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SOME NEW PROPERTIES OF COMPOSITION OPERATORS ASSOCIATED WITH LENS MAPS

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ABSTRACT

We give examples of results on composition operators connected with lens maps. The first two concern the approximation numbers of those operators acting on the usual Hardy space H^2. The last ones are connected with Hardy–Orlicz and Bergman–Orlicz spaces H^ψ and B^ψ, and provide a negative answer to the question of knowing if all composition operators which are weakly compact on a non-reflexive space are norm-compact.

1. Introduction

We first recall the context of this work, which appears as a continuation of [9], [10], [11], [14] and [15].

Let $\mathbb{D}$ be the open unit disk of the complex plane and $\mathcal{H}(\mathbb{D})$ be the space of holomorphic functions on $\mathbb{D}$. To every analytic self-map $\varphi\colon \mathbb{D} \to \mathbb{D}$ (also called **Schur function**), a linear map $C_\varphi\colon \mathcal{H}(\mathbb{D}) \to \mathcal{H}(\mathbb{D})$ can be associated by $C_\varphi(f) = f \circ \varphi$. This map is called the **composition operator of symbol** φ. A basic fact of the theory ([21], page 13, or [4], Theorem 1.7) is **Littlewood's subordination principle** which allows one to show that every composition operator induces a bounded linear map from the Hardy space H^p into itself, $1 \leq p < \infty$.

In this work, we are specifically interested in a one-parameter family (a semi-group) of Schur functions: **lens maps** φ_θ, $0 < \theta < 1$, whose definition is given below. They turn out to be very useful in the general theory of composition operators because they provide non-trivial examples (for example, they generate compact and even Hilbert–Schmidt operators on the Hardy space H^2 [21], page 27). The aim of this work is to illustrate that fact by new examples.

We show in Section 2 that, as operators on H^2, the approximation numbers of C_{φ_θ} behave as $\mathrm{e}^{-c_\theta\sqrt{n}}$. In particular, the composition operator C_{φ_θ} is in all Schatten classes S_p, $p > 0$. In Section 3, we show that, when one "spreads" these lens maps, their approximation numbers become greater, and the associated composition operator $C_{\tilde{\varphi}_\theta}$ is in S_p if and only if $p > 2\theta$. In Section 4, we answer in the negative a question of H.-O. Tylli: is it true that every weakly compact composition operator on a non-reflexive Banach function space is actually compact? We show that there are composition operators on a (non-reflexive) Hardy–Orlicz space, which are weakly compact and Dunford–Pettis, though not compact, and that there are composition operators on a non-reflexive Bergman–Orlicz space which are weakly compact but not compact. We also show that there are composition operators on a non-reflexive Hardy–Orlicz space which are weakly compact but not Dunford–Pettis.

We give now the definition of lens maps (see [21], page 27).

Definition 1.1 (Lens maps): The **lens map** $\varphi_\theta \colon \mathbb{D} \to \mathbb{D}$ **with parameter** θ, $0 < \theta < 1$, is defined by:

$$\varphi_\theta(z) = \frac{(1+z)^\theta - (1-z)^\theta}{(1+z)^\theta + (1-z)^\theta}, \quad z \in \mathbb{D}. \tag{1.1}$$

In a more explicit way, φ_θ is defined as follows. Let $\mathbb{H}$ be the open right half-plane, and $T \colon \mathbb{D} \to \mathbb{H}$ be the (involutive) conformal mapping given by

$$T(z) = \frac{1-z}{1+z}. \tag{1.2}$$

We denote by γ_θ the self-map of $\mathbb{H}$ defined by

$$\gamma_\theta(w) = w^\theta = \mathrm{e}^{\theta \log w}, \tag{1.3}$$

where log is the principal value of the logarithm and finally $\varphi_\theta \colon \mathbb{D} \to \mathbb{D}$ is defined by

$$\varphi_\theta = T^{-1} \circ \gamma_\theta \circ T. \tag{1.4}$$

Those lens maps form a continuous curve of analytic self-maps from $\mathbb{D}$ into itself, and an abelian semi-group for the composition of maps since we obviously have from (1.4) and the rules on powers that $\varphi_\theta(0) = 0$ and

$$\varphi_\theta \circ \varphi_{\theta'} = \varphi_{\theta'} \circ \varphi_\theta = \varphi_{\theta\theta'}. \tag{1.5}$$

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2. Approximation numbers of lens maps

For every operator $A \colon H^2 \to H^2$, we denote by

$$a_n(A) = \inf_{\operatorname{rank} R < n} \|A - R\|, \quad n = 1, 2, \dots$$

its n**-th approximation number**. We refer to [2] for more details on those approximation numbers.

Recall ([24], page 18) that the Schatten class S_p on H^2 is defined by

$$S_p = \{A\colon H^2 \to H^2\,;\ (a_n(A))_n \in \ell^p\}, \quad p > 0;$$

S_2 is the Hilbert–Schmidt class and the quantity $\|A\|_p = \big(\sum_{n=1}^{\infty}(a_n(A))^p\big)^{1/p}$ is a Banach norm on S_p for $p \geq 1$.

We can now state the following theorem:

Theorem 2.1: *Let $0 < \theta < 1$ and φ_θ be the lens map defined in* (1.1)*. There are positive constants a, b, a', b' depending only on θ such that*

$$a'\,\mathrm{e}^{-b'\sqrt{n}} \leq a_n(C_{\varphi_\theta}) \leq a\,\mathrm{e}^{-b\sqrt{n}}. \tag{2.1}$$

In particular, C_{φ_θ} lies in all Schatten classes S_p, $p > 0$.

The lower bound in (2.1) was proved in [15], Proposition 6.3. The fact that C_{φ_θ} lies in all Schatten classes was first proved in [22] under a qualitative form (see the very end of that paper).

The upper bound will be obtained below as a consequence of a result of O. G. Parfenov ([18]). However, an idea of infinite divisibility, which may be used in other contexts, leads to a simpler proof, though it gives a worse estimate in (2.1): $\sqrt{n}$ is replaced by $n^{1/3}$. We shall begin by giving this proof, because it is quite short. It relies on the semi-group property (1.5) and on an estimate of the Hilbert–Schmidt norm $\|C_{\varphi_\alpha}\|_2$ in terms of α, as follows:

Lemma 2.2: *There exist numerical constants K_1, K_2 such that*

$$\frac{K_1}{1-\alpha} \leq \|C_{\varphi_\alpha}\|_2 \leq \frac{K_2}{1-\alpha}\,, \quad \textit{for all } 0 < \alpha < 1. \tag{2.2}$$

In particular, we have

$$a_n(C_{\varphi_\alpha}) \leq \frac{K_2}{\sqrt{n}(1-\alpha)}\,. \tag{2.3}$$

Proof. The relation (2.3) is an obvious consequence of (2.2) since

$$n\big[a_n(C_{\varphi_\alpha})\big]^2 \leq \sum_{j=1}^{n}\big[a_j(C_{\varphi_\alpha})\big]^2 \leq \sum_{j=1}^{\infty}\big[a_j(C_{\varphi_\alpha})\big]^2 = \|C_{\varphi_\alpha}\|_2^2 \leq \frac{K_2^2}{(1-\alpha)^2}\,.$$

For the first part, let $a = \cos(\alpha\pi/2) = \sin((1-\alpha)\pi/2) \geq 1-\alpha$ and let $\sigma = T(m)$ (m is the normalized Lebesgue measure $dm(t) = dt/2\pi$ on the unit circle) be the probability measure carried by the imaginary axis which satisfies

$$\int_{\mathbb{H}} f\, d\sigma = \int_{-\infty}^{\infty} f(iy) \frac{dy}{\pi(1+y^2)}\,.$$

