Hindawi Publishing Corporation
ISRN Combinatorics
Volume 2013, Article ID 201654, 4 pages
http://dx.doi.org/10.1155/2013/201654

Research Article

Some New Perspectives on Global Domination in Graphs

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Received 10 May 2013; Accepted 20 July 2013

Academic Editors: P. Garcia-Vazquez and P. E. Jorgensen

A dominating set is called a global dominating set if it is a dominating set of a graph G and its complement $\overline{G}$. Here we explore the possibility to relate the domination number of graph G and the global domination number of the larger graph obtained from G by means of various graph operations. In this paper we consider the following problem: Does the global domination number remain invariant under any graph operations? We present an affirmative answer to this problem and establish several results.

1. Introduction

The domination in graphs is one of the concepts in graph theory which has attracted many researchers to work on it. Many variants of dominating sets are available in the existing literature. This paper is focused on global domination in graphs.

We begin with simple, finite, and undirected graph $G = (V, E)$ with $|V| = n$. The set $S \subseteq V$ is called a dominating set if $N[S] = V$. A dominating set S is called a minimal dominating set (MDS) if no proper subset S' of S is a dominating set.

The minimum cardinality of a dominating set in G is called the domination number of G denoted by $\gamma(G)$, and the corresponding dominating set is called a γ-set of G.

The complement $\overline{G}$ of G is the graph with vertex set V in which two vertices are adjacent in $\overline{G}$ if and only if they are not neighbors in G.

A dominating set S of G is called a global dominating set if it is also a dominating set of $\overline{G}$. The global domination number $\gamma_g(G)$ is the minimum cardinality of a global dominating set of G. The concept of global domination in a graph was introduced by Sampathkumar [1]. This concept is remained in focus of many researchers. For example, the global domination number of boolean function graph is discussed by Janakiraman et al. [2]. The NP completeness of global domination problems is discussed by Carrington [3] and by Carrington and Brigham [4]. The global domination number for the larger graphs obtained from the given graph is discussed by Vaidya and Pandit [5] while Kulli and Janakiram [6] have introduced the concept of total global dominating sets. The discussion on global domination in graphs of small diameters is carried out by Gangadharappa and Desai [7].

The wheel W_n is defined to be the join $C_{n-1}+K_1$ where $n \geq 4$. The vertex corresponding to K_1 is known as apex vertex, and the vertices corresponding to cycle C_{n-1} are known as rim vertices.

Duplication of an edge $e = uv$ of a graph G produces a new graph G_1 by adding an edge $e' = u'v'$ such that $N(u) = N(u')$ and $N(v) = N(v')$.

The shadow graph $D_2(G)$ of a connected graph G is constructed by taking two copies of G, say G' and G''. Join each vertex u' in G' to the neighbors of the corresponding vertex u'' in G''.

A vertex switching G_v of a graph G is the graph obtained by taking a vertex v of G, removing all the edges incident to v and adding edges joining v to every vertex not adjacent to v in G.

For the various graph theoretic notations and terminology we follow West [8] while the terms related to the concept of domination are used in the sense of Haynes et al. [9].

Here we consider the problem: Does the global domination number remain invariant under any graph operations? We present here an affirmative answer to this question for the graphs obtained by various graph operations on C_n and W_n. Moreover, we obtain the domination number and the global domination number for the shadow graph of P_n and the graphs obtained by switching of a vertex in C_n as well as in W_n.

2. Main Results

Theorem 1. *If C_n' is a graph obtained by duplication of an edge in C_n ($n \neq 3$, $n \neq 5$) by an edge then every γ-set of C_n' is a global dominating set of C_n' and $\gamma(C_n') = \gamma_g(C_n') = \lceil n/3 \rceil$. Furthermore, the global domination number remains invariant under the operation of duplication of an edge for C_n.*

Proof. Let C_n' be a graph obtained by duplication of an edge in C_n ($n \neq 3$, $n \neq 5$). Without loss of generality, let the edge $e = v_1 v_2$ of C_n be duplicated by an edge $e' = v_1' v_2'$.

