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Research Article

Some New Refinements of Hermite–Hadamard-Type Inequalities Involving ψ_k-Riemann–Liouville Fractional Integrals and Applications

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The main objective of this article is to establish some new fractional refinements of Hermite–Hadamard-type inequalities essentially using new ψ_k-Riemann–Liouville fractional integrals, where $k > 0$. Using this new fractional integral, we also derive two new fractional integral identities. Applications of the obtained results are also discussed.

1. Introduction and Preliminaries

Let $f: I = [a, b] \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function; then,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \quad (1)$$

The above inequality is known as Hermite–Hadamard's inequality [1–5]. This inequality provides us a necessary and sufficient condition for a function to be convex. It can be considered as one of the most extensively studied results pertaining to convexity. Since the appearance of this result in the literature, it gained popularity, and many new generalizations for this classical result have been obtained. This can be attributed to its applications in various other fields such as in numerical analysis and in mathematical statistics. For more details on generalizations of convexity, Hermite–Hadamard-like inequalities, and its applications, see [6–14].

Fractional calculus is a calculus in which we study about the integrals and derivatives of any arbitrary real or complex order. The history of fractional calculus is not very much old, but in the short span of time, it experienced a rapid development. Recently, the generalizations [15–25], extensions [26–32], and applications [33–46] for fractional calculus have been made by many researchers. The Riemann–Liouville fractional integrals are defined as follows.

Definition 1 (see [47]). Let $f \in L_1[a, b]$. Then, Riemann–Liouville integrals $J_{a^+}^{\alpha} f$ and $J_{b^-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, \quad x > a, \quad (2)$$

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\mathrm{d}t, \quad x < b, \quad (3)$$

where

$$\Gamma(\alpha) = \int_0^{\infty} e^{-x} x^{\alpha-1}\mathrm{d}x, \quad (4)$$

is the well-known gamma function.

Sarikaya et al. [10] elegantly utilized this concept in establishing fractional analogue of Hermite–Hadamard's inequality. This idea motivated other researchers, and consequently, many new generalizations of Hermite–Hadamard's inequality have been obtained using the concept of Riemann–Liouville fractional integrals.

Sarikaya and Karaca [12] introduced k-analogue of Riemann–Liouville fractional integrals and discussed some of its basic properties. They defined this concept in the following way: to be more precise, let f be piecewise continuous on $I^* = (0,\infty)$ and integrable on any finite subinterval of $I = [0,\infty]$. Then, for $t > 0$, we consider k-Riemann–Liouville fractional integral of f of order α as

$$ {}_kJ_a^{\alpha} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{(\alpha/k)-1} f(t)\mathrm{d}t, \quad x > a, k > 0. \tag{5}$$

If $k \longrightarrow 1$, then k-Riemann–Liouville fractional integrals reduce to classical the Riemann–Liouville fractional integral. It is worth to mention here that the concept of the k-Riemann–Liouville fractional integral is a significant generalization of Riemann–Liouville fractional integrals; as for $k \neq 1$, the properties of k-Riemann–Liouville fractional integrals are quite different from the classical Riemann–Liouville fractional integrals.

Another important generalization of Riemann–Liouville fractional integrals is ψ_k-Riemann–Liouville fractional integrals.

Definition 2 (see [6]). Let (a,b) be a finite interval of the real line $\mathbb{R}$ and $\alpha > 0$. Also, let $\psi(x)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $\psi'(x)$ on (a,b). Then, the left- and right-sided ψ-Riemann–Liouville fractional integrals of a function f with respect to another function ψ on $[a,b]$ are defined as

$$\begin{aligned} I_{a^+}^{\alpha;\psi} f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1} f(t)\mathrm{d}t,\\ I_{b^-}^{\alpha;\psi} f(x) &= \frac{1}{\Gamma(\alpha)} \int_x^b \psi'(t)(\psi(t)-\psi(x))^{\alpha-1} f(t)\mathrm{d}t, \end{aligned} \tag{6}$$

respectively; $\Gamma(\cdot)$ is the gamma function.

For some recent research works, see [48].

Recently, Liu et al. [14] obtained some interesting results pertaining to Hermite–Hadamard's inequality involving ψ_k-Riemann–Liouville fractional integrals. Motivated by the research work of Liu et al. [14], we obtain some new refinements of fractional Hermite–Hadamard's inequality essentially using ψ_k-Riemann–Liouville fractional integrals. We also discuss applications of the obtained results to means. We show that our results represent significant generalization of some previous results.

2. Hermite–Hadamard's Inequality

In this section, we derive a new refinement of Hermite–Hadamard's inequality via the ψ_k-Riemann–Liouville fractional integral.