By definition, T, defined in (1.2), is a unitary operator from $H^2(\mathbb{D}, m)$ into $H^2(\mathbb{H}, \sigma)$, and we easily obtain, setting $\gamma(y) = \gamma_\alpha(iy) = \mathrm{e}^{i(\pi/2)\alpha\, \mathrm{sign}\,(y)}|y|^\alpha$ (where sign is the sign of y and γ_α is defined in (1.3)), that (see [21], Section 2.3)

$$\begin{aligned}\|C_{\varphi_\alpha}\|_2^2 &= \int_{\mathbb{T}} \frac{dm}{1-|\varphi_\alpha|^2} = \int_{\mathbb{H}} \frac{d\sigma}{1-\left|\frac{1-\gamma}{1+\gamma}\right|^2} = \int_{\mathbb{H}} \frac{|1+\gamma|^2}{4\,\mathfrak{Re}\,\gamma}\, d\sigma \\ &= \int_{-\infty}^{+\infty} \frac{|1+\gamma(y)|^2}{4a\,|y|^\alpha}\, \frac{dy}{\pi(1+y^2)} \\ &\leq \frac{K}{1-\alpha}\int_0^{+\infty} \frac{1+y^{2\alpha}}{y^\alpha}\, \frac{dy}{1+y^2} = \frac{2K}{1-\alpha}\int_0^{+\infty} \frac{y^\alpha}{1+y^2}\, dy \\ &\leq \frac{4K}{(1-\alpha)^2}\,,\end{aligned}$$

where K is a numerical constant. This gives the upper bound in (2.2) and the lower one is obtained similarly. ∎

We can now finish the first proof of Theorem 2.1. Let k be a positive integer and let

$$\alpha_k = \theta^{1/k},$$

so that $\alpha_k^k = \theta$.

Now use the well-known sub-multiplicativity $a_{p+q-1}(vu) \leq a_p(v)\, a_q(u)$ of approximation numbers ([19], page 61), as well as the semi-group property (1.5) (which implies $C_{\varphi_\theta} = C_{\varphi_{\alpha_k}}^k$), and (2.3). We see that

$$a_{kn}(C_{\varphi_\theta}) = a_{kn}(C_{\varphi_{\alpha_k}}^k) \leq \big[a_n(C_{\varphi_{\alpha_k}})\big]^k \leq \left[\frac{K_2}{(1-\alpha_k)\sqrt{n}}\right]^k.$$

Observe that

$$1-\alpha_k \geq \frac{1-\alpha_k^k}{k} = \frac{1-\theta}{k}\,.$$

We then get, $c = c_\theta$ denoting a constant which only depends on θ:

$$a_{kn}(C_{\varphi_\theta}) \leq \Big(\frac{k}{c\sqrt{n}}\Big)^k.$$

Set $d = c/\mathrm{e}$ and take $k = d\sqrt{n}$, ignoring the questions of integer part. We obtain

$$a_{dn^{3/2}}(C_{\varphi_\theta}) \leq \mathrm{e}^{-k} = \mathrm{e}^{-d\sqrt{n}}.$$

Setting $N = dn^{3/2}$, we get

$$a_N(C_{\varphi_\theta}) \leq a\,\mathrm{e}^{-bN^{1/3}} \tag{2.4}$$

for an appropriate value of a and b and for any integer $N \geq 1$. This ends our first proof, with an exponent slightly smaller that the right one (1/3 instead of 1/2), yet more than sufficient to prove that $C_{\varphi_\theta} \in \bigcap_{p>0} S_p$. ∎

Remark: Since the estimate (2.3) is rather crude, it might be expected that, using (2.4) and iterating the process, we could obtain a better one. This is not the case, and this iteration leads to (2.4) and the exponent 1/3 again (with different constants a and b).

Proof of Theorem 2.1. This proof will give the correct exponent 1/2 in the upper bound. Moreover, it works more generally for Schur functions whose image lies in polygons inscribed in the unit disk. This upper bound appears, in a different context and under a very cryptic form, in [18]. First note the following simple lemma.

LEMMA 2.3: *Suppose that $a, b \in \mathbb{D}$ satisfy $|a-b| \leq M\min(1-|a|, 1-|b|)$, where M is a constant. Then*

$$d(a,b) \leq \frac{M}{\sqrt{M^2+1}} := \chi < 1.$$

Here d is the pseudo-hyperbolic distance defined by

$$d(a,b) = \left|\frac{a-b}{1-\overline{a}b}\right|, \quad a, b \in \mathbb{D}.$$

Proof. Set $\delta = \min(1-|a|, 1-|b|)$. We have the identity

$$\frac{1}{d^2(a,b)} - 1 = \frac{(1-|a|^2)(1-|b|^2)}{|a-b|^2} \geq \frac{(1-|a|)(1-|b|)}{|a-b|^2} \geq \frac{\delta^2}{M^2\delta^2} = \frac{1}{M^2},$$

hence the lemma. ∎

The second lemma gives an upper bound for $a_N(C_\varphi)$. In this lemma, κ is a numerical constant, $S(\xi, h)$ the usual **pseudo-Carleson window** centred at $\xi \in \mathbb{T}$ (where $\mathbb{T} = \partial\mathbb{D}$ is the unit circle) and of radius h ($0 < h < 1$), defined by

$$S(\xi, h) = \{z \in \mathbb{D};\ |z-\xi| \leq h\}, \tag{2.5}$$

and m_φ is the pull-back measure of m, the normalized Lebesgue measure on $\mathbb{T}$, by φ^*. Recall that if $f \in \mathcal{H}(\mathbb{D})$, one sets $f_r(\mathrm{e}^{it}) = f(r\mathrm{e}^{it})$ for $0 < r < 1$ and, if the limit exists m-almost everywhere, one sets

$$f^*(\mathrm{e}^{it}) = \lim_{r\to 1^-} f(r\mathrm{e}^{it}). \tag{2.6}$$

Actually, we shall do write f instead of f^*. Recall that a measure μ on $\overline{\mathbb{D}}$ is called a **Carleson measure** if there is a constant $c > 0$ such that $\mu\big[\overline{S(\xi,h)}\big] \leq c\,h$ for all $\xi \in \mathbb{T}$. Carleson's embedding theorem says that μ is a Carleson measure if and only if the inclusion map from H^2 into $L^2(\mu)$ is bounded (see [4], Theorem 9.3, for example).

Lemma 2.4: *Let B be a Blaschke product with less than N zeroes (each zero being counted with its multiplicity). Then, for every Schur function φ, one has*

$$a_N^2 := \big[a_N(C_\varphi)\big]^2 \leq \kappa^2 \sup_{0<h<1,\xi\in\mathbb{T}} \frac{1}{h}\int_{\overline{S(\xi,h)}} |B|^2\, dm_\varphi, \tag{2.7}$$

for some universal constant $\kappa > 0$.

Proof. The subspace BH^2 is of codimension $\leq N-1$. Therefore, $a_N = c_N(C_\varphi) \leq \big\|C_{\varphi|BH^2}\big\|$, where the c_N's are the Gelfand numbers (see [2]), and where we used the equality $a_N = c_N$ occurring in the Hilbertian case (see [2]). Now, since $\|Bf\|_{H^2} = \|f\|_{H^2}$ for any $f\in H^2$, we have

$$\begin{aligned}\big\|C_{\varphi|BH^2}\big\|^2 &= \sup_{\|f\|_{H^2}\leq 1}\int_{\mathbb{T}} |B\circ\varphi|^2\,|f\circ\varphi|^2\,dm = \sup_{\|f\|_{H^2}\leq 1}\int_{\overline{\mathbb{D}}} |B|^2|f|^2\,dm_\varphi\\ &= \|R_\mu\|^2,\end{aligned}$$

where $\mu = |B|^2 m_\varphi$ and where $R_\mu\colon H^2 \to L^2(\mu)$ is the restriction map. Of course, μ is a Carleson measure for H^2 since $\mu \leq m_\varphi$. Now, Carleson's embedding theorem tells us that

$$\|R_\mu\|^2 \leq \kappa^2 \sup_{0<h<1,\xi\in\mathbb{T}} \frac{\mu[\overline{S(\xi,h)}]}{h}$$

(see [4], Remark after the proof of Theorem 9.3, at the top of page 163; actually, in that book, Carleson's windows $W(\xi,h)$ are used instead of pseudo-Carleson's windows $S(\xi,h)$, but that does not matter, since $W(\xi,h)\subseteq S(\xi,2h)$: if $r\geq 1-h$ and $|t-t_0|\leq h$, then $|r\mathrm{e}^{it}-\mathrm{e}^{it_0}| \leq |r\mathrm{e}^{it}-\mathrm{e}^{it}| + |\mathrm{e}^{it}-\mathrm{e}^{it_0}| \leq 2h$). That ends the proof of Lemma 2.4. ∎

The following lemma takes into account the behaviour of $\varphi_\theta(\mathrm{e}^{it})$, and will be useful to us in Section 3 as well. The notation $u(t)\approx v(t)$ means that $a\,u(t)\leq v(t)\leq b\,u(t)$, for some positive constants a, b.

