}|^q$ for $q > 1$ is r -convex function on $[a, b]$, then the following inequality holds:

$$|I(a, b, n, f)| \leq \frac{b-a}{n!} L_{np}^n(a, b) L_{\frac{1}{r}}^{\frac{1}{r}} \left(|f^{(n)}(a)|^{qr}, |f^{(n)}(b)|^{qr} \right)$$

Proof. If $|f^{(n)}|^q$ for $q > 1$ is r -convex function on $[a, b]$ and $r > 0$, using Lemma 2, the Hölder integral inequality and

$$|f^{(n)}(x)|^q = \left| f^{(n)} \left(\frac{x-a}{b-a} b + \frac{b-x}{b-a} a \right) \right|^q$$

$$\leq \left[\frac{x-a}{b-a} |f^{(n)}(b)|^{qr} + \frac{b-x}{b-a} |f^{(n)}(a)|^{qr} \right]^{\frac{1}{r}},$$

we have

$$\begin{aligned} & |I(a, b, n, f)| \\ & \leq \frac{1}{n!} \int_a^b x^n |f^{(n)}(x)| dx \\ & \leq \frac{1}{n!} \left(\int_a^b x^{np} dx \right)^{\frac{1}{p}} \left(\int_a^b |f^{(n)}(x)|^q dx \right)^{\frac{1}{q}} \\ & \leq \frac{1}{n!} \left(\int_a^b x^{np} dx \right)^{\frac{1}{p}} \left(\int_a^b \left[\frac{x-a}{b-a} |f^{(n)}(b)|^{qr} + \frac{b-x}{b-a} |f^{(n)}(a)|^{qr} \right]^{\frac{1}{r}} dx \right)^{\frac{1}{q}} \\ & = \frac{1}{n!} \left(\int_a^b x^{np} dx \right)^{\frac{1}{p}} \left(\int_{|f^{(n)}(a)|^{qr}}^{|f^{(n)}(b)|^{qr}} u^{\frac{1}{r}} \frac{b-a}{|f^{(n)}(b)|^{qr} - |f^{(n)}(a)|^{qr}} du \right)^{\frac{1}{q}} \\ & = \frac{1}{n!} (b-a) \left(\frac{b^{np+1} - a^{np+1}}{(np+1)(b-a)} \right)^{\frac{1}{p}} \left(\frac{|f^{(n)}(b)|^{qr(\frac{1}{r}+1)} - |f^{(n)}(a)|^{qr(\frac{1}{r}+1)}}{\left(\frac{1}{r} + 1 \right) \left(|f^{(n)}(b)|^{qr} - |f^{(n)}(a)|^{qr} \right)} \right)^{\frac{1}{q}} \\ & = \frac{1}{n!} (b-a) L_{np}^n(a, b) \left(\frac{|f^{(n)}(b)|^{qr(\frac{1}{r}+1)} - |f^{(n)}(a)|^{qr(\frac{1}{r}+1)}}{\left(\frac{1}{r} + 1 \right) \left(|f^{(n)}(b)|^{qr} - |f^{(n)}(a)|^{qr} \right)} \right)^{\frac{1}{q}} \\ & = \frac{1}{n!} (b-a) L_{np}^n(a, b) L_{\frac{1}{r}}^{\frac{1}{qr}} \left(|f^{(n)}(a)|^{qr}, |f^{(n)}(b)|^{qr} \right). \end{aligned}$$

This completes the proof of theorem. \square

Remark 1. The results obtained in this paper reduces to the results of [14] in case of $r = 1$.

Corollary 1. Under the conditions Theorem 3 for $n = 1$ we have the following inequality:

$$\left| \frac{f(b)b - f(a)a}{b-a} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq L_p(a, b) L_{\frac{1}{r}}^{\frac{1}{qr}} \left(|f'(a)|^{qr}, |f'(b)|^{qr} \right).$$

Proposition 1. Let $a, b \in (0, \infty)$ with $a < b$, $q > 1$ and $m \geq 1$, $r \geq 1$ we have

$$L_{\frac{m}{q}+1}^{\frac{m}{q}+1}(a, b) \leq L_p(a, b) L_{\frac{1}{r}}^{\frac{1}{qr}}(a^{mr}, b^{mr}).$$

Proof. Under the assumption of the Proposition, let $f(x) = \frac{q}{m+q}x^{\frac{m}{q}+1}$, $x \in (0, \infty)$. Then $|f'(x)|^q = x^m$ is r -convex on $(0, \infty)$ and the result follows directly from Corollary 1. \square

Remark 2. Under the assumption of the Proposition 2.1, If $r = 1$, $m = 1$, then the results obtained in this paper reduces to the results of [14].