Let S be a γ-set of C_n'. Then S is a dominating set of C_n'. Now, any two distinct vertices v_i and v_j with $d(v_i, v_j) \neq 2$ in C_n' are enough to dominate $\overline{C_n'}$ because the vertices which are not in $N[v_i]$ must belong to $N[v_j]$ in $\overline{C_n'}$. Since every γ-set of C_n' contains at least two such vertices v_i and v_j, it is a dominating set of $\overline{C_n'}$. Hence, every γ-set of C_n' is a dominating set of C_n' as well as of $\overline{C_n'}$. Consequently, every γ-set of C_n' is a global dominating set of C_n'.

Now, consider a γ-set of C_n':

$$S = \begin{cases} \{v_3, v_6, v_9, \dots, v_{3j}\} & \text{if } n \equiv 0 \,(\text{mod} 3), \\ \{v_3, v_6, v_9, \dots, v_{3j}\} \cup \{v_n\} & \text{if } n \equiv 1 \text{ or } 2 \,(\text{mod} 3) \end{cases} \quad \text{for } 1 \leq j \leq \left\lfloor \frac{n}{3} \right\rfloor. \tag{1}$$

Since $d(v_3) = 3 = d(v_n) = \Delta(C_n')$, S must contain v_3 and v_n for minimum cardinality. Also, to retain the minimum cardinality of S, it must contain the vertices v_{3j} ($2 \leq j \leq \lfloor n/3 \rfloor$). That is, $|S| \geq \lceil n/3 \rceil$. Now, S being a γ-set of C_n' is a global dominating set of C_n' with minimum cardinality $\lceil n/3 \rceil$. This implies that $\gamma(C_n') = \gamma_g(C_n') = \lceil n/3 \rceil$. Also, as reported in Sampathkumar [1], $\gamma_g(C_n) = \lceil n/3 \rceil$ ($n \neq 3$, $n \neq 5$). Hence, $\gamma_g(C_n) = \gamma_g(C_n') = \lceil n/3 \rceil$ ($n \neq 3$, $n \neq 5$). Thus, the global domination number remains invariant under the operation of duplication of an edge by an edge in C_n. □

Theorem 2. *If G is the graph obtained by duplicating each edge of C_n by an edge then every γ-set of G is a global dominating set of G and $\gamma(G) = \gamma_g(G) = n$.*

Proof. Let S be a γ-set of G. Then S is a dominating set of G. Now, any two adjacent vertices v_i and v_j in G are enough to dominate $\overline{G}$ because the vertices which are not in $N[v_i]$ in $\overline{G}$ must belong to $N[v_j]$ in $\overline{G}$. Since every γ-set of G contains at least two such vertices v_i and v_j, every γ-set of G is a dominating set of $\overline{G}$. Hence, every γ-set of G is a dominating set of G as well as of $\overline{G}$. This implies that every γ-set of G is a global dominating set of G.

Let $v_1, v_2, \dots, v_n$ be the vertices of C_n. Consider a γ-set of G, $S = \{v_1, v_2, \dots, v_n\}$. S being a γ-set of G is a dominating set of G with minimum cardinality. Moreover, since $v_1, v_2, \dots, v_n$ are the vertices of maximum degree in G and from the nature of the graph G, it is clear that $S = \{v_1, v_2, \dots, v_n\}$ is of minimum cardinality. Hence, S is a γ-set of G with minimum cardinality n. Therefore, $\gamma(G) = n$. Now, S being a γ-set of G is a global dominating set of G with minimum cardinality n which implies that $\gamma(G) = \gamma_g(G) = n$ as required. □

The following Theorem 3 can be proved by the arguments analogous to the above Theorem 2.

Theorem 3. *If G is the graph obtained by duplicating each edge of P_n by an edge then every γ-set of G is a global dominating set of G and $\gamma(G) = \gamma_g(G) = n$.*

Theorem 4. *If W_n' is a graph obtained by duplicating an edge of W_n by an edge then*

$$\gamma_g(W_n) = \gamma_g(W_n') = \begin{cases} 4 & if\ n = 4 \\ 3 & otherwise. \end{cases} \tag{2}$$

That is, the global domination number remains invariant under the operation of duplication of an edge in W_n.