LEMMA 2.5: *Set* $\gamma(t) = \varphi_\theta(\mathrm{e}^{it}) = |\gamma(t)|\,\mathrm{e}^{iA(t)}$, *with* $-\pi \le t \le \pi$, *and* $-\pi \le A(t) \le \pi$. *Then, for* $0 \le |t|, |t'| \le \pi/2$, *one has*

$$|1-\gamma(t)| \approx 1-|\gamma(t)| \approx |t|^\theta \quad \text{and} \quad |\gamma(t)-\gamma(t')| \le K\,|t-t'|^\theta. \tag{2.8}$$

Moreover, we have for $|t| \le \pi/2$

$$A(t) \approx |t|^\theta \quad \text{and} \quad A'(t) \approx |t|^{\theta-1}. \tag{2.9}$$

Proof. First, recall that

$$\varphi_\theta(z) = \frac{(1+z)^\theta - (1-z)^\theta}{(1+z)^\theta + (1-z)^\theta}\,,$$

so that $\varphi_\theta(\overline{z}) = \overline{\varphi_\theta(z)}$ and $\varphi_\theta(-z) = -\varphi_\theta(z)$. It follows that $\gamma(-t) = \overline{\gamma(t)}$ and $\gamma(t+\pi) = -\gamma(t)$, so that we may assume $0 \le t, t' \le \pi/2$. Then, we have more precisely, setting $c = \mathrm{e}^{-i\theta\pi/2}$, $s = \sin(\theta\pi/2)$ and $\tau = \big(\tan(t/2)\big)^\theta$,

$$\gamma(t) = \frac{(\cos t/2)^\theta - \mathrm{e}^{-i\theta\pi/2}(\sin t/2)^\theta}{(\cos t/2)^\theta + \mathrm{e}^{-i\theta\pi/2}(\sin t/2)^\theta} = \frac{1-c\tau}{1+c\tau} = \frac{1-\tau^2}{|1+c\tau|^2} + \frac{2is\tau}{|1+c\tau|^2}\,,$$

after a simple computation, since $(1+\mathrm{e}^{it})^\theta = \mathrm{e}^{it\theta/2}(2\cos t/2)^\theta$ and $(1-\mathrm{e}^{it})^\theta = \mathrm{e}^{-i\theta\pi/2}\,\mathrm{e}^{it\theta/2}(2\sin t/2)^\theta$. Note by the way that

$$\varphi_\theta(1) = 1\,; \quad \varphi_\theta(i) = i\tan(\theta\pi/4)\,; \quad \varphi_\theta(-1) = -1\,; \quad \varphi_\theta(-i) = -i\tan(\theta\pi/4).$$

Now, observe that $2 \ge |1+c\tau| \ge \mathfrak{Re}\,(1+c\tau) \ge 1$ and therefore that

$$|1-\gamma(t)| = \left|\frac{2c\tau}{1+c\tau}\right| \approx \tau \approx t^\theta\,,$$

and similarly for $1-|\gamma(t)|$ since

$$1-|\gamma(t)|^2 = \frac{4(\mathfrak{Re}\,c)\,\tau}{|1+c\tau|^2}.$$

The relation (2.8) clearly follows. To prove (2.9), we just have to note that, for $0 \le t \le \pi/2$, we have $A(t) = \arctan\frac{2s\tau}{1-\tau^2}$. ∎

Now, we prove Theorem 2.1 in the following form (in which $q = q_\theta$ denotes a positive constant smaller than one), which is clearly sufficient:

$$a_{4N^2+1} \le Kq^N. \tag{2.10}$$

The proof will come from an adequate choice of a Blaschke product of length $4N^2$, with zeroes on the curve $\gamma(t) = \varphi_\theta(\mathrm{e}^{it})$, $-\pi \le t \le \pi$. Let $t_k = \pi 2^{-k}$ and

$p_k = \gamma(t_k)$, with $1 \leq k \leq N$, so that the points p_k are all in the first quadrant. We reflect them through the coordinate axes, setting

$$q_k = \overline{p_k}, \quad r_k = -p_k, \quad s_k = -q_k, \qquad 1 \leq k \leq N.$$

Let now B be the Blaschke product having a zero of order N at each of the points p_k, q_k, r_k, s_k, namely

$$B(z) = \prod_{k=1}^{N} \left[\frac{z - p_k}{1 - \overline{p_k} z} \cdot \frac{z - q_k}{1 - \overline{q_k} z} \cdot \frac{z - r_k}{1 - \overline{r_k} z} \cdot \frac{z - s_k}{1 - \overline{s_k} z} \right]^N .$$

This Blaschke product satisfies, by construction, the symmetry relations

$$B(\overline{z}) = \overline{B(z)}\,, \quad B(-z) = B(z). \tag{2.11}$$

Of course, $|B| = 1$ on the boundary of $\mathbb{D}$, but $|B|$ is small on a large portion of the curve γ, as expressed by the following lemma.

Lemma 2.6: *For some constant $\chi = \chi_\theta < 1$, the following estimate holds:*

$$t_N \leq t \leq t_1 \quad \Longrightarrow \quad |B(\gamma(t))| \leq \chi^N. \tag{2.12}$$

Proof. Let $t_N \leq t \leq t_1$ and k be such that $t_{k+1} \leq t \leq t_k$. Let

$$B_k(z) = \frac{z - p_k}{1 - \overline{p_k} z}.$$

Then, with the help of Lemma 2.5, we see that the assumptions of Lemma 2.3 are satisfied with $a = \gamma(t)$ and $b = \gamma(t_k)$, since $|t - t_k| \leq t_k - t_{k+1} = \pi 2^{-k-1}$, so that $\min(1 - |a|, 1 - |b|) \approx t_k^\theta \approx 2^{-k\theta}$ and hence, for some constant M,

$$|a - b| \leq K\, |t - t_k|^\theta \leq K 2^{-k\theta} \leq M\, \min(1 - |a|, 1 - |b|)\,.$$

We therefore have, by definition, and by Lemma 2.3, where we set $\chi = M/\sqrt{M^2 + 1}$,

$$|B_k(\gamma(t))| = d(\gamma(t), p_k) \leq \chi < 1.$$

It then follows from the definition of B that

$$|B(\gamma(t))| \leq |B_k(\gamma(t))|^N \leq \chi^N,$$

and that ends the proof of Lemma 2.6. ∎

Now fix $\xi \in \mathbb{T}$ and $0 < h \leq 1$. By interpolation, we may assume that $h = 2^{-n\theta}$. By symmetry, we may assume that $\mathfrak{Re}\, \xi \geq 0$ and $\mathfrak{Re}\, \gamma(t) \geq 0$, i.e., $|t| \leq \pi/2$. Then, since $\varphi_\theta(\mathbb{D})$ is contained in the symmetric angular sector of vertex 1 and opening $\theta\pi < \pi$, there is a constant $K > 0$ such that