Theorem 4. For $n \in \mathbb{N}$; let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be n -times differentiable function on I° , $r > 0$ and $a, b \in I^\circ$ with $a < b$. If $f^{(n)} \in L[a, b]$ and $|f^{(n)}|^q$ for $q \geq 1$ is r -convex function on $[a, b]$, then the following inequality holds:

$$\begin{aligned} & |I(a, b, n, f)| \\ & \leq \begin{cases} \frac{1}{n!}(b-a)^{1-\frac{1}{q}-\frac{1}{qr}} L_n^{n(\frac{q-1}{q})}(a, b) \left[C_1 |f^{(n)}(b)|^q + C_2 |f^{(n)}(a)|^q \right]^{\frac{1}{q}}, & r \geq 1 \\ \frac{1}{n!}(b-a)^{1-\frac{1}{q}-\frac{1}{qr}} L_n^{n(\frac{q-1}{q})}(a, b) \left[C_1^r |f^{(n)}(b)|^{qr} + C_2^r |f^{(n)}(a)|^{qr} \right]^{\frac{1}{qr}}, & r \leq 1 \end{cases} \end{aligned}$$

where

$$C_1 = C_1(a, b, r, n) = \int_a^b x^n (x-a)^{\frac{1}{r}} dx, C_2 = C_2(a, b, r, n) = \int_a^b x^n (b-x)^{\frac{1}{r}} dx.$$

Proof. From Lemma 2 and Power-Mean integral inequality, we get

$$\begin{aligned} & |I(a, b, n, f)| \\ & \leq \frac{1}{n!} \int_a^b x^n |f^{(n)}(x)| dx \\ & \leq \frac{1}{n!} \left(\int_a^b x^n dx \right)^{1-\frac{1}{q}} \left(\int_a^b x^n |f^{(n)}(x)|^q dx \right)^{\frac{1}{q}} \\ & \leq \frac{1}{n!} \left(\int_a^b x^n dx \right)^{1-\frac{1}{q}} \left(\int_a^b x^n \left[\frac{x-a}{b-a} |f^{(n)}(b)|^{qr} + \frac{b-x}{b-a} |f^{(n)}(a)|^{qr} \right]^{\frac{1}{r}} dx \right)^{\frac{1}{q}}. \end{aligned}$$

Here, using Lemma 1 we obtain respectively,

For $r \geq 1$

$$\begin{aligned} & |I(a, b, n, f)| \\ & \leq \frac{1}{n!} \left(\int_a^b x^n dx \right)^{1-\frac{1}{q}} \times \left(\int_a^b x^n \left\{ \left[\frac{x-a}{b-a} |f^{(n)}(b)|^{qr} \right]^{\frac{1}{r}} + \left[\frac{b-x}{b-a} |f^{(n)}(a)|^{qr} \right]^{\frac{1}{r}} \right\} dx \right)^{\frac{1}{q}} \\ & = \frac{1}{n!} \left(\frac{1}{b-a} \right)^{\frac{1}{qr}} \left(\frac{b^{n+1} - a^{n+1}}{n+1} \right)^{1-\frac{1}{q}} \end{aligned}$$

$$\begin{aligned}
& \times \left(\int_a^b x^n (x-a)^{\frac{1}{r}} |f^{(n)}(b)|^q dx + \int_a^b x^n (b-x)^{\frac{1}{r}} |f^{(n)}(a)|^q dx \right)^{\frac{1}{q}} \\
& = \frac{1}{n!} \left(\frac{1}{b-a} \right)^{\frac{1}{qr}} (b-a)^{1-\frac{1}{q}} \left(\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)} \right)^{1-\frac{1}{q}} \\
& \quad \times \left[C_1(a, b, r, n) |f^{(n)}(b)|^q + C_2(a, b, q, n) |f^{(n)}(a)|^q \right]^{\frac{1}{q}} \\
& = \frac{1}{n!} (b-a)^{1-\frac{1}{q}-\frac{1}{qr}} L_n^{n(\frac{q-1}{q})}(a, b) \left[C_1 |f^{(n)}(b)|^q + C_2 |f^{(n)}(a)|^q \right]^{\frac{1}{q}},
\end{aligned}$$

