

# Some New Results on the Number of Paths

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Received 23 April 2015; accepted 1 June 2015; published 5 June 2015

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## Abstract

Khidr and El-Desouky [1] derived a symmetric sum involving the Stirling numbers of the first kind through the process of counting the number of paths along a rectangular array  $n \times m$  denoted by  $A_{nm}$ . We investigate the generating function for the general case and hence some special cases as well. The probability function of the number of paths along  $A_{nm}$  is obtained. Moreover, the moment generating function of the random variable  $X$  and hence the mean and variance are obtained. Finally, some applications are introduced.

## Keywords

Stirling Numbers, Generating Function, Moment Generating Function, Comtet Numbers, Maple Program

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## 1. Introduction

Let  $\{a_i\}_{i=1}^n$  be a sequence of natural numbers  $0 \leq a_i \leq n, i = 1, \dots, n$ , and  $A_{nm}$  be an  $n \times m$  array associated with this sequence, whose entries  $\alpha_{ij} = 0, 1$  such that

$$\sum_{j=1}^m \alpha_{ij} = a_i, i = 1, 2, \dots, n, j = 1, \dots, m.$$

The path of order  $k$  along  $A_{nm}$  is defined to be a sequence of entries  $\alpha_{1j_1}, \dots, \alpha_{nj_n}$  as follows

$$\sum_{i=1}^n \alpha_{ij_i} = k, k = 0, 1, \dots, n, j_i = 1, \dots, m, i = 1, \dots, n.$$

The number of paths of order  $k$  will be denoted by

$$g^k(a_1, \dots, a_n; m), k = 0, 1, \dots, n.$$

By neglecting the last row in  $A_{nm}$  and then reconsidering it, we get the recurrence

$$g^k(a_1, \dots, a_n; m) = a_n g^{k-1}(a_1, \dots, a_{n-1}; m) + (m - a_n) g^k(a_1, \dots, a_{n-1}; m). \tag{1}$$

When  $a_i = a$ ,  $a$  is a constant,  $i = 1, \dots, n$ , then

$$g^k(a, \dots, a; m) \cdots g^k(\bar{a}, m) = \binom{n}{k} a^k (m - a)^{n-k}, \tag{2}$$

and

$$\sum_{k=0}^n g^k(\bar{a}, m) = m^n.$$

Khidr and El-Desouky [1] proved that, when  $m = n$

$$g^k(a_1, a_2, \dots, a_n; n) = (-1)^k \sum_{l=k}^n \binom{l}{k} s_{\bar{a}}(n, n-l) n^{n-l}, \tag{3}$$

where  $s_{\bar{a}}(n, i)$  are the generalized Stirling numbers of the first kind associated with the sequence of real numbers  $\bar{a} := (a_1, a_2, \dots, a_n)$ , defined by [1]-[6],

$$(x - a_1)(x - a_2) \cdots (x - a_n) = \sum_{i=0}^n s_{\bar{a}}(n, i) x^i. \tag{4}$$

These numbers satisfy the recurrence relation

$$s_{\bar{a}}(n, i) = s_{\bar{a}}(n-1, i-1) - a_n s_{\bar{a}}(n-1, i), s_{\bar{a}}(n, i) = 0 \text{ for } i > n. \tag{5}$$

And

$$\sum_{k=0}^n g^k(\bar{a}, n) = n^n.$$

Moreover, they introduced a special case of (3), when  $a_i = i, i = 1, 2, \dots, n$ , then the number of paths of order  $k$ ,  $g^k(1, \dots, n; n)$  is denoted by  $g_n^k(n)$ ; and proved that

$$g_n^k(n) = (-1)^k \sum_{l=k}^n \binom{l}{k} s(n+1, n+1-l) n^{n-l}, \tag{6}$$

where  $s(n, l)$  are the Stirling numbers of the first kind defined by, see [2] [3]