$|1-\gamma(t)| \leq K(1-|\gamma(t)|)$. The only pseudo-windows $S(\xi,h)$ giving an integral not equal to zero in the estimation (2.7) of Lemma 2.4 satisfy $|\xi-1| \leq (K+1)h$. Indeed, suppose that $|\gamma(t)-\xi| \leq h$. Then $1-|\gamma(t)| \leq |\gamma(t)-\xi| \leq h$ and $|1-\gamma(t)| \leq K(1-|\gamma(t)|) \leq Kh$. If $|\xi-1| > (K+1)h$, we should have $|\gamma(t)-\xi| \geq |\xi-1| - |\gamma(t)-1| > (K+1)h - Kh = h$, which is impossible. Now, for such a window, we have by definition of m_φ

$$\int_{S(\xi,h)} |B|^2\, dm_{\varphi_\theta} = \int_{|\gamma(t)-\xi|\leq h} |B(\gamma(t))|^2 \frac{dt}{2\pi} \leq \int_{|\gamma(t)-1|\leq (K+2)h} |B(\gamma(t))|^2 \frac{dt}{2\pi}$$
$$\leq \int_{|t|\leq Dt_n} |B(\gamma(t))|^2 \frac{dt}{2\pi} \overset{def}{=} I_h,$$

since $|\gamma(t)-1| \leq |\gamma(t)-\xi| + |\xi-1| \leq h + (K+1)h$ and since $|\gamma(t)-1| \geq a|t|^\theta$ and $|\gamma(t)-1| \leq (K+2)h$ together imply $|t| \leq Dt_n$, where $D>1$ is another constant (recall that $h = 2^{-n\theta} = (t_n/\pi)^\theta$).

To finish the discussion, we separate two cases.

1) If $n \geq N$, we simply majorize $|B|$ by 1. We set $q_1 = 2^{\theta-1} < 1$ and get

$$\frac{1}{h} I_h \leq \frac{1}{h}\int_{-Dt_n}^{Dt_n} |B(\gamma(t))|^2 \frac{dt}{2\pi} \leq \frac{2Dt_n}{2\pi h} = Dq_1^n \leq D\, q_1^N.$$

2) If $n \leq N-1$, we write

$$\frac{1}{h} I_h = \frac{2}{h}\int_0^{Dt_N} |B(\gamma(t))|^2 \frac{dt}{2\pi} + \frac{2}{h}\int_{Dt_N}^{Dt_n} |B(\gamma(t))|^2 \frac{dt}{2\pi} := J_N + K_N.$$

The term J_N is estimated above: $J_N \leq D\, q_1^N$. The term K_N is estimated through Lemma 2.6, which gives us

$$K_N \leq 2^{n\theta} \frac{2Dt_n}{2\pi} \chi^{2N} \leq D\, \chi^{2N},$$

since $t_n 2^{n\theta} \leq \pi$, due to the fact that $\theta < 1$.

If we now apply Lemma 2.4 with $q = \max(q_1, \chi^2)$ and with N changed into $4N^2+1$, we obtain (2.10), by changing the value of the constant K once more. This ends the proof of Theorem 2.1. ∎

Theorem 2.1 has the following consequence (as in [21], page 29).

Proposition 2.7: *Let φ be a univalent Schur function and assume that $\varphi(\mathbb{D})$ contains an angular sector centred on the unit circle and with opening $\theta\pi$, $0<\theta<1$. Then $a_n(C_\varphi) \geq a\,\mathrm{e}^{-b\sqrt{n}}$, $n = 1, 2, \ldots$, for some positive constants a and b, depending only on θ.*

Proof. We may assume that this angular sector is centred at 1. By hypothesis, $\varphi(\mathbb{D})$ contains the image of the "reduced" lens map defined by $\tilde{\varphi}_\theta(z) = \varphi_\theta((1+z)/2)$. Since φ is univalent, there is a Schur function u such that $\tilde{\varphi}_\theta = \varphi \circ u$. Hence $C_{\tilde{\varphi}_\theta} = C_u \circ C_\varphi$ and $a_n(C_{\tilde{\varphi}_\theta}) \leq \|C_u\|\, a_n(C_\varphi)$. Theorem 2.1 gives the result, since the calculations for $\tilde{\varphi}_\theta$ are exactly the same as for φ_θ (because they are equivalent as z tends to 1). ∎

The same is true if φ is univalent and $\varphi(\mathbb{D})$ contains a polygon with vertices on $\partial\mathbb{D}$.

3. Spreading the lens map

In [9], we studied the effect of the multiplication of a Schur function φ by the singular inner function $M(z) = \mathrm{e}^{-\frac{1+z}{1-z}}$, and observed that this multiplication spreads the values of the radial limits of the symbol and lessens the maximal occupation time for Carleson windows. In some cases this improves the compactness or membership to Schatten classes of C_φ. More precisely, we proved the following result.

Theorem 3.1 ([9], Theorem 4.2): *For every $p > 2$, there exist two Schur functions φ_1 and $\varphi_2 = \varphi_1 M$ such that $|\varphi_1^*| = |\varphi_2^*|$ and $C_{\varphi_1}\colon H^2 \to H^2$ is not compact, but $C_{\varphi_2}\colon H^2 \to H^2$ is in the Schatten class S_p.*

Here, we will meet the opposite phenomenon: the symbol φ_1 will have a fairly big associated maximal function ρ_{φ_1}, but will belong to all Schatten classes since it "visits" a bounded number of windows (meaning that there exists an integer J such that, for fixed n, at most J of the $W_{n,j}$ are visited by $\varphi^*(\mathrm{e}^{it})$). The spread symbol will have an improved maximal function, but will visit all windows, so that its membership in Schatten classes will be degraded. More precisely, we will prove that

Theorem 3.2: *Fix $0 < \theta < 1$. Then there exist two Schur functions φ_1 and φ_2 such that:*

(1) *$C_{\varphi_1}\colon H^2 \to H^2$ is in all Schatten classes S_p, $p > 0$, and even $a_n(C_{\varphi_1}) \leq a\,\mathrm{e}^{-b\sqrt{n}}$;*
(2) *$|\varphi_1^*| = |\varphi_2^*|$;*
(3) *$C_{\varphi_2} \in S_p$ if and only if $p > 2\theta$;*
(4) *$a_n(C_{\varphi_2}) \leq K\,(\log n/n)^{1/2\theta}$, $n = 2, 3, \ldots$.*

Of course, it would be better to have a good lower bound for $a_n(C_{\varphi_2})$, but we have not yet succeeded in finding it.

Proof. First observe that $C_{\varphi_1} \in S_2$, so that $C_{\varphi_2} \in S_2$ too, since $|\varphi_1^*| = |\varphi_2^*|$ and since the membership of C_φ in S_2 only depends on the modulus of φ^* because it amounts to ([21], page 26)

$$\int_{-\pi}^{\pi} \frac{dt}{1 - |\varphi^*(\mathrm{e}^{it})|} < \infty.$$

Theorem 3.2 says that we can hardly have more. We first prove a lemma. Recall (see [9], for example) that the **maximal Carleson function** ρ_φ of a Schur function φ is defined, for $0 < h < 1$, by

$$\rho_\varphi(h) = \sup_{|\xi|=1} m_\varphi[S(\xi, h)]. \tag{3.1}$$

Lemma 3.3: *Let $0 < \theta < 1$. Then, the maximal function ρ_{φ_θ} of φ_θ satisfies $\rho_{\varphi_\theta}(h) \leq K^{1/\theta}(1-\theta)^{-1/\theta} h^{1/\theta}$ and, moreover,*

$$\rho_{\varphi_\theta}(h) \approx h^{1/\theta}. \tag{3.2}$$

Proof of Lemma 3.3. Let $0 < h < 1$ and $\gamma(t) = \varphi_\theta(\mathrm{e}^{it})$; K and δ will denote constants which can change from a formula to another. We have, for $|t| \leq \pi/2$,

$$\begin{aligned} 1 - |\gamma(t)|^2 = \frac{4(\Re\mathrm{e}\, c)\tau}{|1 + c\tau|^2} &\geq \delta \cos(\theta\pi/2) \frac{\tau}{|1 + c\tau|^2} \geq \delta(1-\theta)\frac{\tau}{|1+c\tau|^2} \\ &\geq \delta(1-\theta)|t|^\theta. \end{aligned}$$