Definition 3. Let $k > 0$, (a,b) be a finite interval of the real line $\mathbb{R}$, and $\alpha > 0$. Also, let $\psi(x)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $\psi'(x)$ on (a,b). Then, the left- and right-sided ψ_k-Riemann–Liouville fractional integrals of a function f with respect to another function ψ on $[a,b]$ are defined as

$$\begin{aligned} {}_kI_{a^+}^{\alpha;\psi} f(x) &= \frac{1}{k\Gamma_k(\alpha)} \int_a^x \psi'(t)(\psi(x)-\psi(t))^{(\alpha/k)-1} f(t)\mathrm{d}t,\\ {}_kI_{b^-}^{\alpha;\psi} f(x) &= \frac{1}{k\Gamma_k(\alpha)} \int_x^b \psi'(t)(\psi(t)-\psi(x))^{(\alpha/k)-1} f(t)\mathrm{d}t, \end{aligned} \tag{7}$$

respectively;

$$\Gamma_k(x) = \int_0^{\infty} t^{x-1} e^{-\left(t^k/k\right)}\mathrm{d}t, \quad \Re(x) > 0, \tag{8}$$

is the k-analogue of gamma function.

The k-analogues of beta function and incomplete beta function are, respectively, defined as

$$B_k(x,y) = \frac{1}{k}\int_0^1 t^{(x/k)-1}(1-t)^{(x/k)-1}\mathrm{d}t, \tag{9}$$

$$B_k(z;x,y) = \frac{1}{k}\int_0^z t^{(x/k)-1}(1-t)^{(b/y)-1}\mathrm{d}t. \tag{10}$$

We now derive the main result of this section.

Theorem 1. *Let $0 \le e < f$ and g: $[e,f] \longrightarrow \mathbb{R}$ be a positive function and $g \in L_1[e,f]$. Also, suppose that g is a convex function on $[e,f]$, $\psi(x)$ is an increasing and positive monotone function on $(e,f]$, having a continuous derivative $\psi'(x)$ on (e,f), and $\alpha \in (0,1)$. Then, for $k > 0$, the following k-fractional integral inequalities hold:*

$$\begin{aligned} g\left(\frac{e+f}{2}\right) &\le \frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}\left[{}_kI_{\psi^{-1}(e)^+}^{\alpha;\psi}(g\circ\psi)\left(\psi^{-1}(f)\right)\right.\\ &\quad \left. + {}_kI_{\psi^{-1}(f)^-}^{\alpha;\psi}(g\circ\psi)\left(\psi^{-1}(e)\right)\right] \le \frac{g(e)+g(f)}{2}. \end{aligned} \tag{11}$$

Proof. Using the convexity of g, we have

$$2g\left(\frac{e+f}{2}\right) \le g(tc+(1-t)f) + g((1-t)e+td). \tag{12}$$

Multiplying both sides by $t^{(\alpha/k)-1}$ and then integrating with respect to t on $[0,1]$, we have

$$\begin{aligned} \frac{2k}{\alpha} g\left(\frac{e+f}{2}\right) \le& \int_0^1 t^{(\alpha/k)-1} g(tc+(1-t)f)\mathrm{d}t\\ &+ \int_0^1 t^{(\alpha/k)-1} g((1-t)e+td)\mathrm{d}t. \end{aligned} \tag{13}$$

Now, making the substitution $t = (\psi(v) - f/e - f)$, $s = (\psi(v) - e/f - e)$, we have

$$\begin{aligned}
&\frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(e)^+}(g\circ\psi)\left(\psi^{-1}(f)\right)+{}_kI^{\alpha;\psi}_{\psi^{-1}(f)^-}(g\circ\psi)\left(\psi^{-1}(e)\right)\right]\\
&=\frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}\frac{1}{k\Gamma_k(\alpha)}\left[\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}(f-\psi(v))^{(\alpha/k)}(g\circ\psi)(v)\psi'(v)\mathrm{d}v\right.\\
&\quad\left.+\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}(\psi(v)-e)^{(\alpha/k)}(g\circ\psi)(v)\psi'(v)\mathrm{d}v\right]\\
&=\frac{\alpha}{2k}\left[\int_0^1 t^{(\alpha/k)-1}g(tc+(1-t)f)\mathrm{d}t+\int_0^1 t^{(\alpha/k)-1}g((1-t)e+td)\mathrm{d}t\right]\\
&\geq g\left(\frac{e+f}{2}\right).
\end{aligned}\tag{14}$$

Also, using the convexity property of g, we have

$$g(tc+(1-t)f)+g((1-t)e+td)\leq g(e)+g(f).\tag{15}$$

Multiplying both sides by $t^{(\alpha/k)-1}$ and then integrating it with respect to t on $[0,1]$, we obtain

$$\begin{aligned}
&\int_0^1 t^{(\alpha/k)-1}g(tc+(1-t)f)\mathrm{d}t+\int_0^1 t^{(\alpha/k)-1}g((1-t)e+td)\mathrm{d}t\\
&\leq\frac{k}{\alpha}[g(e)+g(f)].
\end{aligned}\tag{16}$$

