Some New Notes on Mersenne Primes and Perfect Numbers

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Abstract

Mersenne primes are a specific type of prime number that can be derived using the formula $M_p = 2^p - 1$, where p is a prime number. A perfect number is a positive integer of the form $P(p) = 2^{p-1}(2^p - 1)$, where $2^p - 1$ is a Mersenne prime and can be written as the sum of its proper divisor, that is, a number which is half the sum of all of its positive divisor. In this paper, some concepts relating to Mersenne primes and perfect numbers were revisited. Mersenne primes and perfect numbers were evaluated using triangular numbers. Further, this paper discussed how to partition perfect numbers into odd cubes for odd prime p. The formula that partition perfect numbers in terms of its proper divisors were developed. The results of this study are useful to understand the mathematical structures of Mersenne primes and perfect numbers.

Keywords: Mersenne primes, perfect numbers, triangular numbers

INTRODUCTION

According to Euclid's proposition (Niven & Zuckerman, 1980), if as many numbers as we please beginning from a unit be set out continuously in double proportion until the sum of all becomes a prime, and if the sum multiplied into the last make some numbers, the product will be perfect. In this context, *double proportion* means that each number is twice the preceding number, as in 1, 2, 4, 8, ... For example, $1 + 2 + 4 = 7$ is prime; therefore, $7 \times 4 = 28$ is a perfect number. Put simply, if $1 + 2 + 4 + 8 + \cdots + p = P$, where P is prime, then $P \times p$ is a perfect number. We can use the fact that $1 + 2 + 4 + 8 + \cdots + 2^{p-1} = 2^p - 1$ to rewrite Euclid's results in a modern form. A theorem states that every perfect number is in the form of $2^{p-1}(2^p - 1)$, where $2^p - 1$ is a Mersenne prime (Ore, 1948). In order to prove this theorem, a helpful function $\sigma(n)$, where n is a positive integer, is used to analyze perfect numbers (Erickson & Vazzana, 2008). Let $\sigma(n)$ be the sum of all the positive divisors of n. Thus, $\sigma(n) = \sum_{d|n} d$, where d is a divisor of n.

Furthermore, if n is perfect, then $\sigma(n) = 2n$. If n is prime, say $n = p$, then $\sigma(p) = p + 1$. If n is a prime power, say $n = p^k$, then $\sigma(p^k) = 1 + p + p^2 + \cdots + p^k = \frac{p^{k+1}-1}{p-1}$. If n is the product of two distinct primes, say $n = pq$, then $\sigma(pq) = 1 + p + q + pq = (1 + p)(1 + q)$. Hence, $\sigma(n)$ is a multiplicative function, which is to say, $\sigma(pq) = \sigma(p)\sigma(q)$.

Niven and Zuckerman (1980) stated that the sum of proper divisors of a positive integer gives various other kinds of numbers. Numbers in which the sum is less than the number itself are called deficient, and that the sum is greater than the number are called abundant. A semi-perfect number is a natural number that is equal to the sum of all or some of its proper divisors. A semi-perfect number that is equal to the sum of all its proper divisors is a perfect number. Most abundant numbers are also semi-perfect; abundant numbers which are not semi-perfect are called weird numbers (Dickson, 1971). These terms, together with *perfect* itself, come from Greek numerology. In solving a perfect number, there

is a one-to-one correspondence between perfect number and of a prime number of the form $2^p - 1$. These prime integers are called Mersenne primes (Rosen, 1993).

