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Research Article

Some New Results on Global Dominating Sets

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A dominating set is called a global dominating set if it is a dominating set of a graph G and its complement $\overline{G}$. A natural question arises: are there any graphs for which it is possible to relate the domination number and the global domination number? We have found an affirmative answer to this question and obtained some graphs having such characteristic.

1. Introduction

We begin with finite and undirected simple graph $G = (V, E)$ of order n. The set $S \subseteq V$ of vertices in a graph G is called a dominating set if every vertex $v \in V$ is either an element of S or is adjacent to an element of S. A dominating set S is a minimal dominating set (MDS) if no proper subset $S' \subset S$ is a dominating set.

The minimum cardinality of a dominating set of G is called the domination number of G which is denoted by $\gamma(G)$ and the corresponding dominating set is called a γ-set of G.

The open neighborhood $N(v)$ of $v \in V$ is the set of vertices adjacent to v and the closed neighborhood of v is the set $N[v] = N(v) \cup \{v\}$.

The complement $\overline{G}$ of G is the graph with vertex set V and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

A subset $D \subseteq V$ is called a global dominating set in G if D is a dominating set of both G and $\overline{G}$. The global domination number $\gamma_g(G)$ is the minimum cardinality of a global dominating set in G. The concept of global domination in graph was introduced by Sampathkumar [1].

The upper bounds of global domination number are investigated by Brigham and Dutton [2] as well as by Poghosyan and Zverovich [3], while the global domination number of Boolean function graph is studied by Janakiraman et al. [4]. The global domination decision problems are NP-complete as discussed by Carrington [5] and by Carrington and Brigham [6]. The edge addition stable property in the context of global domination and connected global domination for cycle C_n and path P_n is discussed by Kavitha and David [7]. The concept of total global dominating set was introduced by Kulli and Janakiram [8] and they have also characterized total global dominating sets.

The wheel W_n is defined to be the join $C_{n-1} + K_1$. The vertex corresponding to K_1 is known as apex vertex and the vertices corresponding to cycle are known as rim vertices.

A shell graph $S_n(n > 3)$ is the graph obtained by taking $(n-3)$ concurrent chords in a cycle C_n. The vertex at which all the chords are concurrent is called the apex. The shell graph S_n is also called fan f_{n-1} that is, $S_n = f_{n-1} = P_{n-1} + K_1$.

Definition 1.1. The one-point union of r cycles of length n denoted by $C_n^{(r)}$ is the graph obtained by identifying one vertex of each cycle.

The one-point union of r cycles C_3 is known as friendship graph which is denoted by F_n.

Definition 1.2 (see Shee and Ho [9]). Let $G = (V, E)$ be a graph and let $G_1, G_2, G_3, \ldots, G_r$, $r > 2$ be r copies of a graph G. Then, the graph obtained by adding an edge from G_i to G_{i+1} for $i = 1, 2, \ldots, r-1$ is called path union of G.

Definition 1.3. Consider two copies of a graph G, namely, $G^{(1)}$ and $G^{(2)}$. Then, $G_1 = \langle G^{(1)} : G^{(2)} \rangle$ is the graph obtained by joining the vertices of degree $(n-1)$ of each graph to a new vertex x.

Definition 1.4. Consider r-copies of a graph G, namely, $G^{(1)}, G^{(2)}, G^{(3)}, \cdots, G^{(r)}$. Then, $G_1 = \langle G^{(1)} : G^{(2)} : G^{(3)} : \cdots : G^{(r)} \rangle$ is the graph obtained by joining the vertices of degree $(n-1)$ of each $G^{(p-1)}$ and $G^{(p)}$ to a new vertex x_{p-1}, where $2 \le p \le k$.

Definition 1.5. Consider two copies of a graph G, namely, $G^{(1)}$ and $G^{(2)}$. Then, $G_1 = \langle G^{(1)} \blacktriangle G^{(2)} \rangle$ is the graph obtained by joining the vertices of degree $(n-1)$ of each graph by an edge as well as to a new vertex x.