Hence, we get, from Lemma 2.5,

$$\begin{aligned} \rho_{\varphi_\theta}(h) &\leq 2\, m(\{1 - |\gamma(t)| \leq h \text{ and } |t| \leq \pi/2\}) \leq 2\, m(\{(1-\theta)\delta|t|^\theta \leq Kh\}) \\ &\leq K^{1/\theta}(1-\theta)^{-1/\theta}h^{1/\theta}. \end{aligned}$$

Similarly, we have

$$\rho_{\varphi_\theta}(h) \geq m_{\varphi_\theta}[S(1,h)] \geq m(\{|1 - \gamma(t)| \leq h\}) \geq m(\{|t|^\theta \leq Kh\}) \geq K h^{1/\theta},$$

and that ends the proof of the lemma. ∎

Going back to the proof of Theorem 3.2, we take $\varphi_1 = \varphi_\theta$ and $\varphi_2(z) = \varphi_1(z)M(z^2)$. We use $M(z^2)$ instead of $M(z)$ in order to treat the points -1 and 1 together.

The first two assertions are clear. For the third one, we define the dyadic Carleson windows, for $n = 1, 2, \ldots, j = 0, 1, \ldots, 2^n - 1$, by

$$W_{n,j} = \{z \in \mathbb{D}\,;\ 1 - 2^{-n} \leq |z| < 1 \text{ and } (2j\pi)2^{-n} \leq \arg(z) < (2(j+1))\pi)2^{-n}\}.$$

Recall (see [9], Proposition 3.3) the following proposition, which is a variant of Luecking's criterion ([16]) for membership in a Schatten class, and which might also be used to give a third proof of the membership of C_{φ_θ} in all Schatten classes S_p, $p > 0$, although the first proof turns out to be more elementary.

Proposition 3.4 ([16], [9]): *Let φ be a Schur function and $p > 0$ a positive real number. Then $C_\varphi \in S_p$ if and only if*

$$\sum_{n=1}^{\infty} \sum_{j=0}^{2^n-1} \big[2^n m_\varphi(W_{n,j})\big]^{p/2} < \infty.$$

We apply this proposition with $\varphi = \varphi_2$, which satisfies, for $0 < |t| \leq \pi/2$, the following relation:

$$\varphi(\mathrm{e}^{it}) = |\gamma(t)|\mathrm{e}^{i[A(t) - \cot(t)]} \stackrel{def}{=} |\gamma(t)|\mathrm{e}^{iB(t)},$$

where $\gamma(t) = \varphi_1(\mathrm{e}^{it})$ and (using Lemma 2.5)

(3.3)
$$0 < |t| \leq \pi/2 \implies B(t) = \Gamma(t) - \frac{1}{t}\,, \quad \text{with } \Gamma(t) \approx |t|^\theta \text{ and } \Gamma'(t) \approx |t|^{\theta - 1}.$$

It clearly follows from (3.3) that the function B is increasing on some interval $[-\delta, 0[$ where δ is a positive numerical constant. Let us fix a positive integer q_0 such that $-\pi/2 \leq t < 0$ and

$$B(t) \geq 2q_0\pi \quad \implies \quad t \geq -\delta.$$

Fix a Carleson window $W_{n,j}$ and let us analyze the set $E_{n,j}$ of those t's such that $\varphi(\mathrm{e}^{it})$ belongs to $W_{n,j}$. Recall that $m_\varphi(W_{n,j}) = m(E_{n,j})$. The membership in $E_{n,j}$ gives two constraints.

1) Modulus constraint. We must have $|\gamma(t)| \geq 1 - 2^{-n}$, and therefore $|t| \leq K2^{-n/\theta}$.

2) Argument constraint. Let us set $\theta_{n,j} = (2j+1)\pi 2^{-n}$, $h = \pi 2^{-n}$ and $I_{n,j} = (\theta_{n,j} - h, \theta_{n,j} + h)$. The angular constraint $\arg\varphi(\mathrm{e}^{it}) \in I_{n,j}$ will be satisfied if $t < 0$ and

$$B(t) \in \bigcup_{q \geq q_0} \big[\theta_{n,j} - h + 2q\pi,\, \theta_{n,j} + h + 2q\pi\big] := \bigcup_{q \geq q_0} J_q(h) := F.$$

We have $F \subset [2q_0\pi, \infty[$, and so $B(t) \in F$ and $t < 0$ imply $t \geq -\delta$. Set

$$E = \bigcup_{q \geq q_0} \left[B^{-1}(\theta_{n,j} - h + 2q\pi),\, B^{-1}(\theta_{n,j} + h + 2q\pi)\right] := \bigcup_{q \geq q_0} I_q(h) \subset [-\delta, 0[.$$

The intervals I_q's are disjoint, since $\theta_{n,j} + 2(q+1)\pi - h > \theta_{n,j} + 2q\pi + h$ and since B increases on $[-\delta, 0[$. Moreover, $t \in E$ implies that $B(t) \in F$, which in turn implies that $\arg \varphi(\mathrm{e}^{it}) \in I_{n,j}$. Using Lemma 2.5, we can find positive constants c_1, c_2 such that

$$q \geq q_0 \quad \Longrightarrow \quad -c_1/q \leq \min I_q(h) \leq \max I_q(h) \leq -c_2/q.$$

Now, by the mean-value theorem, $I_q(h)$ has length $2h/|B'(t_q)|$ for some $t_q \in I_q(h)$. But, using (3.3), we get

$$B(t) \approx \frac{1}{t} \quad \text{and} \quad |B'(t)| \approx \frac{1}{t^2}\,,$$

so that $I_q(h)$ has length approximately $ht_q^2 \approx h/q^2$ since $|t_q| \approx 1/q$. Because of the modulus constraint, the only involved q's are those for which $q \geq q_1$, where $q_1 \approx 2^{n/\theta}$. Taking n numerically large enough, we may assume that $q_1 > q_0$. We finally see that, for any $0 \leq j \leq 2^n - 1$, we have the lower bound

$$m_\varphi(W_{n,j}) = m(E_{n,j}) \gtrsim \sum_{q \geq q_1} m(I_q(h)) \gtrsim \sum_{q \geq q_1} \frac{h}{q^2} \gtrsim \frac{h}{q_1} \gtrsim 2^{-n(1+1/\theta)}.$$

It follows that

$$\sum_{n=1}^{\infty} \sum_{j=0}^{2^n-1} \left[2^n m_\varphi(W_{n,j})\right]^{p/2} \gtrsim \sum_{n=1}^{\infty} \sum_{j=0}^{2^n-1} \left[2^n 2^{-n(1+1/\theta)}\right]^{p/2} = \sum_{n=1}^{\infty} \sum_{j=0}^{2^n-1} \left[2^{-np/2\theta}\right]$$
$$= \sum_{n=1}^{\infty} 2^{n(1-p/2\theta)} = \infty,$$

if $p \leq 2\theta$. Hence $C_{\varphi_2} \notin S_p$ for $p \leq 2\theta$ by Proposition 3.4.

A similar upper bound, and the membership of C_{φ_2} in S_p for $p > 2\theta$, would easily be proved along the same lines (and we will make use of that fact in Section 4). But this will also follow from the more precise result on approximation numbers. To that effect, we shall borrow the following result from [15].