For $r \leq 1$, using Minkowski inequality, we have

$$\begin{aligned}
& |I(a, b, n, f)| \\
& \leq \frac{1}{n!} \left(\int_a^b x^n dx \right)^{1-\frac{1}{q}} \left(\frac{1}{b-a} \right)^{\frac{1}{qr}} \\
& \quad \times \left(\int_a^b \left[x^{nr} (x-a) |f^{(n)}(b)|^{qr} + x^{nr} (b-x) |f^{(n)}(a)|^{qr} \right]^{\frac{1}{r}} dx \right)^{\frac{1}{q}} \\
& = \frac{1}{n!} \left(\frac{b^{n+1}-a^{n+1}}{n+1} \right)^{1-\frac{1}{q}} \\
& \quad \times \left(\left\{ \left[\int_a^b x^n (x-a)^{\frac{1}{r}} |f^{(n)}(b)|^q dx \right]^r + \left[\int_a^b x^n (b-x)^{\frac{1}{r}} |f^{(n)}(a)|^q dx \right]^r \right\}^{\frac{1}{r}} \right)^{\frac{1}{q}} \\
& = \frac{1}{n!} (b-a)^{1-\frac{1}{q}-\frac{1}{qr}} L_n^{n(\frac{q-1}{q})}(a, b) \left[C_1^r |f^{(n)}(b)|^{qr} + C_2^r |f^{(n)}(a)|^{qr} \right]^{\frac{1}{qr}}.
\end{aligned}$$

This completes the proof of theorem. \square

Corollary 2. Under the conditions Theorem 4 for $n = 1$ we have the following inequalities:

$$\begin{aligned}
& |J(a, b, f)| \\
& \leq \begin{cases} A^{1-\frac{1}{q}}(a, b) \left\{ \frac{r(b-a)}{2r+1} [|f'(b)|^q - |f'(a)|^q] + \frac{r[a|f'(b)|^q + b|f'(a)|^q]}{r+1} \right\}^{\frac{1}{q}}, & r \geq 1 \\ A^{1-\frac{1}{q}}(a, b) \left\{ \left(\frac{r^2(a+b)+br}{(r+1)(2r+1)} \right)^r |f'(b)|^{qr} + \left(\frac{r^2(a+b)+ar}{(r+1)(2r+1)} \right)^r |f'(a)|^{qr} \right\}^{\frac{1}{qr}}, & r \leq 1 \end{cases}
\end{aligned}$$

where $J(a, b, f) = \frac{I(a, b, 1, f)}{b-a}$.

Proposition 2. Let $a, b \in (0, \infty)$ with $a < b$, $q \geq 1$ and $m \geq 1$, we have the following inequalities:

$$L_{\frac{m}{q}+1}^{\frac{m}{q}+1}(a, b) \leq \begin{cases} A^{1-\frac{1}{q}}(a, b) \left[\frac{2rA(a^{m+1}, b^{m+1})}{2r+1} + \frac{2r^2G^2(a, b)A(a^{m-1}, b^{m-1})}{(r+1)(2r+1)} \right]^{\frac{1}{q}}, & r \geq 1 \\ A^{1-\frac{1}{q}}(a, b) \left\{ \left(\frac{r^2(a+b)+br}{(r+1)(2r+1)} \right)^r b^{mr} + \left(\frac{r^2(a+b)+ar}{(r+1)(2r+1)} \right)^r a^{mr} \right\}^{\frac{1}{qr}}, & r \leq 1. \end{cases}$$

Proof. The result follows directly from Corollary 2 for function $f(x) = \frac{q}{m+q}x^{\frac{m}{q}+1}$, $x \in (0, \infty)$. \square

Corollary 3. Using Proposition 2 for $m = 1$, we have following inequalities:

$$L_{\frac{1}{q}+1}^{\frac{1}{q}+1}(a, b) \leq \begin{cases} A^{1-\frac{1}{q}}(a, b) \left[\frac{2r}{2r+1} A(a^2, b^2) + \frac{2r^2}{(r+1)(2r+1)} G^2(a, b) \right]^{\frac{1}{q}}, & r \geq 1 \\ A^{1-\frac{1}{q}}(a, b) \left\{ \left(\frac{r^2(a+b)+br}{(r+1)(2r+1)} \right)^r b^r + \left(\frac{r^2(a+b)+ar}{(r+1)(2r+1)} \right)^r a^r \right\}^{\frac{1}{qr}}, & r \leq 1. \end{cases}$$