This implies

$$\begin{aligned}
&\frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(e)^+}(g\circ\psi)\left(\psi^{-1}(f)\right)\right.\\
&\quad\left.+{}_kI^{\alpha;\psi}_{\psi^{-1}(f)^-}(g\circ\psi)\left(\psi^{-1}(e)\right)\right]\leq\frac{g(e)+g(f)}{2}.
\end{aligned}\tag{17}$$

The proof is completed. □

3. Some More Fractional Inequalities of Hermite–Hadamard Type

We now derive two new fractional integral identities involving ψ_k-Riemann–Liouville fractional integrals. These results will serve as auxiliary results for obtaining our next results.

Lemma 1. *Let $e<f$ and g: $[e,f]\longrightarrow\mathbb{R}$ be a differentiable mapping on (e,f). Also, suppose that $g'\in L[e,f]$, $\psi(x)$ is an increasing and positive monotone function on $(e,f]$, having a continuous derivative $\psi'(x)$ on (e,f), and $\alpha\in(0,1)$. Then, for $k>0$, the following identity holds:*

$$\begin{aligned}
&\frac{g(e)+g(f)}{2}-\frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(e)^+}(g\circ\psi)\left(\psi^{-1}(f)\right)\right.\\
&\quad\left.+{}_kI^{\alpha;\psi}_{\psi^{-1}(f)^-}(g\circ\psi)\left(\psi^{-1}(e)\right)\right]\\
&=\frac{1}{2(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}\left[(\psi(v)-e)^{(\alpha/k)}\right.\\
&\quad\left.-(f-\psi(v))^{(\alpha/k)}\right](g'\circ\psi)(v)\psi'(v)\mathrm{d}v.
\end{aligned}\tag{18}$$

Proof. Consider $J_1=(\Gamma_k(\alpha+k)/2(f-e)^{(\alpha/k)})_kI^{\alpha;\psi}_{\psi^{-1}(e)^+}(g\circ\psi)(\psi^{-1}(f))$ and $J_2=(\Gamma_k(\alpha+k)/2(f-e)^{(\alpha/k)})_kI^{\alpha;\psi}_{\psi^{-1}(f)^-}(g\circ\psi)(\psi^{-1}(e))$.

Now,

$$\begin{aligned}
J_1&=\frac{\alpha}{2k(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}(f-\psi(v))^{(\alpha/k)-1}(g\circ\psi)(v)\psi'(v)\mathrm{d}v\\
&=-\frac{1}{2k(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}(g\circ\psi)(v)\mathrm{d}(f-\psi(v))^{(\alpha/k)}\\
&=\frac{g(e)}{2}+\frac{1}{2(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}(f-\psi(v))^{(\alpha/k)}(g'\circ\psi)(v)\psi'(v)\mathrm{d}v.
\end{aligned}\tag{19}$$

Similarly,

$$J_2 = \frac{\alpha}{2k(f-e)^{(\alpha/k)}} \int_{\psi^{-1}(e)}^{\psi^{-1}(f)} (\psi(v)-e)^{(\alpha/k)-1} \cdot (g\circ\psi)(v)\psi'(v)\mathrm{d}v$$
$$= \frac{1}{2k(f-e)^{(\alpha/k)}} \int_{\psi^{-1}(e)}^{\psi^{-1}(f)} (g\circ\psi)(v)\mathrm{d} \cdot (\psi(v)-e)^{(\alpha/k)}$$
$$= \frac{g(f)}{2} - \frac{1}{2(f-e)^{(\alpha/k)}} \int_{\psi^{-1}(e)}^{\psi^{-1}(f)} \cdot (\psi(v)-e)^{(\alpha/k)} (g'\circ\psi)(v)\psi'(v)\mathrm{d}v. \tag{20}$$

It follows that

$$\frac{g(e)+g(f)}{2} - (J_1+J_2) = \frac{1}{2(f-e)^{(\alpha/k)}} \int_{\psi^{-1}(e)}^{\psi^{-1}(f)} \cdot \left[(\psi(v)-e)^{(\alpha/k)} - (f-\psi(v))^{(\alpha/k)} \right] \cdot (g'\circ\psi)(v)\psi'(v)\mathrm{d}v. \tag{21}$$

□

Example 1. Let $c=2, d=3, \alpha=(1/2), k=2, g(x)=x^2, \psi(x)=x$. Then, all the assumptions in Lemma 1 are satisfied. Observe that $(g(c)+g(d)/2)=(13/2)$.