Generally, numbers of the form $M_n = 2^n - 1$ without primarily requirement conditions are called Mersenne numbers. Mersenne numbers are sometimes defined to have the additional requirement that n must be prime, equivalently that they called pernicious Mersenne numbers, namely those numbers whose binary representation contains a prime number of ones and no zeros. The smallest composite pernicious Mersenne number is $2^{11} - 1 = 2047 = 23 * 89$. Dickson (1971) stated that the story of Mersenne numbers started in the 16th Century, with the French Monk, Father Marin Mersenne (1588 – 1648). Mersenne had gained an interest in the numbers of the form $2^n - 1$ (mainly from Fermat's new tools, like his Little Theorem), and in 1644 produced *Cogitata Physica-Mathematica*, in which Mersenne stated that $2^p - 1$ is prime for the following values of *p:* 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, 257 and composite for all other values of *p* < 257. It was clear to Mersenne's peers that he could not have possibly tested all these values, and as it is, his assertion was incorrect. Not every prime value of *p* in $2^p - 1$ results in a prime, but the chances of $2^p - 1$ being prime are much greater than for a randomly selected number. It took some 300 years before the details of this assertion could be checked completely, with the following outcome: M_p is not a prime for $p = 67$ and $p = 257$, and M_p is a prime for $p = 61, p = 89$ and 107. Mersenne made 5 mistakes. Thus, there are 12 primes $p < 257$ such that M_p is a prime. The triangular number T_n is the number of dots in the triangular array with n rows that has j dots in the jth row (Rosen, 1993). For instance, $T_1 = 1$, $T_2 = 3$, $T_3 = 6$, and $T_4 = 10$. This is defined as $T_n = \dfrac{n(n+1)}{2}, n \in \{1, 2, 3, ...\}.$ Triangular numbers, in fact, is a family of numbers (Montalbo et al., 2015). This study

intended to expose the new structures of Mersenne primes, perfect numbers, and triangular numbers. Specifically, it aimed to develop new claims relating to the said numbers. We also partitioned perfect numbers into odd cubes and derived a formula for odd prime *p*. Further, we construct a formula on how to partition perfect numbers in terms of its proper divisors and determined the number of primes in the partition.

METHOD

This study is exploratory in nature. The formula that generates Mersenne primes and perfect numbers were presented using the concept of Euclid's proposition. Furthermore, different useful functions related to perfect numbers in proving theorems were considered for the analysis in partitioning these numbers. Mersenne primes and perfect numbers were evaluated according to its structures, existence, and characteristics. Also, those numbers were evaluated by a triangular number. Calculations of perfect numbers were then developed, after which partitioning these numbers to its proper divisors were done, and then it leads to the partitioning formula. Figure 1 presents the schematic diagram of the flow of the study.
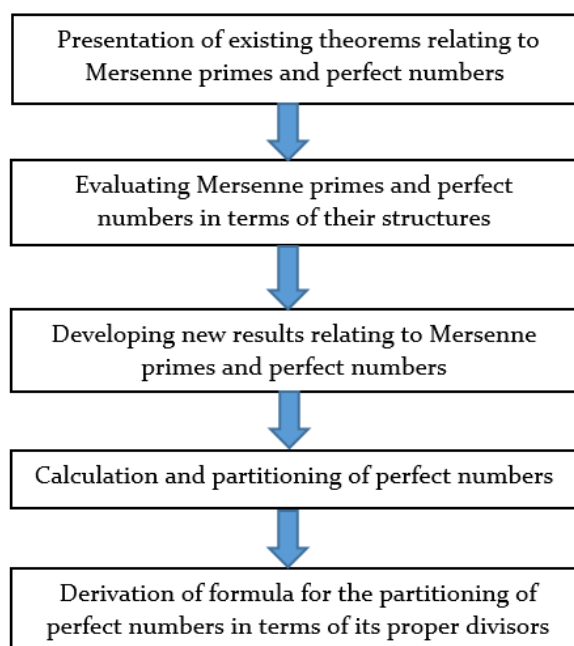

Figure 1. Schematic Diagram of the Research Flow

RESULTS AND DISCUSSION

From the definition of Mersenne prime above, the following Remark is immediate.

Remark 1. If $2^n - 1$ is prime, then it is a Mersenne prime.

The next Theorem determines all even perfect numbers stated by Euclid and later developed by Euler into modern form (Ore, 1948; Rosen, 1993).