Corollary 4. Using Proposition 2 for $q = 1$, we have following inequalities:

$$L_{m+1}^{m+1}(a, b) \leq \begin{cases} \frac{2rA(a^{m+1}, b^{m+1})}{2r+1} + \frac{2r^2G^2(a, b)A(a^{m-1}, b^{m-1})}{(r+1)(2r+1)}, & r \geq 1 \\ \left\{ \left(\frac{r^2(a+b)+br}{(r+1)(2r+1)} \right)^r b^{mr} + \left(\frac{r^2(a+b)+ar}{(r+1)(2r+1)} \right)^r a^{mr} \right\}^{\frac{1}{r}}, & r \leq 1. \end{cases}$$

Corollary 5. Using Corollary 4. for $m = 1$, we have following inequalities:

$$L_2^2(a, b) \leq \begin{cases} \frac{2r}{2r+1} A(a^2, b^2) + \frac{2r^2}{(r+1)(2r+1)} G^2(a, b), & r \geq 1 \\ \left\{ \left(\frac{r^2(a+b)+br}{(r+1)(2r+1)} \right)^r b^r + \left(\frac{r^2(a+b)+ar}{(r+1)(2r+1)} \right)^r a^r \right\}^{\frac{1}{r}}, & r \leq 1. \end{cases}$$

Corollary 6. Under the conditions Theorem 4 for $q = 1$ we have the following inequalities:

$$|I(a, b, n, f)| \leq \begin{cases} \frac{1}{n!} (b-a)^{-\frac{1}{r}} \left[C_1 |f^{(n)}(b)| + C_2 |f^{(n)}(a)| \right], & r \geq 1 \\ \frac{(b-a)^{-\frac{1}{r}}}{n!} \left[C_1^r |f^{(n)}(b)|^r + C_2^r |f^{(n)}(a)|^r \right]^{\frac{1}{r}}, & r \leq 1 \end{cases}$$

Theorem 5. For $n \in \mathbb{N}$; let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be n -times differentiable function on I° , $r > 0$ and $a, b \in I^\circ$ with $a < b$. If $f^{(n)} \in L[a, b]$ and $|f^{(n)}|^q$ for $q > 1$ is r -convex function on $[a, b]$, then the following inequality holds:

$$|I(a, b, n, f)| \leq \begin{cases} \frac{1}{n!} (b-a)^{\frac{1}{p}-\frac{1}{qr}} \left(|f^{(n)}(b)|^q D_1 + |f^{(n)}(a)|^q D_2 \right)^{\frac{1}{q}}, & r \geq 1 \\ \frac{1}{n!} (b-a)^{\frac{1}{p}-\frac{1}{qr}} \left(|f^{(n)}(b)|^{qr} D_1^r + |f^{(n)}(a)|^{qr} D_2^r \right)^{\frac{1}{qr}}, & r \leq 1 \end{cases}$$

where

$$D_1 = D_1(a, b, r, n, q) = \int_a^b x^{nq} (x-a)^{\frac{1}{r}} dx$$

$$D_2 = D_2(a, b, r, n, q) = \int_a^b x^{nq} (b-x)^{\frac{1}{r}} dx.$$

Proof. Since $|f^{(n)}|^q$ for $q > 1$ is r -convex function on $[a, b]$, using Lemma 2 and the Hölder integral inequality, we have the following inequality:

$$\begin{aligned} & |I(a, b, n, f)| \\ & \leq \frac{1}{n!} \int_a^b 1 \cdot x^n |f^{(n)}(x)| dx \\ & \leq \frac{1}{n!} \left(\int_a^b 1^p dx \right)^{\frac{1}{p}} \left(\int_a^b x^{nq} |f^{(n)}(x)|^q dx \right)^{\frac{1}{q}} \\ & \leq \frac{1}{n!} \left(\int_a^b 1^p dx \right)^{\frac{1}{p}} \left(\int_a^b x^{nq} \left[\frac{x-a}{b-a} |f^{(n)}(b)|^{qr} + \frac{b-x}{b-a} |f^{(n)}(a)|^{qr} \right]^{\frac{1}{r}} dx \right)^{\frac{1}{q}}. \end{aligned}$$