Definition 1.6. Consider r-copies of a graph G, namely, $G^{(1)}, G^{(2)}, G^{(3)}, \cdots, G^{(r)}$. Then, $G_1 = \langle G^{(1)} \blacktriangle G^{(2)} \blacktriangle G^{(3)} \blacktriangle \ldots \blacktriangle G^{(r)} \rangle$ is the graph obtained by joining the vertices of degree $(n-1)$ of each $G^{(p-1)}$ and $G^{(p)}$ by an edge as well as to a new vertex x_{p-1}, where $2 \le p \le k$.

For the various graph theoretic notations and terminology we follow West [10], while the terms related with the concept of domination are used in the sense of Haynes et al. [11].

In the present paper we investigate some results on global dominating sets in the context of one-point union of cycles and path union of some graph families.

2. Main Results

Theorem 2.1. *If G is a one-point union of finite number of copies of cycle C_3, then $\gamma_g(G) = \gamma(G) + 2$.*

Proof. Consider r-copies of cycle C_3 in G and let c be the vertex in common for these r-copies. Because c is adjacent to every other vertex in G, c is not adjacent to any other vertex in the complement of G. Therefore, any dominating set for the complement of G must contain c. Hence, any global dominating set of G must contain c.

Let v_1 and v_2 be the vertices of a cycle C_3 other than c in G. Now, since vertices both c and v_1 are adjacent to third vertex v_2 of a cycle C_3 in G, at least three vertices are essential to dominate $\overline{G}$, which also dominate G. Therefore, $\gamma_g(G) = 3$. Moreover, $S = \{c\}$ is the only γ-set of G as c is the only vertex of degree $(n-1)$ in G. Hence, $\gamma(G) = 1$.

Thus, $\gamma_g(G) = \gamma(G) + 2$ as required. □

Theorem 2.2. *If G is a one-point union of finite number of copies of cycle C_n, then $\gamma(G) = \gamma_g(G) = (\lceil n/3 \rceil - 1)r + 1$ for $n > 3$.*

Proof. Consider r-copies of cycle C_n in G and let c be the vertex in common for these r-copies. Since c is the vertex with maximum degree, it must belong to any global dominating set corresponding to G for minimum cardinality.

Now, it is easy to observe that $\gamma_g(C_n) = \lceil n/3 \rceil$ for $n \neq 3$, $n \neq 5$. Therefore, any global dominating set corresponding to G must contain $(\lceil n/3 \rceil - 1)$ vertices from each copy of $C_n (n \neq 3, n \neq 5)$ other than the vertex c for the minimum cardinality. Thus, $\gamma_g(G) = (\lceil n/3 \rceil - 1)r + 1$ for $n \neq 3, n \neq 5$.

For $n = 5$, we argue that any two nonadjacent vertices of C_5 dominate C_5. If G is a one-point union of r- copies of C_5, then the vertex c, being a common vertex of G, and a vertex v nonadjacent to c from each r-copy of C_5 is the minimum vertices which dominate G. These vertices also dominate $\overline{G}$ because a vertex v nonadjacent to c from any two copies of C_5 are enough to dominate $\overline{G}$. Therefore, $\gamma_g(G) = r + 1 = (1)r + 1 = (\lceil 5/3 \rceil - 1)r + 1$.

Thus, $\gamma_g(G) = (\lceil n/3 \rceil - 1)r + 1$ for $n > 3$.

The set, being a global dominating set with minimum cardinality, is an MDS with minimum cardinality. that is, the set is also a γ-set of G. Thus, $\gamma(G) = \gamma_g(G) = (\lceil n/3 \rceil - 1)r + 1$ for $n > 3$ as required. □

Theorem 2.3. *Let G be a path union of finite number of copies of the graph of order n having at least one vertex of degree $(n-1)$. Then, S is a γ-set of G if and only if S is a global dominating set of G. Also, $\gamma_g(G) = \gamma(G)$.*

Proof. Let $v_1, v_2, v_3, \ldots, v_r$ be the vertices of degree $(n-1)$ from each $G_1, G_2, G_3, \ldots, G_r$, respectively.