$$\frac{\Gamma_k(\alpha+k)}{2(d-c)^{(\alpha/k)}} \left[{}_kI^{\alpha;\psi}_{\psi^{-1}(c)^+} (g\circ\psi)\left(\psi^{-1}(d)\right) + {}_kI^{\alpha;\psi}_{\psi^{-1}(d)^-} (g\circ\psi)\left(\psi^{-1}(c)\right) \right]$$
$$= \frac{\Gamma_{(2)}(1/2)}{2} \left[\frac{1}{\Gamma_{(2)}(1/2)} \int_2^3 v^2 (3-v)^{-(3/4)}\mathrm{d}v + \frac{1}{\Gamma_{(2)}(1/2)} \int_2^3 v^2 (v-2)^{-(3/4)}\mathrm{d}v \right] = \frac{577}{90}. \tag{22}$$

This implies

$$\frac{g(c)+g(d)}{2} - \frac{\Gamma_k(\alpha+k)}{2(d-c)^{(\alpha/k)}} \left[{}_kI^{\alpha;\psi}_{\psi^{-1}(c)^+} (g\circ\psi)\left(\psi^{-1}(d)\right) + {}_kI^{\alpha;\psi}_{\psi^{-1}(d)^-} (g\circ\psi)\left(\psi^{-1}(c)\right) \right] = \frac{4}{45}. \tag{23}$$

Also,

$$\frac{1}{2(d-c)^{(\alpha/k)}} \int_{\psi^{-1}(c)}^{\psi^{-1}(d)} \left[(\psi(v)-c)^{(\alpha/k)} - (d-\psi(v))^{(\alpha/k)} \right] \cdot (g'\circ\psi)(v)\psi'(v)\mathrm{d}v$$
$$= \int_2^3 v(v-2)^{(1/4)}\mathrm{d}v - \int_2^3 v(3-v)^{(1/4)}\mathrm{d}v = \frac{4}{45}. \tag{24}$$

Example 2. Let $c=2, d=3, \alpha=(1/2), k=(1/2), g(x)=x^2, \psi(x)=x$. Then, all the assumptions in Lemma 1 are satisfied. Observe that $(g(c)+g(d)/2)=(13/2)$.

$$\frac{\Gamma_k(\alpha+k)}{2(d-c)^{(\alpha/k)}} \left[{}_kI^{\alpha;\psi}_{\psi^{-1}(c)^+} (g\circ\psi)\left(\psi^{-1}(d)\right) + {}_kI^{\alpha;\psi}_{\psi^{-1}(d)^-} (g\circ\psi)\left(\psi^{-1}(c)\right) \right]$$
$$= \frac{\Gamma_{(1/2)}(1/2)}{2} \left[\frac{1}{\Gamma_{(1/2)}(1/2)} \int_2^3 v^2\mathrm{d}v + \frac{1}{\Gamma_{(1/2)}(1/2)} \int_2^3 v^2\mathrm{d}v \right] = \frac{19}{3}. \tag{25}$$

This implies

$$\frac{g(c)+g(d)}{2} - \frac{\Gamma_k(\alpha+k)}{2(d-c)^{(\alpha/k)}} \left[{}_kI^{\alpha;\psi}_{\psi^{-1}(c)^+} (g\circ\psi)\left(\psi^{-1}(d)\right) + {}_kI^{\alpha;\psi}_{\psi^{-1}(d)^-} (g\circ\psi)\left(\psi^{-1}(c)\right) \right] = \frac{1}{6}. \tag{26}$$

Also,

$$\frac{1}{2(d-c)^{(\alpha/k)}} \int_{\psi^{-1}(c)}^{\psi^{-1}(d)} \left[(\psi(v)-c)^{(\alpha/k)} - (d-\psi(v))^{(\alpha/k)} \right] \cdot (g'\circ\psi)(v)\psi'(v)\mathrm{d}v$$
$$= \int_2^3 v(v-2)\mathrm{d}v - \int_2^3 v(3-v)\mathrm{d}v = \frac{1}{6}. \tag{27}$$

Lemma 2. *Let $e<f$ and $g\colon [e,f] \longrightarrow \mathbb{R}$ be a differentiable mapping on (e,f). Also, suppose that $g' \in L[e,f]$, $\psi(x)$ is an increasing and positive monotone function on $(e,f]$, having a continuous derivative $\psi'(x)$ on (e,f), and $\alpha \in (0,1)$. Then, for $k>0$, the following identity holds:*

$$\frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(e)^+}(g\circ\psi)\left(\psi^{-1}(f)\right)\right.$$
$$\left.+{}_kI^{\alpha;\psi}_{\psi^{-1}(f)^-}(g\circ\psi)\left(\psi^{-1}(e)\right)\right]-g\left(\frac{e+f}{2}\right)$$
$$=\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}h\left(g'\circ\psi\right)(v)\psi'(v)\mathrm{d}v \tag{28}$$
$$+\frac{1}{2(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}\left[(\psi(v)-e)^{(\alpha/k)}\right.$$
$$\left.-(f-\psi(v))^{(\alpha/k)}\right]\left(g'\circ\psi\right)(v)\psi'(v)\mathrm{d}v,$$

where

$$h=\begin{cases}\frac{1}{2}, & \text{for } \psi^{-1}\left(\frac{e+f}{2}\right)\leq v\leq\psi^{-1}(f),\\ -\frac{1}{2}, & \text{for } \psi^{-1}(e)\leq v\leq\psi^{-1}\left(\frac{e+f}{2}\right).\end{cases} \tag{29}$$