Here, using Lemma 1 we obtain respectively,

For $r \geq 1$,

$$\begin{aligned} & |I(a, b, n, f)| \\ & \leq \frac{1}{n!} \left(\int_a^b 1^p dx \right)^{\frac{1}{p}} \left(\int_a^b x^{nq} \left[\left(\frac{x-a}{b-a} \right)^{\frac{1}{r}} |f^{(n)}(b)|^q + \left(\frac{b-x}{b-a} \right)^{\frac{1}{r}} |f^{(n)}(a)|^q \right] dx \right)^{\frac{1}{q}} \\ & = \frac{(b-a)^{-\frac{1}{qr}}}{n!} (b-a)^{\frac{1}{p}} \left(|f^{(n)}(b)|^q D_1 + |f^{(n)}(a)|^q D_2 \right)^{\frac{1}{q}} \\ & = \frac{1}{n!} (b-a)^{\frac{1}{p}-\frac{1}{qr}} \left(|f^{(n)}(b)|^q D_1 + |f^{(n)}(a)|^q D_2 \right)^{\frac{1}{q}}, \end{aligned}$$

For $r \leq 1$, using Minkowski inequality, we have

$$\begin{aligned} & |I(a, b, n, f)| \\ & \leq \frac{1}{n!} (b-a)^{\frac{1}{p}-\frac{1}{qr}} \left(\int_a^b \left[x^{nqr} (x-a) |f^{(n)}(b)|^{qr} + x^{nqr} (b-x) |f^{(n)}(a)|^{qr} \right]^{\frac{1}{r}} dx \right)^{\frac{1}{q}} \\ & = \frac{1}{n!} (b-a)^{\frac{1}{p}-\frac{1}{qr}} \\ & \quad \times \left\{ \left(|f^{(n)}(b)|^q \int_a^b x^{nq} (x-a)^{\frac{1}{r}} dx \right)^r + \left(|f^{(n)}(a)|^q \int_a^b x^{nq} (b-x)^{\frac{1}{r}} dx \right)^r \right\}^{\frac{1}{qr}} \\ & = \frac{1}{n!} (b-a)^{\frac{1}{p}-\frac{1}{qr}} \left(|f^{(n)}(b)|^{qr} D_1^r + |f^{(n)}(a)|^{qr} D_2^r \right)^{\frac{1}{qr}}. \end{aligned}$$

This completes the proof of theorem. \square

Corollary 7. *Under the conditions Theorem 5 for $n = 1$ we have the following inequalities:*

$$\begin{aligned} & \left| \frac{f(b)b - f(a)a}{b-a} - \frac{1}{b-a} \int_a^b f(x)dx \right| \\ & \leq \begin{cases} (b-a)^{\frac{1}{p}-\frac{1}{qr}-1} (|f'(b)|^q D_1 + |f'(a)|^q D_2)^{\frac{1}{q}}, & r \geq 1 \\ (b-a)^{\frac{1}{p}-\frac{1}{qr}-1} (|f'(b)|^{qr} D_1^r + |f'(a)|^{qr} D_2^r)^{\frac{1}{qr}}, & r \leq 1. \end{cases} \end{aligned}$$

Proposition 3. *Let $a, b \in (0, \infty)$ with $a < b$, $q > 1$ and $m \geq 1$, we have*

$$L_{\frac{m}{q}+1}(a, b) \leq \begin{cases} (b-a)^{\frac{1}{p}-\frac{1}{qr}-1} (b^m D_1 + a^m D_2)^{\frac{1}{q}}, & r \geq 1 \\ (b-a)^{\frac{1}{p}-\frac{1}{qr}-1} (b^{mr} D_1^r + a^{mr} D_2^r)^{\frac{1}{q}}, & r \leq 1. \end{cases}$$

Proof. The result follows directly from Corollary 7 for $f(x) = \frac{q}{m+q} x^{\frac{m}{q}+1}$, $x \in (0, \infty)$. \square

Corollary 8. *For $m = 1$ from Proposition 3, we obtain the following inequality:*

$$L_{\frac{1}{q}+1}(a, b) \leq \begin{cases} (b-a)^{\frac{1}{p}-\frac{1}{qr}-1} (b D_1 + a D_2)^{\frac{1}{q}}, & r \geq 1 \\ (b-a)^{\frac{1}{p}-\frac{1}{qr}-1} (b^r D_1^r + a^r D_2^r)^{\frac{1}{q}}, & r \leq 1. \end{cases}$$