Theorem 2. (Euclid-Euler) The positive integer n is an even perfect number if and only if $n = 2^{p-1}(2^p - 1)$ where p is an integer such that $p \geq 2$ and $2^p - 1$ is Mersenne prime.

Proof. ($\Rightarrow$) Let n be even perfect number. Write $n = 2^s t$, where s and t are positive integers and t is odd. Since $(2^s, t) = 1$, then we have $\sigma(n) = \sigma(2^s t) = \sigma(2^s)\sigma(t) = (2^{s+1} - 1)\sigma(t)$ (eq'n 1). Since n is perfect, then we have $\sigma(n) = 2n = 2^{s+1}t$ (eq'n 2). Combining eq'n 1 and 2 shows that $(2^{s+1} - 1)\sigma(t) = 2^{s+1}t$ (eq'n 3). Since $(2^{s+1}, 2^{s+1} - 1) = 1$, then $2^{s+1} | \sigma(t)$. Hence, there is an integer q such that $\sigma(t) = 2^{s+1}q$. Inserting this expression for $\sigma(t)$ into eq'n 3 tells us that $(2^{s+1} - 1)2^{s+1}q = 2^{s+1}t$, and therefore $(2^{s+1} - 1)q = 1$ (eq'n 4). So, $q|t$ and $q \neq t$. Replacing t by the expression on the left-hand side of eq'n 4, we find that $t + q = (2^{s+1} - 1)q + q = 2^{s+1}q = \sigma(t)$ (eq'n 5).

Next, we will show that $q = 1$. Note that if $q \neq 1$, then there are at least three distinct positive divisors of t, namely $1, q,$ and t. This implies that $\sigma(t) \geq t + q + 1$, which contradicts eq'n 5. Hence, $q = 1$ and from eq'n 4, we conclude that $t = 2^{s+1} - 1$. Also, from eq'n 5, we see that $\sigma(t) = t + 1$, so that t must be prime since its only positive divisors are 1 and t. Thus, $n = 2^s(2^{s+1} - 1)$, where $2^{s+1} - 1$ is prime.

($\Leftarrow$) We show that if $n = 2^{p-1}(2^p - 1)$ where $2^p - 1$ is Mersenne prime, then n is perfect. Note that since $2^p - 1$ is odd, we have $(2^{p-1}, 2^p - 1) = 1$. Since σ is a multiplicative function, it follows that

$\sigma(n) = \sigma(2^{p-1})\sigma(2^p - 1)$. Note that $\sigma(2^{p-1}) = 2^p - 1$ and $\sigma(2^p - 1) = 2^p$, since we are assuming that $2^p - 1$ is prime. Thus, $\sigma(n) = (2^p - 1)2^p = 2n$, demonstrating that n is a perfect number. $\square$

Suppose we have a sequence that satisfies a certain recurrence relation and initial conditions. It is often helpful to know an explicit formula for the sequence, especially if we need to compute terms with very large subscripts or if we need to examine general properties for the sequence. The explicit formula is called a solution to the recurrence relation. The following result for Mersenne primes and perfect numbers involving the concepts of recurrence relation stated as Theorem 3, is presented as follows.

Theorem 3. Let $a_1 = 1$. If $a_n = 2a_{n-1} + 1$ is a prime for some positive integer $n \geq 2$, then

  i.) a_n is a Mersenne prime; and

  ii.) $P = a_n 2^{n-1}$ is an even perfect number.

Proof (i). Suppose that $a_n = 2a_{n-1} + 1$ is prime for some positive integer $n \geq 2$.