Suppose that S is a γ-set of G. Now from the adjacency of the vertices in G, it is easy to observe that $S = \{v_1, v_2, v_3, \ldots, v_r\}$ is the only γ-set of G except for the path union of W_4.

In case of W_4, a set comprises of one vertex of degree $(n-1)$ from each $G_1, G_2, G_3, \ldots, G_r$ or a vertex from each $G_1, G_2, G_3, \ldots, G_r$ will be a γ-set. Because each vertex v_i $(i = 1, 2, \ldots, r)$ of G_i dominates all the vertices of $\overline{G}$ except its adjacent vertices and these adjacent vertices are dominated by v_{i-1} or v_{i+1}, consequently any two adjacent vertices v_i are enough to dominate all the vertices of $\overline{G}$. Hence, S is a dominating set of $\overline{G}$.

Now, S being a γ-set of G, it is also a dominating set of G. Thus, S is a dominating set of both G and $\overline{G}$, which implies that S is a global dominating set of G.

Conversely, suppose that S is a global dominating set of G, that is, S is a dominating set of both G and $\overline{G}$.

Consider $S = \{v_1, v_2, v_3, \ldots, v_r\}$. On removing any of the vertex v_i $(i = 1, 2, \ldots, r)$ from S, the set $S - \{v_i\}$ will not dominate $N[v_i]$. Therefore, $S - \{v_i\}$ is not a dominating set of G. This implies that S is an MDS of G. Each v_i being a vertex of degree $(n-1)$, clearly S is of minimum cardinality. Hence, S is a γ-set of G.

Thus, we have proved that S is a γ-set of G if and only if it is a global dominating set of G.

Since S being a γ-set of G, it is an MDS with minimum cardinality which implies that $\gamma_g(G) = \gamma(G)$ as required. □

Theorem 2.4. *Let $G_1 = \langle G^{(1)} : G^{(2)} \rangle$. Then,*

(i) $\gamma_g(G_1) = \gamma(G_1)$ *where G can be W_4, K_n and*

(ii) $\gamma_g(G_1) = \gamma(G_1) + 1$ *where G can be $W_n (n > 4)$, S_n, F_n, $K_{1,n}$.*

Proof. Let $G^{(1)}$ and $G^{(2)}$ denote the two copies of the graph G. Let v_1 and v_2 be the vertices of degree $(n-1)$ in $G^{(1)}$ and $G^{(2)}$ respectively. Suppose that x is the vertex adjacent to both v_1 and v_2.

Consider $G_1 = \langle G^{(1)} : G^{(2)} \rangle$.

(i) Let G be either W_4 or K_n. Then in order to dominate vertex x in G_1, either v_1 or v_2 is required. If $v_1 \in S$, then v_1 and any other vertex v from $G^{(2)}$ will dominate G_1. Clearly, $S = \{v_1, v\}$ is a γ-set of G_1. Hence, $\gamma(G_1) = 2$.

For $v \neq v_2$, S is also a dominating set of $\overline{G_1}$ which implies that S is a global dominating set of G_1 and is of minimum cardinality. Consequently, $\gamma_g(G_1) = 2$. Thus, $\gamma_g(G_1) = \gamma(G_1)$ as required.

(ii) Let G be $W_n(n > 4)$, S_n, F_n, or $K_{1,n}$. Since v_1 and v_2 are the vertices of degree $(n-1)$ and they are adjacent to the vertex x, $S = \{v_1, v_2\}$ is the γ-set of G_1 which implies that $\gamma(G_1) = 2$.

$S = \{v_1, v_2\}$ being a γ-set and x is adjacent to both v_1 and v_2, $S_g = \{v_1, v_2, x\}$ is a global dominating set of G_1 which is of minimum cardinality. This implies that $\gamma_g(G_1) = 3$.

Hence, $\gamma_g(G_1) = \gamma(G_1) + 1$ as required. □

Theorem 2.5. *If $G_1 = \langle G^{(1)} : G^{(2)} : \cdots : G^{(r)} \rangle$ with $r > 2$ where G can be W_n, S_n, F_n, $K_{1,n}$, or K_n, then $\gamma_g(G_1) = \gamma(G_1)$.*

Proof. Let $G^{(j)}$ $(j = 1, 2, \ldots, r)$ denote r-copies of graph G and let v_j be the vertex of degree $(n-1)$ in G. Suppose that $x_1, x_2, x_3, \ldots, x_{r-1}$ are the vertices such that v_{p-1} and v_p are adjacent to x_{p-1} where $2 \leq p \leq r$.