Theorem 3.5 ([15]): *Let φ be a Schur function. Then the approximation numbers of $C_\varphi\colon H^2 \to H^2$ have the upper bound*

$$a_n(C_\varphi) \leq K \inf_{0<h<1} \left[(1-h)^n + \sqrt{\frac{\rho_\varphi(h)}{h}}\right], \quad n = 1, 2, \dots . \tag{3.4}$$

Applying this theorem to φ_2, which satisfies $\rho_{\varphi_2}(h) \leq Kh^{1+1/\theta}$ as is clear from the preceding computations, would provide upper bounds for $m_\varphi(W_{n,j})$ of the same order as the lower bounds obtained. Then choosing $h = H \log n/n$, where H is a large constant ($H = 1/2\theta$ will do) and using $1 - h \leq \mathrm{e}^{-h}$, we get from (3.4)

$$a_n(C_{\varphi_2}) \leq K\left[n^{-H} + \left(\frac{\log n}{n}\right)^{1/2\theta}\right] \leq K\left(\frac{\log n}{n}\right)^{1/2\theta}.$$

This ends the proof of Theorem 3.2. ∎

Remark: Theorem 3.5 of [15] gives a very imprecise estimate on the approximation numbers of lens maps, as we noted in that paper. On the other hand, when we apply it to a lens map spread by multiplication by the inner function M, we obtain an estimate which is close to being optimal, up to a logarithmic factor. This indicates that many phenomena have still to be understood concerning approximation numbers of composition operators.

4. Lens maps as counterexamples

Recall that an operator $T\colon X \to Y$ between Banach spaces is said to be Dunford–Pettis (in short DP) or completely continuous, if for any sequence (x_n) which is weakly convergent to 0, the sequence (Tx_n) is norm-convergent to 0. It is called weakly compact (in short w-compact) if the image $T(B_X)$ of the unit ball in X is (relatively) weakly compact in Y. The identity map $i_1\colon \ell_1 \to \ell_1$ is DP and not w-compact, by the Schur property of ℓ_1 and its non-reflexivity. If $1 < p < \infty$, the identity map $i_p\colon \ell_p \to \ell_p$ is w-compact and not DP by the reflexivity of ℓ_p and the fact that the canonical basis (e_n) of ℓ_p converges weakly to 0, whereas $\|e_n\|_p = 1$. Therefore, the two notions, clearly weaker than that of compactness, are not comparable in general. Moreover, when X is reflexive, any operator $T\colon X \to Y$ is w-compact and any Dunford–Pettis operator $T\colon X \to Y$ is compact.

Yet, in the context of composition operators $T = C_\varphi\colon X \to X$, with X a non-reflexive Banach space of analytic functions, several results say that weak compactness of C_φ implies its compactness. Let us quote some examples

- $X = H^1$; this was proved by D. Sarason in 1990 ([20]);

- $X = H^\infty$ and the disk algebra $X = A(\mathbb{D})$ (A. Ülger [23] and R. Aron, P. Galindo and M. Lindström [1], independently; the first-named of us also gave another proof in [7]);
- X is the little Bloch space $\mathscr{B}_0$ (K. Madigan and A. Matheson [17]);
- X is the Hardy–Orlicz space $X = H^\psi$, when the Orlicz function ψ grows more rapidly than power functions, namely when it satisfies the condition Δ^0 ([11], Theorem 4.21, page 55);
- $X = BMOA$ and $X = VMOA$ (J. Laitila, P. J. Nieminen, E. Saksman and H.-O. Tylli [6]).

Moreover, in some cases, C_φ is compact whenever it is Dunford–Pettis ([7] for $X = H^\infty$ and [11], Theorem 4.21, page 55, for $X = H^\psi$, when the conjugate function of ψ satisfies the condition Δ_2).

The question naturally arises whether for any non-reflexive Banach space X of analytic functions on $\mathbb{D}$, every weakly compact (resp. Dunford–Pettis) composition operator $C_\varphi\colon X \to X$ is actually compact. The forthcoming theorems show that the answer is negative in general. Our spaces X will be Hardy–Orlicz and Bergman–Orlicz spaces, so we first recall some definitions and facts about Orlicz spaces ([11], Chapters 2 and 3).

An **Orlicz function** is a nondecreasing convex function $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\psi(0) = 0$ and $\psi(\infty) = \infty$. Such a function is automatically continuous on $\mathbb{R}^+$. If $\psi(x)$ is not equivalent to an affine function, we must have $\psi(x)/x \underset{x\to\infty}{\longrightarrow} \infty$. The Orlicz function ψ is said to satisfy the Δ_2-condition if $\psi(2x)/\psi(x)$ remains bounded. The **conjugate function** $\tilde{\psi}$ of an Orlicz function ψ is the Orlicz function defined by

$$\tilde{\psi}(x) = \sup_{y\geq 0} \big(xy - \psi(y)\big).$$

For the conjugate function, one has the following characterization of Δ_2 (see [11], page 7): $\tilde{\psi}$ has Δ_2 if and only if, for some $\beta > 1$ and $x_0 > 0$,

$$\psi(\beta x) \geq 2\beta\psi(x)\,, \quad \text{for all } x \geq x_0. \tag{4.1}$$

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and L^0 the space of measurable functions $f\colon \Omega \to \mathbb{C}$. The **Orlicz space** $L^\psi = L^\psi(\Omega, \mathcal{A}, \mathbb{P})$ is defined by

$$L^\psi(\Omega, \mathcal{A}, \mathbb{P}) = \Big\{ f \in L^0\,;\ \int_\Omega \psi(|f|/K)\, d\,\mathbb{P} < \infty \text{ for some } K > 0 \Big\}.$$

This is a Banach space for the **Luxemburg norm**:

$$\|f\|_{L^\psi} = \inf\Big\{K > 0\,;\ \int_\Omega \psi(|f|/K)\,d\,\mathbb{P} \leq 1\Big\}.$$

The **Morse–Transue space** M^ψ (see [11], page 9) is the subspace of functions f in L^ψ for which $\int_\Omega \psi(|f|/K)\,d\,\mathbb{P} < \infty$ for every $K > 0$. It is the closure of L^∞. One always has $(M^\psi)^* = L^{\tilde\psi}$ and $L^\psi = M^\psi$ if and only if ψ has Δ_2. When the conjugate function $\tilde\psi$ of ψ has Δ_2, the bidual of M^ψ is then (isometrically isomorphic to) L^ψ.

Now, we can define the **Hardy–Orlicz space** H^ψ attached to ψ as follows. Take the probability space $(\mathbb{T}, \mathcal{B}, m)$ and, recalling that $f_r(\mathrm{e}^{it}) = f(r\mathrm{e}^{it})$,

$$H^\psi = \Big\{f \in \mathcal{H}(\mathbb{D})\,;\ \sup_{0<r<1} \|f_r\|_{L^\psi(m)} := \|f\|_{H^\psi} < \infty\Big\}.$$

We refer to [11] for more information on H^ψ. Similarly, we define (see [11]) the **Bergman–Orlicz space** B^ψ, using this time the normalized area measure A, by

$$B^\psi = \{f \in \mathcal{H}(\mathbb{D})\,;\ \|f\|_{B^\psi} := \|f\|_{L^\psi(A)} < +\infty\}.$$

If $\psi(x) = x^p$, $p \geq 1$, we get the usual Hardy and Bergman spaces H^p and B^p. Those spaces are Banach spaces for any ψ, and Hilbert spaces for $\psi(x) = x^2$. The **Hardy–Morse–Transue space** HM^ψ and **Bergman–Morse–Transue space** BM^ψ are defined by $HM^\psi = H^\psi \cap M^\psi$ and $BM^\psi = B^\psi \cap M^\psi$. When the conjugate function of ψ has Δ_2, the bidual of HM^ψ is (isometrically isomorphic to) H^ψ ([11], page 10).

We can now state our first theorem.