Proof. Suppose

$$\begin{aligned}I_1&=-\frac{1}{2}\int_{\psi^{-1}(e)}^{\psi^{-1}(e+f/2)}\left(g'\circ\psi\right)(v)\psi'(v)\mathrm{d}v=-\frac{1}{2}g\left(\frac{e+f}{2}\right)+\frac{g(e)}{2},\\ I_2&=\frac{1}{2}\int_{\psi^{-1}(e+f/2)}^{\psi^{-1}(f)}\left(g'\circ\psi\right)(v)\psi'(v)\mathrm{d}v=-\frac{1}{2}g\left(\frac{e+f}{2}\right)+\frac{g(f)}{2},\\ I_3&=\frac{1}{2(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}(f-\psi(v))^{(\alpha/k)}\left(g'\circ\psi\right)(v)\psi'(v)\mathrm{d}v\\ &=-\frac{g(e)}{2}+\frac{\alpha}{2k(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}(f-\psi(v))^{(\alpha/k)-1}(g\circ\psi)(v)\psi'(v)\mathrm{d}v\\ &=-\frac{g(e)}{2}+\frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}{}_kI^{\alpha;\psi}_{\psi^{-1}(e)^+}(g\circ\psi)\left(\psi^{-1}(f)\right),\end{aligned} \tag{30}$$

$$\begin{aligned}I_4&=-\frac{1}{2(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}(\psi(v)-e)^{(\alpha/k)}\left(g'\circ\psi\right)(v)\psi'(v)\mathrm{d}v\\ &=-\frac{g(f)}{2}+\frac{\alpha}{2k(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}(\psi(v)-e)^{(\alpha/k)-1}(g\circ\psi)(v)\psi'(v)\mathrm{d}v\\ &=-\frac{g(f)}{2}+\frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}{}_kI^{\alpha;\psi}_{\psi^{-1}(f)^-}(g\circ\psi)\left(\psi^{-1}(e)\right).\end{aligned} \tag{31}$$

Summing I_1, I_2, I_3, and I_4, we get the required result. □

Example 3. Let $c=2, d=3, \alpha=(1/2), k=2, g(x)=x^2, \psi(x)=x$. Then, all the assumptions in Lemma 2 are satisfied. Note that $g(c+d/2)=(25/4)$.

$$\begin{aligned}&\frac{\Gamma_k(\alpha+k)}{2(d-c)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(c)^+}(g\circ\psi)\left(\psi^{-1}(d)\right)+{}_kI^{\alpha;\psi}_{\psi^{-1}(d)^-}(g\circ\psi)\left(\psi^{-1}(c)\right)\right]\\ &=\frac{\Gamma_{(1/2)}(1/2)}{8}\left[\frac{1}{\Gamma_{(1/2)}(1/2)}\int_2^3 v^2(3-v)^{-(3/4)}\mathrm{d}v\right.\\ &\left.+\frac{1}{\Gamma_{(1/2)}(1/2)}\int_2^3 v^2(v-2)^{-(3/4)}\mathrm{d}v\right]=\frac{577}{90}.\end{aligned} \tag{32}$$

This implies

$$\frac{\Gamma_k(\alpha+k)}{2(d-c)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(c)^+}(g\circ\psi)\left(\psi^{-1}(d)\right) + {}_kI^{\alpha;\psi}_{\psi^{-1}(d)^-}(g\circ\psi)\left(\psi^{-1}(c)\right)\right] - g\left(\frac{c+d}{2}\right) = \frac{29}{180}. \tag{33}$$

Also,

$$\int_{\psi^{-1}(c)}^{\psi^{-1}(d)} h\left(g'\circ\psi\right)(v)\psi'(v)\mathrm{d}v = \frac{1}{4}, \tag{34}$$

where h is defined in Lemma 2.

$$\frac{1}{2(d-c)^{(\alpha/k)}}\int_{\psi^{-1}(c)}^{\psi^{-1}(d)}\left[(d-\psi(v))^{(\alpha/k)} - (\psi(v)-c)^{(\alpha/k)}\right]\cdot\left(g'\circ\psi\right)(v)\psi'(v)\mathrm{d}v = \int_2^3 v(v-2)^{(1/4)}\mathrm{d}v - \int_2^3 v(3-v)^{(1/4)}\mathrm{d}v = -\frac{4}{45}. \tag{35}$$