Consider $G_1 = \langle G^{(1)} : G^{(2)} : \cdots : G^{(r)} \rangle$. Construct a dominating set S as follows,

Select a set of vertices v_j $(j = 1, 2, \ldots, r)$ of degree $(n-1)$ from each $G^{(1)}, G^{(2)}, \ldots, G^{(r)}$, respectively where v_j dominates $N[v_j]$. Then, $S = \{v_1, v_2, v_3, \ldots, v_r\}$ is a dominating set of G_1.

If we remove any of v_j from S, then $S - \{v_j\}$ will not dominate $N[v_j]$ which implies that $S - \{v_j\}$ is not a dominating set of G_1. Consequently, S is a minimal dominating set. Each v_j being a vertex of degree $(n-1)$, clearly S is an MDS with minimum cardinality r. Therefore, $\gamma(G_1) = r$.

We claim that S is also a dominating set of $\overline{G_1}$ because any two vertices v_i and v_j with $d(v_i, v_j) > 2$ are enough to dominate $\overline{G_1}$. Hence, S is a dominating set of both G_1 and $\overline{G_1}$, which implies that $\gamma_g(G_1) = r$.

Thus, $\gamma_g(G_1) = \gamma(G_1)$ as required. □

The following Theorem 2.6 can be proved by the arguments analogous to the above Theorem 2.5.

Theorem 2.6. *If* $G_1 = \langle G^{(1)} \blacktriangle G^{(2)} \blacktriangle \cdots \blacktriangle G^{(r)} \rangle$ *with* $r > 2$ *where* G *can be* W_n, S_n, F_n, $K_{1,n}$, *or* K_n, *then* $\gamma_g(G_1) = \gamma(G_1)$.

On considering the two copies in Theorem 2.6, we have the following result.

Theorem 2.7. *If* $G_1 = \langle G^{(1)} \blacktriangle G^{(2)} \rangle$ *where* G *can be* W_n, S_n, F_n, $K_{1,n}$, *or* K_n, *then* $\gamma_g(G_1) = \gamma(G_1)+1$.

Proof. Let $G^{(1)}$ and $G^{(2)}$ denote the two copies of the graph G. Let v_1 and v_2 be the vertices of degree $(n-1)$ in $G^{(1)}$ and $G^{(2)}$, respectively and let x be the vertex adjacent to v_1 and v_2 both.

Consider the graph $G_1 = \langle G^{(1)} \blacktriangle G^{(2)} \rangle$. Then as v_1 and v_2 both being the vertices of degree $(n-1)$ in $G^{(1)}$ and $G^{(2)}$ respectively, the set $S = \{v_1, v_2\}$ is an MDS with minimum cardinality, that is, S is a γ-set of G_1 which implies that $\gamma(G_1) = 2$.

Now, S is not a dominating set of $\overline{G_1}$ because the vertex x is adjacent to both v_1 and v_2. But $S = \{v_1, v_2, x\}$ is a dominating set of $\overline{G_1}$ which is also a dominating set of G_1, that is, S is a global dominating set of G_1. To dominate the vertex x in G_1, either v_1 or v_2 is required in S which is also in a global dominating set of G_1. To dominate $\overline{G_1}$, if $v_1 \in S$, then a vertex will be adjacent to both v_1 and any vertex from $G^{(2)}$. Therefore, at least three vertices are required to dominate $\overline{G_1}$. Consequently, $S = \{v_1, v_2, x\}$ is a global dominating set of G_1 with minimum cardinality, that is, $\gamma_g(G_1) = 3$.