Then, we have
$$a_n = 2a_{n-1} + 1$$
$$= 2(2a_{n-2} + 1) + 1$$
$$= 2^2 a_{n-1} + 2 + 1$$
$$= 2^2(2a_{n-3} + 1) + 2 + 1$$
$$a_n = 2^3 a_{n-3} + 2^2 + 2 + 1.$$

Continuing the process, we obtain
$$a_n = 1 + 2^1 + 2^2 + 2^3 + \cdots + 2^{n-1}a_{n-(n-1)}$$
$$= 1 + 2^1 + 2^2 + 2^3 + \cdots + 2^{n-1}a_1.$$

Since $a_1 = 1$, it follows that,
$$a_n = 1 + 2^1 + 2^2 + 2^3 + \cdots + 2^{n-1}$$
$$a_n = \sum_{i=0}^{n-1} 2^i.$$

(Consider this as equation 1)

Multiplying equation 1 with -2, we have
$$-2a_n = -\sum_{i=0}^{n-1} 2^{i+1}.$$

(Consider this as equation 2)

Combining equation 1 and equation 2, it follows that

$$a_n - 2a_n = \sum_{i=0}^{n-1} 2^i - \sum_{i=0}^{n-1} 2^{i+1} = \sum_{i=0}^{n-1}(2^i - 2^{i+1}).$$

Thus, $a_n = \sum_{i=0}^{n-1}(2^{i+1} - 2^i)$

$$= (2^1 - 1) + (2^2 - 2^1) + (2^3 - 2^2) + \cdots + (2^n - 2^{n-1}).$$

By telescoping we obtain,

$$a_n = 2^n - 1.$$

Since $a_n = 2^n - 1$ is prime, it follows that $a_n = 2^n - 1$ is a Mersenne prime by *Remark 1*.

Proof (ii). Suppose that a_n is prime for some positive integer $n \geq 2$. Then, a_n is a Mersenne prime and it follows that n is a prime number by definition. Thus, by Theorem 2, $P = a_n 2^{n-1}$ is an even perfect number. □

Lemma 4. (Leithold, 1996) Let $a, b, c \in \mathbb{N}$ and F is a function. Then,

$$\sum_{i=a}^{b} F(i) = \sum_{i=a-c}^{b-c} F(i+c).$$

The next result on Mersenne primes and perfect numbers involves sigma notation that concern with the sums of many terms and it is a direct consequence of Lemma 4 above. This sigma notation is to facilitate writing these sums and use of the symbol $\sum$.

Theorem 5. If $S_n = \sum_{i=1}^{n}(2^i - 2^{i-1})$ is a prime for some positive integer $n \geq 2$, then
   i.) S_n is a Mersenne prime; and
   ii.) $P = S_n 2^{n-1}$ is an even perfect number.

Proof (i). Suppose that $S_n = \sum_{i=1}^{n}(2^i - 2^{i-1})$ is a prime for some positive integer $n \geq 2$. Then, by Lemma 4, we have

$$S_n = \sum_{i=1}^{n}(2^i - 2^{i-1}) = \left(\sum_{i=1}^{n-1} 2^i + 2^n\right) - \left(\sum_{i=1-1}^{n-1} 2^{(i+1)-1}\right)$$

$$= \sum_{i=1}^{n-1} 2^i + 2^n - \sum_{i=0}^{n-1} 2^i$$

$$= \sum_{i=1}^{n-1} 2^i + 2^n - \left(\sum_{i=1}^{n-1} 2^i + 2^0\right)$$

$$= \sum_{i=1}^{n-1} 2^i + 2^n - \sum_{i=1}^{n-1} 2^i - 1$$

$$S_n = 2^n - 1.$$

Since $S_n = 2^n - 1$ is a prime, then by Remark 1, S_n is a Mersenne prime.

Proof (ii). Since S_n is a Mersenne prime for some positive integer $n \geq 2$, then it follows that n is a prime number by definition. Thus, $P = S_n 2^{n-1}$ is an even perfect number by Theorem 2. □

Theorem 6. (Dickson, 1971) If p is prime and a is a positive integer with $\gcd(a, p) = 1$, then $a^{p-1} \equiv 1 (mod\ p)$.

For our next result involving Mersenne prime, this is obtained using the Fermat's last theorem above. This Theorem shows that if a Mersenne prime is subtracted by 1, then it is divisible by $2p$ whenever p is a prime number.