THEOREM 4.1: *There exists a Schur function φ and an Orlicz function ψ such that H^ψ is not reflexive and the composition operator $C_\varphi\colon H^\psi \to H^\psi$ is weakly-compact and Dunford–Pettis, but is not compact.*

Proof. First take for φ the lens map $\varphi_{1/2}$ which in view of (3.2) of Lemma 3.3 satisfies, for some constant $K > 1$,

$$\rho_\varphi(h) \geq K^{-1} h^2, \quad 0 < h < 1. \tag{4.2}$$

We now recall the construction of an Orlicz function made in [13]. Let (x_n) be the sequence of positive numbers defined as follows: $x_1 = 4$ and then, for every integer $n \geq 1$, $x_{n+1} > 2x_n$ is the abscissa of the second intersection point of the parabola $y = x^2$ with the straight line containing (x_n, x_n^2) and

$(2x_n, x_n^4)$; equivalently, $x_{n+1} = x_n^3 - 2x_n$. We now define our Orlicz function ψ by $\psi(x) = 4x$ for $0 \leq x \leq 4$ and, for $n \geq 1$, by,

$$\begin{gathered} \psi(x_n) = x_n^2, \\ \psi \text{ affine between } x_n \text{ and } x_{n+1}, \text{ so that } \psi(2x_n) = x_n^4. \end{gathered} \tag{4.3}$$

Observe that ψ does not satisfy the Δ_2-condition, since $\psi(2x_n) = [\psi(x_n)]^2$. It clearly satisfies (since ψ^{-1} is concave)

$$\begin{aligned} & x^2 \leq \psi(x) \leq x^4 && \text{for } x \geq 4, \\ & \psi^{-1}(Kx) \leq K\psi^{-1}(x) && \text{for any } x > 0, K > 1. \end{aligned} \tag{4.4}$$

Therefore, it has a moderate growth, but a highly irregular behaviour, which will imply the results we have in view. Indeed, let $y_n = \psi(x_n)$ and $h_n = 1/y_n$. We see from (4.2), (4.3) and (4.4) that

$$D(h_n) \overset{def}{=} \frac{\psi^{-1}(1/h_n)}{\psi^{-1}(1/\rho_\varphi(h_n))} \geq \frac{\psi^{-1}(1/h_n)}{\psi^{-1}(K/h_n^2)} = \frac{\psi^{-1}(y_n)}{\psi^{-1}(Ky_n^2)} \geq \frac{x_n}{2Kx_n} = \frac{1}{2K}. \tag{4.5}$$

Thus, we have $\limsup_{h\to 0^+} D(h) > 0$. By [11], Theorem 4.11 (see also [12], comment before Theorem 5.2), C_φ is not compact.

On the other hand, let $j_{\psi,2}\colon H^\psi \to H^2$ and $j_{4,\psi}\colon H^4 \to H^\psi$ be the natural injections, which are continuous, thanks to (4.4). We have the following diagram:

$$H^\psi \xrightarrow{j_{\psi,2}} H^2 \xrightarrow{C_\varphi} H^4 \xrightarrow{j_{4,\psi}} H^\psi.$$

The second map is continuous as a consequence of (3.2) and of a result of P. Duren ([3]; see also [4], Theorem 9.4, page 163), which extends Carleson's embedding theorem (see also [11], Theorem 4.18). Hence $C_\varphi = j_{4,\psi} \circ C_\varphi \circ j_{\psi,2}$ factorizes through a reflexive space (H^2 or H^4) and is therefore w-compact.

To prove that C_φ is Dunford–Pettis, we use the following result of [14] (Theorem 2.1):

Theorem 4.2 ([14]): *Let φ be a Schur function and Φ be an Orlicz function. Assume that, for some $A > 0$, one has*

$$\sup_{0<t\leq h} \frac{\rho_\varphi(t)}{t^2} \leq \frac{1/h^2}{\Phi\big(A\Phi^{-1}(1/h^2)\big)}, \quad 0 < h < 1. \tag{4.6}$$

Then, the canonical inclusion $j_{\Phi,\varphi}\colon B^\Phi \to L^\Phi(m_\varphi)$ is continuous.

In particular, it is continuous for any Orlicz function Φ if $\rho_\varphi(h) = O(h^2)$.

Now, let $J_\psi\colon H^\psi \to B^\psi$ be the canonical inclusion, and consider the following diagram:

$$H^\psi \xrightarrow{J_\psi} B^\psi \xrightarrow{j_{\psi,\varphi}} L^\psi(m_\varphi).$$

The first map is Dunford–Pettis, by [13], Theorem 4.1. The second map is continuous by (3.2) and (4.6). Clearly, being Dunford–Pettis is an ideal property (if either u or v is Dunford–Pettis, so is vu). Therefore, $j_{\psi,\varphi} \circ J_\psi$ is Dunford–Pettis, and this amounts to saying that $C_\varphi\colon H^\psi \to H^\psi$ is Dunford–Pettis.

Now, the non-reflexivity of H^ψ follows automatically, since C_φ is Dunford–Pettis but not compact.

This ends the proof of Theorem 4.1. ∎

Theorem 4.1 admits the following variant.

Theorem 4.3: *There exist a Schur function φ and an Orlicz function χ such that H^χ is not reflexive and the composition operator $C_\varphi\colon H^\chi \to H^\chi$ is weakly compact and not Dunford–Pettis; in particular, it is not compact.*

Proof. We use the same Schur function $\varphi = \varphi_{1/2}$, but we replace ψ by the function χ defined by $\chi(x) = \psi(x^2)$. Let $A > 1$. Observe that, in view of (4.4),

$$\frac{\chi(Ax)}{[\chi(x)]^2} = \frac{\psi(A^2x^2)}{[\psi(x^2)]^2} \leq \frac{A^8x^8}{x^8} = A^8.$$

By [13], Proposition 4.4, $J_\chi\colon H^\chi \to B^\chi$ is w-compact, and we can see $C_\varphi\colon H^\chi \to H^\chi$ as the canonical inclusion $j\colon H^\chi \to L^\chi(m_\varphi)$. Hence Theorem 4.2 and the diagram

$$j = j_{\chi,\varphi} \circ J_\chi\colon H^\chi \xrightarrow{J_\chi} B^\chi \xrightarrow{j_{\chi,\varphi}} L^\chi(m_\varphi)$$

show that $C_\varphi\colon H^\chi \to H^\chi$ is w-compact as well.

Now, to prove that C_φ is not Dunford–Pettis, we cannot use [13], as in the proof of Theorem 4.1, but we follow the lines of Proposition 3.1 of [13]. We remark first that, by definition, the function χ satisfies, for $\beta = 2$, the following inequality:

$$\chi(\beta x) = \psi(4x^2) \geq 4\psi(x^2) = 2\beta\chi(x);$$

hence, by (4.1), this implies that the conjugate function of χ verifies the Δ_2-condition.

Let x_n be as in (4.3), and set

$$u_n = \sqrt{x_n} \quad \text{and} \quad A = \sqrt{2}$$

so that

$$\chi(Au_n) = \big[\chi(u_n)\big]^2 = x_n^4. \tag{4.7}$$

Finally, let

$$r_n = 1 - \frac{1}{\chi(u_n)} \quad \text{and} \quad f_n(z) = u_n \bigg(\frac{1 - r_n}{1 - r_n z}\bigg)^2.$$

By ([11], Corollary 3.10), $\|f_n\|_{H^\chi} \leq 1$ and f_n tends to 0 uniformly on compact subsets of $\mathbb{D}$; that implies that $f_n \to 0$ weakly in H^χ since the conjugate function of χ has Δ_2 ([11], Proposition 3.7). On the other hand, if $K_n = \|f_n\|_{L^\chi(m_\varphi)}$, mimicking the computation of ([13], Proposition 3.1), we get

$$1 = \int_{\mathbb{D}} \chi(|f_n|/K_n)\, dm_\varphi \geq (1 - r_n)^2 \chi(\alpha u_n / 4K_n) \tag{4.8}$$

for some $0 < \alpha < 1$ independent of n, where we used the convexity of χ and the fact that the lens map φ satisfies, by (4.2),

$$m_\varphi(\{z \in \mathbb{D}\,;\ |1 - z| \leq 1 - r_n\}) \geq \alpha(1 - r_n)^2.$$

In view of (4.7), (4.8) reads as well

$$\chi(\alpha u_n / 4K_n) \leq \chi^2(u_n) = \chi(Au_n),$$

so that

$$\|j(f_n)\|_{L^\chi(m_\varphi)} = K_n \geq \alpha/4A. \tag{4.9}$$

This shows that $j\colon H^\chi \to L^\chi(m_\varphi)$ and therefore also $C_\varphi\colon H^\chi \to H^\chi$ are not Dunford–Pettis.