Proof. Let $V(W_n) = \{v_1, v_2, \dots, v_{n-1}\} \cup \{c\}$, where c is the apex vertex of W_n.

It is easy to observe that

$$\gamma_g(W_n) = \begin{cases} 4 & \text{if } n = 4 \\ 3 & \text{otherwise.} \end{cases} \tag{3}$$

Case I (When a rim edge of W_n is duplicated by an edge). Without loss of generality, let the rim edge $e = v_1 v_2$ of W_n be duplicated by an edge $e' = v_1' v_2'$.

For $n = 4$, since the vertex c is adjacent to each vertex of W_n', it must belong to any global dominating set of W_n'. Moreover, any two vertices are adjacent to third vertex other than c in W_n'. Therefore, any global dominating set of W_n' must contain at least four vertices including c which implies that $\gamma_g(W_n') = 4$.

For $n > 4$, the vertex c dominates W_n' while the vertex c and any two adjacent rim vertices of W_n are enough to dominate $\overline{W_n'}$. Therefore, any global dominating set of W_n' must contain at least three vertices of W_n'. This shows that $\gamma_g(W_n') = 3$.

Thus,

$$\gamma_g(W_n') = \begin{cases} 4 & \text{if } n = 4 \\ 3 & \text{otherwise.} \end{cases} \tag{4}$$

Case II (When a spoke edge of W_n is duplicated by an edge). Without loss of generality, let the spoke edge $e = cv_1$ be duplicated by an edge $e' = c' v_1'$.

For $n = 4$, any two vertices in W_n' are adjacent to the third vertex and also any three vertices are adjacent to the fourth vertex in W_n'. Therefore, any global dominating set of W_n' must contain at least four vertices which implies that $\gamma_g(W_n') = 4$.

For $n > 4$, clearly any global dominating set of W_n' must contain either c or c' to achieve its minimum cardinality.

Moreover, any two adjacent rim vertices of W_n and the vertex c are enough to dominate $\overline{W_n'}$ and they also dominate W_n'. This implies that $\gamma_g(W_n') = 3$.

Thus,

$$\gamma_g(W_n') = \begin{cases} 4 & \text{if } n = 4 \\ 3 & \text{otherwise.} \end{cases} \quad (5)$$

Hence, we have proved that

$$\gamma_g(W_n) = \gamma_g(W_n') = \begin{cases} 4 & \text{if } n = 4 \\ 3 & \text{otherwise.} \end{cases} \quad (6)$$

That is, the global domination number remains invariant under the operation of duplication of an edge in W_n. □

Theorem 5. *Every γ-set of $D_2(P_n)$ is a global dominating set of $D_2(P_n)$, and*

$$\gamma(D_2(P_n)) = \gamma_g(D_2(P_n)) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 4 \\ \left\lfloor \frac{n}{2} \right\rfloor + 1 & \text{otherwise.} \end{cases} \quad (7)$$

Proof. Consider two copies of P_n. Let $v_1, v_2, \ldots, v_n$ be the vertices of the first copy of P_n and $u_1, u_2, \ldots, u_n$ the vertices of the second copy of P_n.

If S is a γ-set of $D_2(P_n)$ then S is a dominating set of $D_2(P_n)$. Now, any two adjacent vertices v_i and v_j of $D_2(P_n)$ are enough to dominate $\overline{D_2(P_n)}$ because the vertices which are not in $N[v_i]$ in $\overline{D_2(P_n)}$ must belong to $N[v_j]$ in $\overline{D_2(P_n)}$. Since every γ-set of G contains at least two such vertices v_i and v_j, every γ-set of $D_2(P_n)$ is a dominating set of $\overline{D_2(P_n)}$. Hence, every γ-set of $D_2(P_n)$ is a dominating set of $D_2(P_n)$ as well as of $\overline{D_2(P_n)}$. This implies that every γ-set of $D_2(P_n)$ is a global dominating set of $D_2(P_n)$.