This implies

$$\int_{\psi^{-1}(c)}^{\psi^{-1}(d)} h\left(g'\circ\psi\right)(v)\psi'(v)\mathrm{d}v + \frac{1}{2(d-c)^{(\alpha/k)}}\cdot\int_{\psi^{-1}(c)}^{\psi^{-1}(d)}\left[(d-\psi(v))^{(\alpha/k)} - (\psi(v)-c)^{(\alpha/k)}\right]\left(g'\circ\psi\right)(v)\psi'(v)\mathrm{d}v = \frac{29}{180}. \tag{36}$$

Example 4. Let $c=2, d=3, \alpha=(1/2), k=(1/2), g(x)=x^2, \psi(x)=x$. Then, all the assumptions in Lemma 2 are satisfied. Note that $g(c+d/2)=(25/4)$.

$$\frac{\Gamma_k(\alpha+k)}{2(d-c)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(c)^+}(g\circ\psi)\left(\psi^{-1}(d)\right) + {}_kI^{\alpha;\psi}_{\psi^{-1}(d)^-}(g\circ\psi)\left(\psi^{-1}(c)\right)\right] = \frac{\Gamma_{(1/2)}(1/2)}{2}\left[\frac{1}{\Gamma_{(1/2)}(1/2)}\int_2^3 v^2\mathrm{d}v + \frac{1}{\Gamma_{(2)}(1/2)}\int_2^3 v^2\mathrm{d}v\right] = \frac{19}{3}. \tag{37}$$

This implies

$$\frac{\Gamma_k(\alpha+k)}{2(d-c)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(c)^+}(g\circ\psi)\left(\psi^{-1}(d)\right) + {}_kI^{\alpha;\psi}_{\psi^{-1}(d)^-}(g\circ\psi)\left(\psi^{-1}(c)\right)\right] - g\left(\frac{c+d}{2}\right) = \frac{1}{12}. \tag{38}$$

Also,

$$\int_{\psi^{-1}(c)}^{\psi^{-1}(d)} h\left(g'\circ\psi\right)(v)\psi'(v)\mathrm{d}v = \frac{1}{4}, \tag{39}$$

where h is defined in Lemma 2.

$$\frac{1}{2(d-c)^{(\alpha/k)}}\int_{\psi^{-1}(c)}^{\psi^{-1}(d)}\left[(d-\psi(v))^{(\alpha/k)} - (\psi(v)-c)^{(\alpha/k)}\right]\left(g'\circ\psi\right)(v)\psi'(v)\mathrm{d}v = \int_2^3 v(v-2)\mathrm{d}v - \int_2^3 v(3-v)\mathrm{d}v = -\frac{1}{6}. \tag{40}$$

This implies

$$\int_{\psi^{-1}(c)}^{\psi^{-1}(d)} h\left(g'\circ\psi\right)(v)\psi'(v)\mathrm{d}v + \frac{1}{2(d-c)^{(\alpha/k)}}\int_{\psi^{-1}(c)}^{\psi^{-1}(d)}\cdot\left[(d-\psi(v))^{(\alpha/k)} - (\psi(v)-c)^{(\alpha/k)}\right]\cdot\left(g'\circ\psi\right)(v)\psi'(v)\mathrm{d}v = \frac{1}{12}. \tag{41}$$

Before proceeding to next results, let us recall the definition of s-convex function of Breckner type.

Definition 4 (see [49]). A function $g\colon [0,\infty)\longrightarrow[0,\infty)$ is said to be s-convex function of Breckner type if

$$g((1-t)x+ty) \le (1-t)^s g(x) + t^s g(y), \quad \forall x, y \in [0,\infty), t\in[0,1], s\in(0,1]. \tag{42}$$

Theorem 2. *Let $e<f$ and $g\colon [e,f]\longrightarrow\mathbb{R}$ be a differentiable mapping on (e,f). Also, suppose that $|g'|$ is Breckner type of s-convex on $[e,f]$, $\psi(x)$ is an increasing and positive monotone function on $(e,f]$, having a continuous derivative $\psi'(x)$ on (e,f), and $\alpha\in(0,1)$. Then, for $k>0$, the following inequality holds:*

$$\left|\frac{g(e)+g(f)}{2} - \frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(e)^+}(g\circ\psi)\left(\psi^{-1}(f)\right) + {}_kI^{\alpha;\psi}_{\psi^{-1}(f)^-}(g\circ\psi)\left(\psi^{-1}(e)\right)\right]\right| \le \frac{f-e}{2}\left[L_1\left|g'(e)\right| + L_2\left|g'(f)\right|\right], \tag{43}$$

where

$$L_1 \coloneqq 2k\mathrm{B}_k\left(\frac{1}{2};1+s,\frac{k+\alpha}{k}\right) + \frac{k\left(1-2^{-(ks+\alpha/k)}\right)}{k+ks+\alpha} - \mathrm{B}_k\left(1+s,\frac{k+\alpha}{k}\right), \tag{44}$$

$$L_2 \coloneqq \frac{k\left(1-2^{-(ks+\alpha/k)}\right)}{k+ks+\alpha} - 2k\mathrm{B}_k\left(\frac{1}{2};\frac{k+\alpha}{k},1+s\right) - \mathrm{B}_k\left(\frac{k+\alpha}{k},1+s\right), \tag{45}$$

respectively.