$$x(x-1) \cdots (x-n+1) = \sum_{l=0}^n s(n, l) x^l.$$

Also the generating function for  $g_n^k(n)$  is given by

$$G(t) = \prod_{i=1}^n [n - j(1-t)]. \tag{7}$$

In this article, in Section 2, we derive a generalization of some results given in [1], for the number of paths of order  $k$ ,  $g^k(a_1, \dots, a_n; m)$ , when  $m \neq n$ . The generating function of  $g^k(a_1, \dots, a_n; m)$  is given. In Section 3, we find the probability distribution for  $g^k(a_1, \dots, a_n; m)$  and study some of their properties. The moment generating function, skewness and kurtosis for  $g^k(a_1, \dots, a_n; m)$  are investigated. Moreover special case and numerical results are given in Section 4.

## 2. Main Results

**Theorem 1.** The number of paths of order  $k$  is given by

$$g^k(a_1, a_2, \dots, a_n; m) = (-1)^k \sum_{i=k}^n \binom{i}{k} s_{\bar{a}}(n, n-i) m^{n-i}. \quad (8)$$

Proof. Using (5) in (8), we get

$$\begin{aligned} & g^k(a_1, a_2, \dots, a_n; m) \\ &= (-1)^k \sum_{i=k}^n \binom{i}{k} \{s_{\bar{a}}(n-1, n-i-1) - a_n s_{\bar{a}}(n-1, n-i)\} m^{n-i} \\ &= (-1)^k \sum_{i=k}^n \binom{i}{k} s_{\bar{a}}(n-1, n-i-1) m^{n-i} - (-1)^k \sum_{i=k}^n a_n \left\{ \binom{i-1}{k-1} + \binom{i-1}{k} \right\} s_{\bar{a}}(n-1, n-i) m^{n-i} \\ &= (-1)^k \sum_{i=k}^n \binom{i}{k} s_{\bar{a}}(n-1, n-i-1) m^{n-i} - \left\{ (-1)^k a_n \sum_{i=k+1}^n \binom{i-1}{k} \right. \\ &\quad \left. \times s_{\bar{a}}(n-1, n-i) m^{n-i} + (-1)^k a_n \sum_{i=k}^n \binom{i-1}{k-1} s_{\bar{a}}(n-1, n-i) m^{n-i} \right\} \\ &= (-1)^k \sum_{i=k}^n \binom{i}{k} s_{\bar{a}}(n-1, n-i-1) m^{n-i} - (-1)^k a_n \sum_{i=k}^n \binom{i}{k} s_{\bar{a}}(n-1, n-i-1) \\ &\quad \times m^{n-i-1} - (-1)^k a_n \sum_{i=k-1}^n \binom{i}{k-1} s_{\bar{a}}(n-1, n-i-1) m^{n-i-1} \\ &= (m - a_n) g^k(a_1, \dots, a_{n-1}; m) + a_n g^{k-1}(a_1, \dots, a_{n-1}; m). \end{aligned}$$

This by virtue of (1) completes the proof of (8).

**Theorem 2.** The generating function of the number of paths of order  $k$  is given by

$$G^{(n)}(n; m) = \prod_{i=1}^n (m - a_i (1-t)). \quad (9)$$

Proof. Let the generating function of the number of paths of order  $k$  be denoted by

$$G^{(n)}(n; m) = \sum_{k=0}^n g^k(a_1, \dots, a_n; m) t^k. \quad (10)$$

Using (1), we obtain

$$\begin{aligned} G^{(n)}(n; m) &= \sum_{k=1}^n a_n g^{k-1}(n-1; m) t^k + (m - a_n) \sum_{k=0}^n g^k(n-1; m) t^k, \\ &= a_n t G^{(n-1)}(n-1; m) + (m - a_n) G^{(n-1)}(n-2; m) \\ &= (m - a_n + a_n t) G^{(n-1)}(n-1; m), \end{aligned}$$

and hence we get

$$G^{(n)}(n; m) = \prod_{i=1}^n (m - a_i (1-t)) G^{(0)}(0; m),$$

where  $G^{(0)}(0; m) = g^0(0; m) = 1$ . This completes the proof.