Case I ($n = 2, 3$). For $D_2(P_2)$ and $D_2(P_3)$, clearly $S = \{v_1, v_2\}$ and $S = \{v_2, v_3\}$ are γ-sets as well as global dominating sets with minimum cardinality respectively. Therefore, $\gamma(D_2(P_n)) = \gamma_g(D_2(P_n)) = 2 = \lfloor n/2 \rfloor + 1$ for $n = 2, 3$.

Case II ($n \geq 4$). (i) For $n \equiv 0 (\text{mod } 4)$ (i.e., $n = 4k$, $k \in N$), consider a γ-set $S = \{v_{2+4i}, v_{3+4i} / 0 \leq i \leq k-1\}$ where $|S| = n/2$.

(ii) For $n \equiv 1 (\text{mod } 4)$ (i.e., $n = 4k+1$, $k \in N$), consider a γ-set $S = \{v_{2+4i}, v_{3+4i} / 0 \leq i \leq k-1\} \cup \{v_{n-1}\}$ where $|S| = \lfloor n/2 \rfloor + 1$.

(iii) For $n \equiv 2$ or $3 (\text{mod } 4)$ (i.e., $n = 4k+2$ or $n = 4k+3$, $k \in N$), consider a γ-set $S = \{v_{2+4i}, v_{3+4i} / 0 \leq i \leq k-1\} \cup \{v_{n-2}, v_{n-1}\}$ where $|S| = \lfloor n/2 \rfloor + 1$.

Now, S being a γ-set of $D_2(P_n)$ is a global dominating set of $D_2(P_n)$ with minimum cardinality implying that $|S| = \gamma(D_2(P_n)) = \gamma_g(D_2(P_n))$.

Thus, we have proved that

$$\gamma(D_2(P_n)) = \gamma_g(D_2(P_n)) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 4 \\ \left\lfloor \frac{n}{2} \right\rfloor + 1 & \text{otherwise.} \end{cases} \quad (8)$$

□

Theorem 6. *If G_v is a graph obtained by switching of a vertex in cycle C_n ($n \geq 8$) then $\gamma(G_v) = \gamma_g(G_v) = 3$.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the successive vertices of $G = C_n$ and G_v denotes the graph obtained by switching of a vertex v of G. Without loss of generality, let the switched vertex be v_1.

Consider a set $S = \{v_1, v_2, v_n\}$. Then S is a dominating set of G_v as all the vertices except the pendant vertices, namely, v_2 and v_n, are in $N[v_1]$ while v_2 and v_n are already in S. Moreover, the set S is a minimal dominating set of G_v because for any $v_i \in S$, the set $S - \{v_i\}$ does not dominate the vertex v_i of G_v. Furthermore, the vertex v_1 dominates $(n-2)$ vertices of G_v and the remaining two vertices are pendant vertices. Therefore, at least three vertices are required to dominate G_v, and hence $|S| \geq 3$. Thus, S being a minimal dominating set with minimum cardinality is a γ-set of G_v which implies that $\gamma(G_v) = 3$.

Now, we claim that the pendant vertices in G_v are enough to dominate remaining vertices of $\overline{G_v}$. Since the vertex which is not in $N[v_2]$ in $\overline{G_v}$ must belong to $N[v_n]$ in $\overline{G_v}$, any $S \subseteq V$ containing v_2 and v_n will be a dominating set of $\overline{G_v}$. Thus, S is a dominating set of G_v as well as of $\overline{G_v}$. This implies that S is a global dominating set of G_v.

Since S is a γ-set of G_v as above, it is of minimum cardinality. Therefore, $\gamma_g(G_v) = 3$. Thus, $\gamma(G_v) = \gamma_g(G_v) = 3$ as required. □

Theorem 7. *If G_v is a graph obtained by switching of a rim vertex in a wheel W_n then*

$$\gamma_g(G_v) = \begin{cases} 2 & \text{if } n = 4, 5 \\ 3 & \text{otherwise.} \end{cases} \quad (9)$$

Proof. Let $v_1, v_2, \ldots, v_{n-1}$ be the successive rim vertices of $G = W_n$ and G_v denotes the graph obtained by switching of the vertex v of G. Without loss of generality, let the switched vertex be v_1. Let c be the apex vertex of W_n.