Theorem 6. *For $n \in \mathbb{N}$; let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be n -times differentiable function on I° (interior of I), $r > 0$ and $a, b \in I^\circ$ with $a < b$. If $f^{(n)} \in L[a, b]$ and $|f^{(n)}|^q$ for $q > 1$ is r -convex function on $[a, b]$, then the following inequalities holds:*

$$\begin{aligned} & |I(a, b, n, f)| \\ & \leq \begin{cases} 2^{\frac{1}{qr}} \frac{b-a}{n!} \left(\frac{r}{r+1}\right)^{\frac{1}{q}} L_{np}^n(a, b) A^{\frac{1}{qr}} \left(|f^{(n)}(a)|^{qr}, |f^{(n)}(b)|^{qr} \right), & 0 < r \leq 1, \\ \frac{b-a}{n!} L_{np}^n(a, b) A^{\frac{1}{qr}} \left(|f^{(n)}(a)|^{qr}, |f^{(n)}(b)|^{qr} \right), & r \geq 0, \\ \frac{1}{n!} (b-a) L_{np}^n(a, b) \left(L_r \left(|f^{(n)}(a)|^q, |f^{(n)}(b)|^q \right) \right)^{\frac{1}{q}}, & r > 0, \end{cases} \end{aligned}$$

Proof. For $0 < r \leq 1$, since $|f^{(n)}|^q$ for $q > 1$ is r -convex function on $[a, b]$, with respect to Hermite-Hadamard inequality we have

$$\int_a^b |f^{(n)}(x)|^q dx \leq (b-a) \frac{r}{r+1} \left[|f^{(n)}(a)|^{qr} + |f^{(n)}(b)|^{qr} \right]^{\frac{1}{r}}.$$

Using Lemma 2 and the Hölder integral inequality we have

$$|I(a, b, n, f)|$$

$$\begin{aligned}
&\leq \frac{1}{n!} \int_a^b x^n |f^{(n)}(x)| dx \\
&\leq \frac{1}{n!} \left(\int_a^b x^{np} dx \right)^{\frac{1}{p}} \left(\int_a^b |f^{(n)}(x)|^q dx \right)^{\frac{1}{q}} \\
&\leq \frac{1}{n!} \left(\int_a^b x^{np} dx \right)^{\frac{1}{p}} \left((b-a) \frac{r}{r+1} \left[|f^{(n)}(a)|^{qr} + |f^{(n)}(b)|^{qr} \right]^{\frac{1}{r}} \right)^{\frac{1}{q}} \\
&= 2^{\frac{1}{qr}} \frac{b-a}{n!} \left(\frac{r}{r+1} \right)^{\frac{1}{q}} \left[\frac{|f^{(n)}(a)|^{qr} + |f^{(n)}(b)|^{qr}}{2} \right]^{\frac{1}{qr}} \\
&= 2^{\frac{1}{qr}} \frac{b-a}{n!} \left(\frac{r}{r+1} \right)^{\frac{1}{q}} L_{np}^n(a, b) A^{\frac{1}{qr}} \left(|f^{(n)}(a)|^{qr}, |f^{(n)}(b)|^{qr} \right).
\end{aligned}$$

For $r \geq 1$, since $|f^{(n)}|^q$ for $q > 1$ is r -convex function on $[a, b]$, with respect to Theorem 2 we get

$$\int_a^b |f^{(n)}(x)|^q dx \leq (b-a) \left[\frac{|f^{(n)}(a)|^{qr} + |f^{(n)}(b)|^{qr}}{2} \right]^{\frac{1}{r}}.$$

Using Lemma 2 and the Hölder integral inequality we have

$$\begin{aligned}
|I(a, b, n, f)| &\leq \frac{1}{n!} \int_a^b x^n |f^{(n)}(x)| dx \\
&\leq \frac{1}{n!} \left(\int_a^b x^{np} dx \right)^{\frac{1}{p}} \left(\int_a^b |f^{(n)}(x)|^q dx \right)^{\frac{1}{q}} \\
&= \frac{b-a}{n!} \left[\frac{|f^{(n)}(a)|^{qr} + |f^{(n)}(b)|^{qr}}{2} \right]^{\frac{1}{qr}} \\
&= \frac{b-a}{n!} L_{np}^n(a, b) A^{\frac{1}{qr}} \left(|f^{(n)}(a)|^{qr}, |f^{(n)}(b)|^{qr} \right).
\end{aligned}$$

For $r > 0$, using Lemma 2, Theorem 1 and the Hölder integral inequality we have

$$|I(a, b, n, f)| \leq \frac{1}{n!} \int_a^b x^n |f^{(n)}(x)| dx$$

$$\begin{aligned}
&\leq \frac{1}{n!} \left(\int_a^b x^{np} dx \right)^{\frac{1}{p}} \left(\int_a^b |f^{(n)}(x)|^q dx \right)^{\frac{1}{q}} \\
&\leq \frac{1}{n!} \left(\int_a^b x^{np} dx \right)^{\frac{1}{p}} \left((b-a) L_r \left(|f^{(n)}(a)|^q, |f^{(n)}(b)|^q \right) \right)^{\frac{1}{q}} \\
&= \frac{1}{n!} (b-a) L_{np}^n(a, b) \left(L_r \left(|f^{(n)}(a)|^q, |f^{(n)}(b)|^q \right) \right)^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof of theorem. \square