Hence, $\gamma_g(G_1) = \gamma(G_1) + 1$ as required. □

3. Concluding Remarks

We have investigated some results corresponding to the concept of global domination. We have taken up the issue to obtain the global domination number for the larger graph obtained from the given graph. We have established the relations between the domination number and the global domination number for the graphs obtained by some graph operations, namely, one-point union and path union of graphs. More exploration is possible in the context of different graph families.

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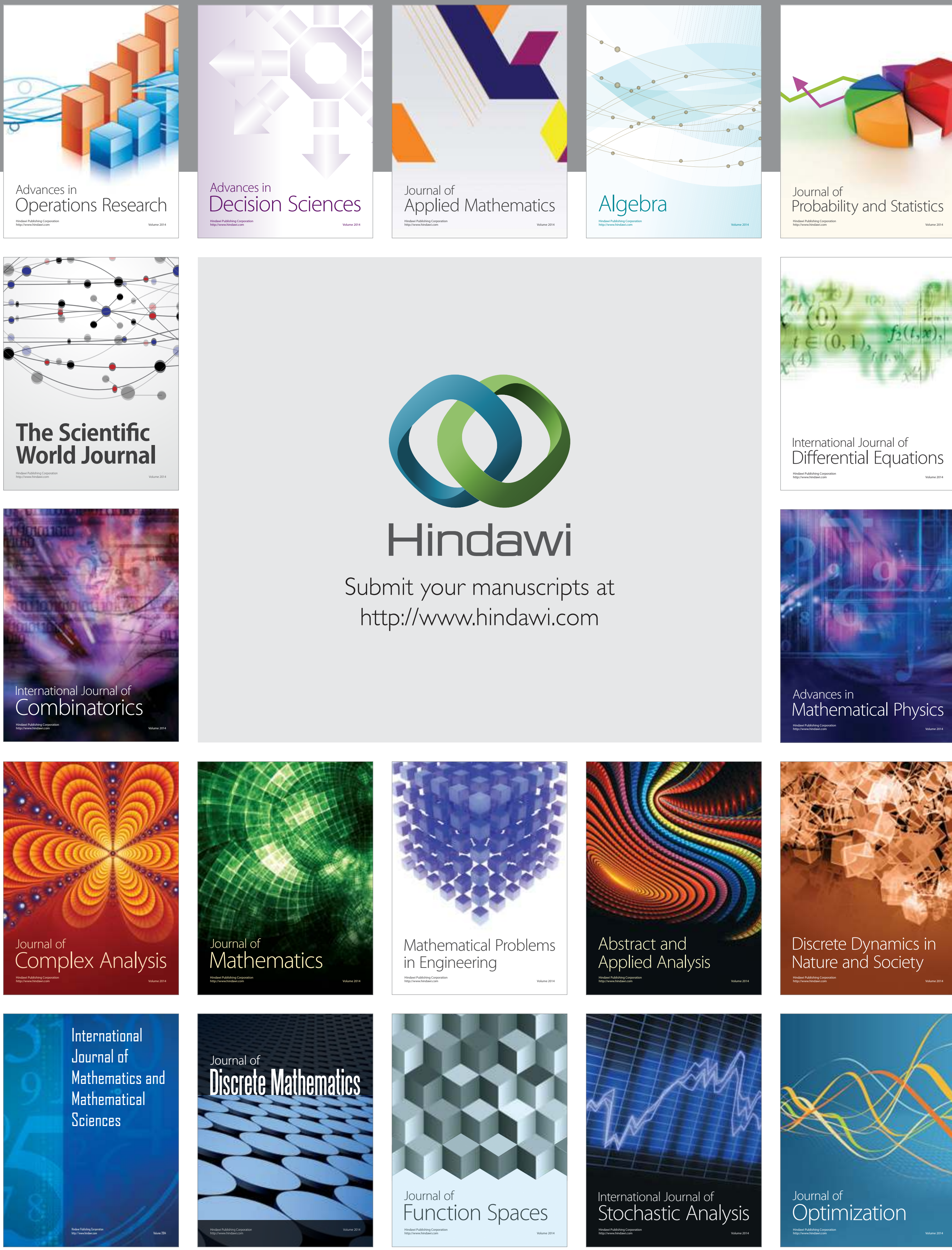
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