Proof. Using Lemma 1 and the fact that $|g'|$ is Breckner type of s-convex function, we have

$$\begin{aligned}
&\left|\frac{g(e)+g(f)}{2}-\frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(e)^+}(g\circ\psi)\left(\psi^{-1}(f)\right)+{}_kI^{\alpha;\psi}_{\psi^{-1}(f)^-}(g\circ\psi)\left(\psi^{-1}(e)\right)\right]\right|\\
&\leq\frac{1}{2(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}\left|(\psi(v)-e)^{(\alpha/k)}-(f-\psi(v))^{(\alpha/k)}\right|\left|(g'\circ\psi)(v)\right|\left|\psi'(v)\right|\mathrm{d}v\\
&=\frac{f-e}{2}\int_0^1\left|(1-t)^{(\alpha/k)}-t^{(\alpha/k)}\right|\left|g'(tc+(1-t)f)\right|\mathrm{d}t\\
&\leq\frac{f-e}{2}\int_0^1\left|(1-t)^{(\alpha/k)}-t^{(\alpha/k)}\right|\left[t^s\left|g'(e)\right|+(1-t)^s\left|g'(f)\right|\right]\mathrm{d}t\\
&=\frac{f-e}{2}\left[\left|g'(e)\right|\int_0^{(1/2)}t^s\left[(1-t)^{(\alpha/k)}-t^{(\alpha/k)}\right]\mathrm{d}t+\left|g'(f)\right|\int_0^{(1/2)}(1-t)^s\left[(1-t)^{(\alpha/k)}-t^{(\alpha/k)}\right]\mathrm{d}t\right.\\
&\quad\left.+\left|g'(e)\right|\int_{(1/2)}^1 t^s\left[t^{(\alpha/k)}-(1-t)^{(\alpha/k)}\right]\mathrm{d}t+\left|g'(f)\right|\int_{(1/2)}^1(1-t)^s\left[t^{(\alpha/k)}-(1-t)^{(\alpha/k)}\right]\mathrm{d}t\right]\\
&=\frac{f-e}{2}\left[L_1\left|g'(e)\right|+L_2\left|g'(f)\right|\right],
\end{aligned}\tag{46}$$

where

$$\begin{aligned}
L_1 &:= H_1+H_3=\int_0^{(1/2)}t^s\left[(1-t)^{(\alpha/k)}-t^{(\alpha/k)}\right]\mathrm{d}t+\int_{(1/2)}^1 t^s\left[t^{(\alpha/k)}-(1-t)^{(\alpha/k)}\right]\mathrm{d}t\\
&=2k\mathrm{B}_k\left(\frac{1}{2};1+s,\frac{k+\alpha}{k}\right)+\frac{k\left(1-2^{-(ks+\alpha/k)}\right)}{k+ks+\alpha}-\mathrm{B}_k\left(1+s,\frac{k+\alpha}{k}\right),
\end{aligned}\tag{47}$$

$$\begin{aligned}
L_2 &:= H_2+H_4=\int_0^{(1/2)}(1-t)^s\left[(1-t)^{(\alpha/k)}-t^{(\alpha/k)}\right]\mathrm{d}t+\int_{(1/2)}^1(1-t)^s\left[t^{(\alpha/k)}-(1-t)^{(\alpha/k)}\right]\mathrm{d}t\\
&=\frac{k\left(1-2^{-(ks+\alpha/k)}\right)}{k+ks+\alpha}-2k\mathrm{B}_k\left(\frac{1}{2};\frac{k+\alpha}{k},1+s\right)-\mathrm{B}_k\left(\frac{k+\alpha}{k},1+s\right).
\end{aligned}\tag{48}$$

This completes the proof. □

Theorem 3. *Let g: $[e,f]\longrightarrow\mathbb{R}$ be a differentiable function on (e,f) with $e<f$. Also, suppose that $|g'|$ is Breckner type of s-convex function. If $\psi(x)$ is an increasing and positive monotone function on $(e,f]$, having a continuous derivative $\psi'(x)$ on (e,f) and $\alpha\in(0,1)$, then for $k>0$, the following inequality holds:*

$$\begin{aligned}
&\left|\frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(e)^+}(g\circ\psi)\left(\psi^{-1}(f)\right)\right.\right.\\
&\quad\left.\left.+{}_kI^{\alpha;\psi}_{\psi^{-1}(f)^-}(g\circ\psi)\left(\psi^{-1}(e)\right)\right]-g\left(\frac{e+f}{2}\right)\right|\\
&\leq\frac{|g(f)-g(e)|}{2}+\frac{f-e}{2}\left[L_1\left|g'(e)\right|+L_2\left|g'(f)\right|\right],
\end{aligned}\tag{49}$$

where L_1 and L_2 are given by (44) and (45), respectively.