Theorem 7. Let p be an odd prime and M_p be a Mersenne prime. Then, $M_p = 2^p - 1$ less by one is divisible by $2p$.

Proof. Since p is an odd prime, then $\gcd(2, p) = 1$. Then, it follows that $2^{p-1} \equiv 1 (mod\ p)$ by Theorem 6. This implies that $2^{p-1} - 1 \equiv 0 (mod\ p)$ and hence $p | (2^{p-1} - 1)$. Then, we have $p | \left(\frac{2^p - 2}{2}\right)$ and it implies that $2p | (2^p - 2)$. Thus, $(2^p - 1) - 1$ is divisible by $2p$. □

Theorem 8. (Muche et. al., 2017) Let $P(p) = 2^{p-1}(2^p - 1)$ be a perfect number. Then, $P(p)$ is a triangular number.

Proof. Consider that $P(p) = 2^{p-1}(2^p - 1)$ is a perfect number. Then,

$$P(p) = 2^{p-1}(2^p - 1)$$
$$= \frac{2^p}{2}(2^p - 1)$$
$$= \frac{(2^{p-1}+1)(2^p-1)}{2}.$$

Let $m = 2^p - 1$ be a Mersenne prime. Then, $P(m) = \frac{m(m+1)}{2}$ and by definition of triangular number, $P(p)$ is a triangular number. $\square$

Since a perfect number is a triangular number by Theorem 8, then a perfect number can be partitioned as the sum of natural numbers from 1 to $2^p - 1$ where p is prime. This is shown in Remark 9 below.

Remark 9. If $2^p - 1$ is a Mersenne prime, then $P(p) = \sum_{i=1}^{2^p-1} i$ is a perfect number.

The following results below are the direct consequences of the definition of a triangular number and Theorem 8. The function $T(n)$ in Theorem 10 and 11 is several ways of putting a dominating set in path graph when the order of graph is $n \equiv 1 (mod\ 3)$ (Casinillo, 2018).

Theorem 10. If $n \equiv 1(mod\ 3)$ and $T(n) = \frac{1}{18}(n^2 + 7n + 10)$, then $T(n)$ is a triangular number.

Proof. Suppose that $n \equiv 1(mod\ 3)$. Then, $n = 3k + 1$ for all positive integer k. So, we have,

$$T(3k+1) = \frac{1}{18}((3k+1)^2 + 7(3k+1) + 10)$$
$$= \frac{1}{18}(9k^2 + 6k + 1 + 21k + 7 + 10)$$
$$= \frac{1}{18}(9k^2 + 27k + 18)$$
$$= \frac{1}{2}(k^2 + 3k + 2)$$
$$= \frac{1}{2}(k+2)(k+1)$$
$$T(3k+1) = \frac{m(m+1)}{2}, \quad \text{where } m = k+1.$$

Thus, by definition of a triangular number, then it follows that $T(n)$ is a triangular number. $\square$

Theorem 11. Let $n = 3(2^p - 1) - 2$ and $T(n) = \frac{1}{18}(n^2 + 7n + 10)$. If $2^p - 1$ is Mersenne prime, then $T(n)$ is a perfect number.

Proof. Suppose that $n = 3(2^p - 1) - 2$ and $2^p - 1$ is Mersenne prime. Then, we obtained,

$$T(3(2^p - 1) - 2)$$
$$= \frac{1}{18}((3(2^p - 1) - 2)^2$$
$$+ 7(3(2^p - 1) - 2) + 10)$$
$$= \frac{1}{18}[9(2^p - 1)^2 - 12(2^p - 1) + 4$$
$$+ 21(2^p - 1) - 14 + 10$$
$$= \frac{1}{2}[(2^p - 1)^2 + (2^p - 1)$$
$$= \frac{1}{2}(2^{2p} - 2(2^p) + 1 + 2^p - 1)$$
$$= \frac{2^p}{2}(2^p - 1)$$
$$T(3(2^p - 1) - 2) = 2^{p-1}(2^p - 1).$$

Thus, by Theorem 2, $T(n)$ is a perfect number. $\square$

Lemma 12. (Santos, 1977) The sum of the first n terms of an arithmetic progression whose general term $a_n = a_1 + (n - 1)d$, is $S_n = \frac{n}{2}(a_1 + a_n)$, where d is a common difference.