It remains to show that H^χ is not reflexive. We shall prove below a more general result, but here, the conjugate function $\tilde{\chi}$ of χ satisfies the Δ_2 condition, as we saw. Hence H^χ is the bidual of HM^χ. Since χ fails to satisfy the Δ_2-condition, we know that $L^\chi \neq M^\chi$. Let $u \in L^\chi \setminus M^\chi$, with $u \geq 1$. Let f be the associated outer function, namely

$$f(z) = \exp\Big(\frac{1}{2\pi} \int_0^{2\pi} \frac{\mathrm{e}^{it} + z}{\mathrm{e}^{it} - z} \log u(t)\, dt\Big).$$

One has $|f^*| = u$ almost everywhere, with the notation of (2.6), and hence $f \in H^\chi \setminus HM^\chi$. It follows that $H^\chi \neq HM^\chi$. Hence HM^χ is not reflexive, and therefore H^χ is not reflexive either. ∎

As promised, we give the general result on non-reflexivity.

Proposition 4.4: *Let ψ be an Orlicz function which does not satisfy the Δ_2-condition. Then neither H^ψ nor B^ψ is reflexive.*

Proof. We only give the proof for B^ψ because it is the same for H^ψ.

Since ψ does not satisfy Δ_2 there is a sequence (x_n) of positive numbers, tending to infinity, such that $\psi(2x_n)/\psi(x_n)$ tends to infinity. Let $r_n \in (0,1)$ such that $(1-r_n)^2 = 1/\psi(2x_n)$ and set

$$q_n(z) = \frac{(1-r_n)^4}{(1-r_n z)^4}.$$

One has

$$\|q_n\|_\infty = 1 \quad \text{and} \quad \|q_n\|_1 = \frac{(1-r_n)^2}{(1+r_n)^2} \leq (1-r_n)^2.$$

On the other hand, on the pseudo-Carleson window $S(1, 1-r_n)$, one has

$$|1-r_n z| \leq (1-r_n) + r_n|1-z| \leq (1-r_n) + r_n(1-r_n) = 1 - r_n^2 \leq 2(1-r_n);$$

hence $|q_n(z)| \geq 1/16$. It follows that

$$\begin{aligned} 1 = \int_{\mathbb{D}} \psi\bigg(\frac{|q_n|}{\|q_n\|_\psi}\bigg)\, dA &\geq \int_{S(1,1-r_n)} \psi\bigg(\frac{|q_n|}{\|q_n\|_\psi}\bigg)\, dA \\ &\geq A[S(1,1-r_n)]\, \psi\bigg(\frac{1}{16\,\|q_n\|_\psi}\bigg) \geq \frac{1}{3}\,(1-r_n)^2\psi\bigg(\frac{1}{16\,\|q_n\|_\psi}\bigg) \\ &\geq (1-r_n)^2\psi\bigg(\frac{1}{48\,\|q_n\|_\psi}\bigg) = \frac{1}{\psi(2x_n)}\,\psi\bigg(\frac{1}{48\,\|q_n\|_\psi}\bigg); \end{aligned}$$

hence $\psi(1/[48\,\|q_n\|_\psi]) \leq \psi(2x_n)$, so $1/(48\,\|q_n\|_\psi) \leq 2x_n$ and $96\,x_n\,\|q_n\|_\psi \geq 1$.

Set now $f_n = q_n/\|q_n\|_\psi$; one has $\|f_n\|_\psi = 1$ and (using that $\psi(x_n\,|q_n(z)|) \leq |q_n(z)|\,\psi(x_n)$, by convexity, since $|q_n(z)| \leq 1$)

$$\begin{aligned} \int_{\mathbb{D}} \psi\bigg(\frac{|f_n|}{96}\bigg)\, dA = \int_{\mathbb{D}} \psi\bigg(\frac{x_n\,|q_n|}{96\,x_n\,\|q_n\|_\psi}\bigg)\, dA &\leq \int_{\mathbb{D}} \psi(x_n\,|q_n|)\, dA \\ &\leq \psi(x_n) \int_{\mathbb{D}} |q_n|\, dA \\ &\leq \psi(x_n)\,(1-r_n)^2 = \frac{\psi(x_n)}{\psi(2x_n)} \underset{n\to\infty}{\longrightarrow} 0. \end{aligned}$$

By [8], Lemma 11, this implies that the sequence (f_n) has a subsequence equivalent to the canonical basis of c_0 and hence B^ψ is not reflexive. ∎

We finish by giving a counterexample using Bergman–Orlicz spaces instead of Hardy–Orlicz spaces.

Theorem 4.5: *There exists a Schur function φ and an Orlicz function ψ such that the space B^ψ is not reflexive and the composition operator $C_\varphi\colon B^\psi \to B^\psi$ is weakly-compact but not compact.*

Proof. We use again the Orlicz function ψ defined by (4.3) and the Schur function $\varphi = \varphi_{1/2}$. The space B^ψ is not reflexive since ψ does not satisfy the condition Δ_2.

We now need an estimate similar to (3.2) for φ_θ, namely

$$\rho_{\varphi,2}(h) := \sup_{|\xi|=1} A[\{z \in \mathbb{D};\ \varphi(z) \in S(\xi, h)\}] \approx h^{2/\theta}. \tag{4.10}$$

The proof of (4.10) is best seen by passing to the right half-plane with the measure A_{γ_θ} which is locally equivalent to the Lebesgue planar measure A; we get $\rho_{\varphi,2}(h) \geq A(\{|z|^\theta \leq h\} \cap \mathbb{H}) \geq K h^{2/\theta}$ and the upper bound in (4.10) is proved similarly.

We now see that $C_\varphi\colon B^\psi \to B^\psi$ is not compact as follows. We use the same x_n as in (4.3) and set $y_n = \psi(x_n)$, $k_n = 1/\sqrt{y_n}$. We notice that, since $\rho_{\varphi,2}(h) \geq K^{-1}h^4$ (with $K > 1$) in view of (4.10), we have

$$E(k_n) \overset{def}{=} \frac{\psi^{-1}(1/k_n^2)}{\psi^{-1}(1/\rho_{\varphi,2}(k_n))} \geq \frac{\psi^{-1}(1/k_n^2)}{\psi^{-1}(K/k_n^4)} = \frac{\psi^{-1}(y_n)}{\psi^{-1}(Ky_n^2)} \geq \frac{x_n}{2Kx_n} = \frac{1}{2K}\,,$$

so that

$$\limsup_{k\to 0^+} E(k) > 0,$$

and this implies that $C_\varphi\colon B^\psi \to B^\psi$ is not compact ([14], Theorem 3.2). To see that $C_\varphi\colon B^\psi \to B^\psi$ is w-compact, we use the diagram

$$B^\psi \xrightarrow{j_{\psi,2}} B^2 \xrightarrow{C_\varphi} B^4 \xrightarrow{j_{4,\psi}} B^\psi$$

as well as (4.10), which gives $\rho_{\varphi,2}(h) \leq Kh^4$. A result of W. Hastings ([5]) now implies the continuity of the second map. This diagram shows that C_φ factors through a reflexive space (B^2 or B^4), and is therefore w-compact. ∎

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