Proof. Using Lemma 2, the property of modulus, and the given hypothesis of the theorem, we have

$$\begin{aligned}
&\left|\frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(e)^+}(g\circ\psi)\left(\psi^{-1}(f)\right)\right.\right.\\
&\quad\left.\left.+{}_kI^{\alpha;\psi}_{\psi^{-1}(f)^-}(g\circ\psi)(\pi)\right]-g\left(\frac{e+f}{2}\right)\right|\\
&\leq\left|\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}h(g'\circ\psi)(v)\psi'(v)\mathrm{d}v\right|\\
&\quad+\left|\frac{1}{2(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}\left[(f-\psi(v))^{(\alpha/k)}\right.\right.\\
&\quad\left.\left.-(\psi(v)-e)^{(\alpha/k)}\right](g'\circ\psi)(v)\psi'(v)\mathrm{d}v\right|\\
&=I_1+I_2.
\end{aligned}\tag{50}$$

Using substitution $t = (\psi(v) - e/f - e)$ and the fact that $|g'|$ is Breckner type of s-convex function, we have

$$I_1 \le \frac{f-e}{2}\left[L_1\left|g'(e)\right| + L_2\left|g'(f)\right|\right], \tag{51}$$

where L_1 and L_2 are given by (44) and (45), respectively. And

$$I_2 = \frac{|g(f) - g(e)|}{2}. \tag{52}$$

This completes the proof. □

4. Applications

In this section, we discuss some applications of Theorem 2 to means by considering a particular example of s-convexity. First of all, we recall some previously known concepts related to means [50].

For arbitrary real numbers α, β, $\alpha \neq \beta$, we define the following:

(1) Arithmetic mean:

$$A(\alpha, \beta) = \frac{\alpha + \beta}{2}, \quad \alpha, \beta \in \mathbb{R}; \tag{53}$$

(2) Logarithmic mean:

$$\overline{L}(\alpha, \beta) = \frac{\beta - \alpha}{\ln|\beta| - \ln|\alpha|}, \quad \alpha, \beta \in \mathbb{R}\backslash\{0\}; \tag{54}$$

(3) Generalized log-mean:

$$L_n(\alpha, \beta) = \left[\frac{\beta^{n+1} - \alpha^{n+1}}{(n+1)(\beta - \alpha)}\right]^{(1/n)}, \quad n \in \mathbb{N}, n \ge 1, \alpha, \beta \in \mathbb{R}, \alpha < \beta. \tag{55}$$

We now give the main results of this section.

Proposition 1. *Let $e, f \in \mathbb{R}^+$ with $e < f$; then,*

$$\left|A\left(e^s, f^s\right) - L_s^s(e, f)\right| \le \frac{s(f-e)}{2}\left[W_1|e|^{s-1} + W_2|f|^{s-1}\right], \tag{56}$$

where

$$W_1 \coloneqq 2\mathrm{B}\left(\frac{1}{2}; 1+s, 2\right) + \frac{1 - 2^{-1-s}}{2+s} - \mathrm{B}(1+s, 2), \tag{57}$$

$$W_2 \coloneqq \frac{1 - 2^{-1-s}}{2+s} - 2\mathrm{B}\left(\frac{1}{2}; 2, 1+s\right) - \mathrm{B}(2, 1+s), \tag{58}$$

respectively.

Proof. Applying Theorem 2 for $g(x) = x^s$, $\psi(x) = x$, and $\alpha = 1 = k$, we obtain the required result. □

Proposition 2. *Let $e, f \in \mathbb{R}^+$ with $e < f$; then,*

$$\left|A\left(e^s, f^s\right) - L_s^s(e, f)\right| \le \frac{\left|f^s - g^s\right|}{2} + \frac{s(f-e)}{2}\left[W_1|e|^{s-1} + W_2|f|^{s-1}\right], \tag{59}$$

where W_1 and W_2 are given by (57) and (58), respectively.

Proof. Applying Theorem 3 for $g(x) = x^s$, $\psi(x) = x$, and $\alpha = 1 = k$, we obtain the required result. □

5. Conclusion

In this article, we obtain some new fractional estimates of Hermite–Hadamard's inequality essentially using a new k-analogue of ψ_k-fractional integrals. We derive two new fractional integral identities in the setting of k-fractional calculus. In order to check the validity of these identities, we discuss some particular examples. In the final section, we have discussed applications of Theorems 2 and 3 to means.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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