We need the Lemma above to prove the next Theorem. This Theorem involves a recurrence relation as a function of n, where n is a Mersenne prime.

Theorem 13. Let $a_n = a_{n-1} + n$, where $a_0 = 0$. If n is a Mersenne prime, then a_n is a triangular number.

Proof. Let $a_n = a_{n-1} + n$ where $a_0 = 0$ and n is a Mersenne prime. Then, we have

$$a_n = a_{n-1} + n$$
$$= (a_{n-2} + (n - 1)) + n$$
$$= (a_{n-3} + (n - 2)) + (n - 1) + n.$$

Continuing the process, we end up with,
$$a_n = a_0 + 1 + 2 + 3 + \cdots + (n-3) +$$
$$(n-2) + (n-1) + n$$
$$= 1 + 2 + 3 + \cdots + (n-3) + (n-2) +$$
$$(n-1) + n, \text{ since } a_0 = 0$$

Now, let $n = 2^p - 1$ be a Mersenne prime, then we obtained
$$a_{2^p-1} = 1 + 2 + \cdots + ((2^p - 1) - 3)$$
$$+ ((2^p - 1) - 2) + ((2^p - 1) - 1)$$
$$+ (2^p - 1).$$

By Lemma 12, it follows that
$$a_{2^p-1} = \frac{2^p - 1}{2}[2(1) + ((2^p - 1) - 1)]$$
$$= 2^{p-1}(2^p - 1).$$

By Theorem 2, $a_{2^p-1} = 2^{p-1}(2^p - 1)$ is a perfect number. Clearly, we have
$$a_{2^p-1} = 2^{p-1}(2^p - 1)$$
is a triangular number by Theorem 8. This completes the proof showing that a_n is a triangular number. □

The following Remark is a direct consequence of Theorem 8 and Theorem 13.

Remark 14. Let p is an odd prime. Then, $P(p) = 1 + 9T_{\left(\frac{2^p-2}{3}\right)}$ is a perfect number.

Lemma 15. (Gallian, 2010) Let $a, b,$ and n be an integer. If $n|ab$ and n is relatively prime to a, then $n|b$.

The next result for a perfect number is obtained using the concepts of congruencies and the generalized Euclid's lemma above.

Theorem 16. Let $P(p) = 2^{p-1}(2^p - 1)$ be a perfect number. Then, one of the following two conditions hold:
i.) $P(p) \equiv 0(mod\ 3)$; or
ii.) $P(p) \equiv 1(mod\ 9)$.

Proof. Let $z = 2^p - 1$. Then, $P(z) = \frac{z(z+1)}{2}$ is a triangular number by Theorem 8.
Then, consider the following cases:
Case 1. Let $z \equiv 0(mod\ 3)$. Then, $z(z+1) \equiv 0(mod\ 3)$ and so we obtained
$P(p) \equiv 0(mod\ 3)$ by Lemma 15.

Case 2. Let $z \equiv 2(mod\ 3)$. Then, $z + 1 \equiv 3 \equiv 0(mod\ 3)$. Then, it follows that
$z(z+1) \equiv 0(mod\ 3)$ and hence, we have
$P(p) \equiv 0(mod\ 3)$.