Corollary 9. Under the conditions Theorem 6 for $n = 1$ we have the following inequalities:

$$\begin{aligned}
&\left| \frac{f(b)b - f(a)a}{b-a} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
&\leq \begin{cases} 2^{\frac{1}{qr}} \left(\frac{r}{r+1} \right)^{\frac{1}{q}} L_p(a, b) A^{\frac{1}{qr}} (|f'(a)|^{qr}, |f'(b)|^{qr}), & 0 < r \leq 1 \\ L_p(a, b) A^{\frac{1}{qr}} (|f'(a)|^{qr}, |f'(b)|^{qr}), & r \geq 1 \\ L_p(a, b) (L_r (|f'(a)|^q, |f'(b)|^q))^{\frac{1}{q}}, & r > 0. \end{cases}
\end{aligned}$$

Proposition 4. Let $a, b \in (0, \infty)$ with $a < b$, $q > 1$ and $m \in [0, 1]$, we have

$$L_{\frac{m}{q}+1}^{\frac{m}{q}+1}(a, b) \leq \begin{cases} 2^{\frac{1}{qr}} \left(\frac{r}{r+1} \right)^{\frac{1}{q}} L_p(a, b) A^{\frac{1}{qr}} (a^{mr}, b^{mr}), & 0 < r \leq 1 \\ L_p(a, b) A^{\frac{1}{qr}} (a^{mr}, b^{mr}), & r \geq 1 \\ L_p(a, b) (L_r (a^m, b^m))^{\frac{1}{q}}, & r > 0 \end{cases}$$

Under the assumption of the Proposition, let $f(x) = \frac{q}{m+q} x^{\frac{m}{q}+1}$, $x \in (0, \infty)$. Then $|f'(x)|^q = x^m$ is r -convex on $(0, \infty)$ and the result follows directly from Corollary 9.

Corollary 10. For $m = 1$ from Proposition 4, we obtain the following inequalities:

$$L_{\frac{1}{q}+1}^{\frac{1}{q}+1}(a, b) \leq \begin{cases} 2^{\frac{1}{qr}} \left(\frac{r}{r+1} \right)^{\frac{1}{q}} L_p(a, b) A^{\frac{1}{qr}} (a^r, b^r), & 0 < r \leq 1 \\ L_p(a, b) A^{\frac{1}{qr}} (a^r, b^r), & r \geq 1 \\ L_p(a, b) (L_r (a, b))^{\frac{1}{q}}, & r > 0 \end{cases}$$

Theorem 7. For $n \in \mathbb{N}$; let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be n -times differentiable function on I° , $r > 0$ and $a, b \in I^\circ$ with $a < b$. If $f^{(n)} \in L[a, b]$ and $|f^{(n)}|^{\frac{q}{r}}$ for $q > 1$ is r -convex function on $[a, b]$, then the following inequality holds:

$$|I(a, b, n, f)| \leq \frac{1}{n!} (b-a) L_{np}^n(a, b) A^{\frac{1}{q}} \left(|f^{(n)}(a)|^q, |f^{(n)}(b)|^q \right).$$

Proof. If $|f^{(n)}|^\frac{q}{r}$ for $q > 1$ is r -convex function on $[a, b]$ and $r > 0$, using (1.1) inequality, Lemma 2 and the Hölder integral inequality respectively, we have

$$\int_a^b |f^{(n)}(x)|^q dx = \int_a^b \left(|f^{(n)}(x)|^\frac{q}{r} \right)^r dx \leq (b-a) \frac{|f^{(n)}(a)|^q + |f^{(n)}(b)|^q}{2}$$

and

$$\begin{aligned} |I(a, b, n, f)| &\leq \frac{1}{n!} \int_a^b x^n |f^{(n)}(x)| dx \\ &\leq \frac{1}{n!} \left(\int_a^b x^{np} dx \right)^{\frac{1}{p}} \left(\int_a^b |f^{(n)}(x)|^q dx \right)^{\frac{1}{q}} \\ &= \frac{1}{n!} \left(\frac{b^{np+1} - a^{np+1}}{np+1} \right)^{\frac{1}{p}} \left((b-a) \frac{|f^{(n)}(a)|^q + |f^{(n)}(b)|^q}{2} \right)^{\frac{1}{q}} \\ &= \frac{1}{n!} (b-a) L_{np}^n(a, b) A^{\frac{1}{q}} \left(|f^{(n)}(a)|^q, |f^{(n)}(b)|^q \right). \end{aligned}$$