Case I ($n = 4, 5$). It is easy to observe that $S = \{v_1, v_3\}$ is a global dominating set of G_v with minimum cardinality. Therefore, $\gamma_g(G_v) = 2$ for $n = 4, 5$.

Case II ($n \geq 6$). Consider a set $S = \{c, v_3, v_4\}$. Then, S is a dominating set of G_v as all the vertices except the vertex v_1 are in $N[c]$ in G_v while v_1 is dominated by v_3. Now, the vertex v_3 dominates all the vertices of $\overline{G_v}$ except the vertices v_1, v_2, v_4, and c. But the vertices v_1 and v_2 are dominated by the vertices c and v_4 in $\overline{G_v}$ respectively while the vertices v_4 and c are already in S. Therefore, S is a dominating set of $\overline{G_v}$. Hence, S is a dominating set of G_v as well as of $\overline{G_v}$. That is,

S is a global dominating set of G_v. Moreover, two vertices, namely, c and either of the vertices from v_3 and v_4 are enough to dominate G_v. Since any two vertices in G_v are adjacent to the third vertex in G_v, at least three vertices are essential to dominate $\overline{G_v}$. This implies that $\gamma_g(G_v) = 3$. Thus, we have proved that

$$\gamma_g(G_v) = \begin{cases} 2 & \text{if } n = 4, 5 \\ 3 & \text{otherwise.} \end{cases} \quad (10)$$

□

Theorem 8. *If G_v is the graph obtained by switching of the apex vertex in wheel W_n then $\gamma(G_v) = \gamma_g(G_v) = \lceil (n-1)/3 \rceil + 1$.*

Proof. Let $v_1, v_2, \ldots, v_{n-1}$ be the successive rim vertices of $G = W_n$ and let G_v denote the graph obtained by switching of the vertex v of G. Let the switched vertex be the apex vertex c of W_n.

Because c is adjacent to every other vertex in G, c is not adjacent to any other vertex in G_v. Therefore, any dominating set for G_v must contain c. This implies that any global dominating set of G_v must contain c.

Case I ($n = 4, 6$). It is easy to observe that $S = \{c, v_1\}$ and $S = \{c, v_1, v_3\}$ are γ-sets as well as global dominating sets of G_v of W_4 and W_6 respectively with minimum cardinality.

Therefore,

$$\gamma(G_v) = \gamma_g(G_v) = \begin{cases} 2 = \left\lceil \dfrac{(4-1)}{3} \right\rceil + 1 & \text{for } n = 4, \\ 3 = \left\lceil \dfrac{(6-1)}{3} \right\rceil + 1 & \text{for } n = 6. \end{cases} \quad (11)$$

Thus, $\gamma(G_v) = \gamma_g(G_v) = \lceil (n-1)/3 \rceil + 1$ for $n = 4, 6$.

Case II ($n \neq 4, 6$). The vertex c will be the isolated vertex in G_v. Hence, $G_v = C_{n-1} \cup K_1$. Therefore, $\gamma(G_v) = \gamma(C_{n-1}) + 1$. But as reported in Sampathkumar [1], $\gamma(C_{n-1}) = \lceil (n-1)/3 \rceil$. This implies that $\gamma(G_v) = \lceil (n-1)/3 \rceil + 1$.

Because every dominating set of G_v must contain c and c is enough to dominate $\overline{G_v}$, $\gamma(G_v) = \gamma_g(G_v)$.

Hence, $\gamma(G_v) = \gamma_g(G_v) = \lceil (n-1)/3 \rceil + 1$ for $n \neq 4, 6$.

Thus, we have proved that $\gamma(G_v) = \gamma_g(G_v) = \lceil (n-1)/3 \rceil + 1$ as required. □

3. Concluding Remarks

The concept of global domination is remarkable as it relates the dominating sets of a graph and its complement. We have explored this concept in the context of some graph operations and also have investigated the domination number and the global domination number for the larger graph obtained by some graph operations on a given graph. The invariance parameter for global domination is also explored.

Acknowledgment

The authors are highly thankful to the anonymous referees for their kind comments and fruitful suggestions on the first draft of this paper.

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