Case 3. Let $z \equiv 1(mod\ 3)$. Then, there exists a positive integer m such that
$z = 3m + 1$. So, we have
$$z(z+1) = (3m+1)(3m+2)$$
$$= 9m^2 + 6m + 3m + 2$$
$$= 9m^2 + 9m + 2$$
$$= 9m(m+1) + 2$$
$$\frac{z(z+1)}{2} = \frac{9m(m+1)}{2} + 1$$

Obviously, it follows that
$P(m) = 9\frac{m(m+1)}{2} + 1 = P(z)$. Thus, we obtained $P(p) \equiv 1(mod\ 9)$. □

Lemma 17. (Santos, 1977) Let n be a natural number. Then, $\displaystyle\sum_{i=1}^{n}(2i-1)^3 = n^2(2n^2 - 1)$.

For the next result that involves how to partition an even perfect number into odd cubes, we need the following Lemma below to prove this result.

Theorem 18. Let $P(p)$ be a perfect number where p is an odd prime. Then,
$$P(p) = \sum_{r=1}^{2^{(p-1)/2}}(2r-1)^3.$$

Proof. Suppose p is an odd prime. Then, it must be shown that
$\displaystyle\sum_{r=1}^{2^{(p-1)/2}}(2r-1)^3 = 2^{p-1}(2^p - 1)$. So, by Lemma 17, we obtain
$$\sum_{r=1}^{2^{(p-1)/2}}(2r-1)^3 = \left(2^{\frac{p-1}{2}}\right)^2\left[2\left(2^{\frac{p-1}{2}}\right)^2 - 1\right]$$
$$= 2^{p-1}[2(2^{p-1}) - 1]$$
$$= 2^{p-1}(2^p - 1).$$

This completes the proof showing that each even perfect number $P(p) = 2^{p-1}(2^p - 1)$, where p is an odd prime can be written as the sum of odd cubes. □

As a direct consequence of Theorem 18, Remark 19 is obtained and presented below.

Remark 19. There are $2^{(p-1)/2}$ distinct odd cube summands in a perfect number of the form $P(p) = 2^{p-1}(2^p - 1)$, where p is an odd prime.

From the definition of perfect numbers above, Remark 20 and Theorem 21 are obtained. The following result shows that perfect numbers are half the sum of all of its positive divisors.

Remark 20. Let P be a perfect number. If $d \leq P$, then $\sum_{d|P} \left(\dfrac{P}{d}\right) = 2P$.

Theorem 21. If P is a perfect number and $d < P$ is a positive integer, then

$$\sum_{d|P} \left(\frac{1}{d}\right) = \frac{2P-1}{P}.$$

Proof. Suppose that P is a perfect number and $d < P$ is a positive integer. If $d|P$, then it implies that d is a proper divisor of P. Note that if $d \leq P$, then by definition,

$$\sigma(P) = \sum_{d|P} d = \sum_{d|P} \left(\frac{P}{d}\right)$$

By Remark 20, we obtained,

$\sum_{d|P} \left(\dfrac{P}{d}\right) = P \sum_{d|P} \left(\dfrac{1}{d}\right) = 2P$. It follows that

$\sum_{d|P} \left(\dfrac{1}{d}\right) = 2$. But $d \neq P$, so $\sum_{d|P} \left(\dfrac{1}{d}\right) = 2 - \dfrac{1}{P}$.

Thus, $\sum_{d|P} \left(\dfrac{1}{d}\right) = \dfrac{2P-1}{P}$. $\qquad \square$

Lemma 22. (Leithold, 1996) If $a_1, a_2, a_4, \ldots, a_n$ is a geometric sequence with common ratio r, and $s_n = a_1 + a_2 + \cdots + a_n$, then $s_n = \dfrac{a_n(1-r^n)}{1-r}, r \neq 1$.

It is worth noting that in partitioning a perfect number P, P is written as a sum of its proper divisors. Hence, Lemma 22 below, Theorem 23 is obtained that presents the formula that partitions a perfect number P into its proper divisors.