This completes the proof of theorem. \square

Corollary 11. *Under the conditions Theorem 7 for $n = 1$ we have the following inequality:*

$$\left| \frac{f(b)b - f(a)a}{b-a} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq L_p(a, b) A^{\frac{1}{q}} \left(|f'(a)|^q, |f'(b)|^q \right).$$

Proposition 5. *Let $a, b \in (0, \infty)$ with $a < b$, $q > 1$ and $m \in [0, 1]$, we have*

$$\left| \frac{f(b)b - f(a)a}{b-a} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq L_p(a, b) A^{\frac{1}{q}} (a^{mr}, b^{mr}).$$

Proof. Under the assumption of the Proposition, let $f(x) = \frac{q}{mr+q} x^{\frac{mr}{q}+1}$, $x \in (0, \infty)$. Then $|f'(x)|^\frac{q}{r} = x^m$ is r -convex on $(0, \infty)$ and the result follows directly from Corollary 11. \square

Corollary 12. *For $m = 1$ from Proposition 5, we obtain the following inequality:*

$$\left| \frac{f(b)b - f(a)a}{b-a} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq L_p(a, b) A^{\frac{1}{q}} (a^r, b^r).$$

Theorem 8. *For $n \in \mathbb{N}$; let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be n -times differentiable function on I° , $r > 0$ and $a, b \in I^\circ$ with $a < b$. If $f^{(n)} \in L[a, b]$ and $|f^{(n)}|^\frac{q}{r}$ for $q > 1$ is*

r-concave function on $[a, b]$, then the following inequality holds:

$$|I(a, b, n, f)| \leq \frac{b-a}{n!} L_{np}^n(a, b) \left| f^{(n)}\left(\frac{a+b}{2}\right) \right|.$$

Proof. If $|f^{(n)}(x)|^{\frac{q}{r}}$ for $q > 1$ is *r*-concave function on $[a, b]$ and $r > 0$, using Lemma 2, the Hölder integral inequality and

$$\int_a^b |f^{(n)}(x)|^q dx = \int_a^b \left(|f^{(n)}(x)|^{\frac{q}{r}} \right)^r dx \leq (b-a) \left| f^{(n)}\left(\frac{a+b}{2}\right) \right|^q$$

we have

$$\begin{aligned} |I(a, b, n, f)| &\leq \frac{1}{n!} \int_a^b x^n |f^{(n)}(x)| dx \\ &\leq \frac{1}{n!} \left(\int_a^b x^{np} dx \right)^{\frac{1}{p}} \left(\int_a^b |f^{(n)}(x)|^q dx \right)^{\frac{1}{q}} \\ &\leq \frac{1}{n!} \left(\int_a^b x^{np} dx \right)^{\frac{1}{p}} \left((b-a) \left| f^{(n)}\left(\frac{a+b}{2}\right) \right|^q \right)^{\frac{1}{q}} \\ &= \frac{1}{n!} (b-a) L_{np}^n(a, b) \left| f^{(n)}\left(\frac{a+b}{2}\right) \right|. \end{aligned}$$

This completes the proof of theorem. \square

Corollary 13. *Under the conditions Theorem 8 for $n = 1$ we have the following inequality:*

$$\left| \frac{f(b)b - f(a)a}{b-a} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq L_p(a, b) \left| f'\left(\frac{a+b}{2}\right) \right|.$$

Proposition 6. *Let $a, b \in (0, \infty)$ with $a < b$, $q > 1$ and $m \in [0, 1]$, we have*

$$\left| \frac{f(b)b - f(a)a}{b-a} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq L_p(a, b) A^{\frac{mr}{q}}(a, b).$$

Proof. Under the assumption of the Proposition, let $f(x) = \frac{q}{mr+q} x^{\frac{mr}{q}+1}$, $x \in (0, \infty)$. Then $|f'(x)|^{\frac{q}{r}} = x^m$ is *r*-concave on $(0, \infty)$ and the result follows directly from Corollary 13. \square

Corollary 14. *For $m = 1$ from Proposition 6, we obtain the following inequality:*

$$\left| \frac{f(b)b - f(a)a}{b-a} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq L_p(a, b) A^{\frac{r}{q}}(a, b).$$

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