Theorem 23. Let $P(p) = 2^{p-1}(2^p - 1)$ be a perfect number. Then, the formula for partitioning an even perfect number in terms of proper divisors is given by

$$P(p) = \sum_{i=1}^{p} 2^{i-1} + \sum_{i=1}^{p-1} 2^{i-1}(2^p - 1).$$

Proof. Suppose that $P(p) = 2^{p-1}(2^p - 1)$ is a perfect number. Then, we have

$$\begin{aligned} P(p) &= 2^{p-1}(2^p - 1) \\ &= 2[2^{p-2}(2^p - 1)] \\ &= 2^{p-2}(2^p - 1) + 2^{p-2}(2^p - 1) \\ &= 2^{p-3}(2^p - 1) + 2^{p-3}(2^p - 1) + \\ &\quad 2^{p-2}(2^p - 1) \end{aligned}$$

Continuing the process, we obtained

$$\begin{aligned} P(p) &= 2(2^p - 1) + 2(2^p - 1) + \\ &\quad 2^2(2^p - 1) + \cdots + 2^{p-2}(2^p - 1) \\ &= (2^p - 1) + (2^p - 1) + 2(2^p - 1) \\ &\quad + 2^2(2^p - 1) + \cdots + 2^{p-2}(2^p - 1) \end{aligned}$$

Applying Lemma 22, we have

$$\begin{aligned} P(p) &= 1 + 2 + \cdots + 2^{p-1} + (2^p - 1) + \\ &\quad 2(2^p - 1) + 2^2(2^p - 1) + \cdots + \\ &\quad 2^{p-2}(2^p - 1). \end{aligned}$$

Clearly, we end up with,

$$P(p) = \sum_{i=1}^{p} 2^{i-1} + \sum_{i=1}^{p-1} 2^{i-1}(2^p - 1). \qquad \square$$

Definition 24. (Voight, 1998) The number of proper divisors of a perfect number P is the number of terms in the partitioned perfect number and it is denoted by $D(P)$.

The next Theorem is immediate from Theorem 23 and Definition 24.

Theorem 25. The number of proper divisors of a perfect number of the form $P(p) = 2^{p-1}(2^p - 1)$ is given by $D(P(p)) = 2p - 1$.

Proof. Let $P(p) = 2^{p-1}(2^p - 1)$ be a perfect number. Then, we have

$P(p) = \sum_{i=1}^{p} 2^{i-1} + \sum_{i=1}^{p-1} 2^{i-1}(2^p - 1)$ by Theorem 23.

By Definition 24, it follows that,

$$D(P(p)) = D\left(\sum_{i=1}^{p} 2^{i-1} + \sum_{i=1}^{p-1} 2^{i-1}(2^p - 1) \right).$$

Clearly, we have $D\big(P(p)\big) = p + p - 1$.

Hence, $D\big(P(p)\big) = 2p - 1$. □

The following Remarks are direct consequences of Theorem 23 and 25. These determine the number of composite and prime numbers in the summands of a partitioned perfect number.

Remark 26. Let $P(p) = 2^{p-1}(2^p - 1)$ be a perfect number. Then, $P(p)$ has $2p - 4$ composite proper divisors.

Remark 27. Let $P(p) = 2^{p-1}(2^p - 1)$ be a perfect number. Then, 2 and $2^p - 1$ are the only primes among its proper divisors.

CONCLUSION

New claims on Mersenne primes, even perfect numbers and triangular were obtained relating to recurrence relation and summation notations. The result shows that a Mersenne prime of the form $M_p = 2^p - 1$ less by one is divisible by $2p$. The study also shows that all even perfect numbers are triangular numbers, thus, the form $P(p) = 2^{p-1}(2^p - 1)$ can be partitioned as the sum of the integers from 1 to Mersenne prime of the form $M_p = 2^p - 1$. Further, the study shows that a perfect number can be written as the sum of $2^{(p-1)/2}$ odd cubes and $P(p) \equiv 0 \pmod 3$ or $P(p) \equiv 1 \pmod 9$. Also, perfect numbers can be partitioned as a sum of $2p - 1$ proper divisors with $2p - 4$ composite proper divisors and two prime divisors namely: even prime number 2 and Mersenne prime $2^p - 1$, where p is a prime number.

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