From (9), we get

$$G^{(n)}(n; m) = a_1 \left( t - \frac{a_1 - m}{a_1} \right) a_2 \left( t - \frac{a_2 - m}{a_2} \right) \cdots a_n \left( t - \frac{a_n - m}{a_n} \right) = \prod_{i=1}^n a_i \sum_{k=0}^n s_{\bar{\alpha}}(n, k) t^k,$$

where  $\alpha_i = -\left( \frac{m - a_i}{a_i} \right)$ ,  $\bar{\alpha} := (\alpha_1, \alpha_2, \dots, \alpha_n)$  and hence we have

$$g^k(a_1, \dots, a_n; m) = \prod_{i=1}^n a_i s_{\bar{\alpha}}(n, k), \quad (11)$$

where  $\alpha_i = -\left(\frac{m-a_i}{a_i}\right), i = 1, \dots, n.$

For the special case  $a_i = i, m = n$ , we get

$$g_n^k(n) = n! s_{\bar{\alpha}}(n, k), \quad (12)$$

where  $\alpha_i = -\left(\frac{n-i}{i}\right), i = 1, \dots, n.$

From (6) and (12), we have the identity

$$(-1)^k \sum_{m=k}^n \binom{m}{k} s(n+1, n+1-m) n^{n-m} = n! s_{\bar{\alpha}}(n, k), \quad (13)$$

where  $\alpha_i = -\left(\frac{n-i}{i}\right), i = 1, \dots, n.$

### 3. Some Applications

Let  $X$ , be the number of paths along  $A_{nm}$ , then by virtue of (8) we have

$$\begin{aligned} P(X = k) &= (-1)^k \sum_{i=k}^n \binom{i}{k} s_{\bar{\alpha}}(n, n-i) \frac{m^{n-i}}{m^n} \\ &= (-1)^k \sum_{i=k}^n \binom{i}{k} \frac{s_{\bar{\alpha}}(n, n-i)}{m^i}, \quad k = 0, 1, \dots, n. \end{aligned} \quad (14)$$

On the other hand the moment generating function of the random variable  $X$  denoted by  $M_X(t)$ , is given by the following theorem.

**Theorem 3.** The moment generating function of  $X$ , is given by

$$M_X(t) = \sum_{n=0}^{\infty} \sum_{i=0}^n \frac{(-1)^i i!}{n! m^i} S(n, i) s_{\bar{\alpha}}(n, n-i) t^n. \quad (15)$$

Proof. We begin by the definition of the moment generating function as follows.

$$\begin{aligned} M_X(t) &= \sum_{k=0}^n e^{kt} f(k) = \sum_{k=0}^n e^{kt} (-1)^k \sum_{i=k}^n \binom{i}{k} \frac{s_{\bar{\alpha}}(n, n-i)}{m^i} \\ &= \sum_{i=0}^n \sum_{k=0}^i (-1)^k \binom{i}{k} e^{kt} \frac{s_{\bar{\alpha}}(n, n-i)}{m^i} = \sum_{i=0}^n (1-e^t)^i \frac{s_{\bar{\alpha}}(n, n-i)}{m^i} \\ &= \sum_{i=0}^n \sum_{n=i}^n \frac{(-1)^i i!}{n! m^i} S(n, i) s_{\bar{\alpha}}(n, n-i) t^n \\ &= \sum_{n=0}^{\infty} \sum_{i=0}^n \frac{(-1)^i i!}{n! m^i} S(n, i) s_{\bar{\alpha}}(n, n-i) t^n, \end{aligned}$$

This completes the proof.

**Corollary 1.** The  $j$ th moments of  $X$  is

$$E[X^j] = \sum_{i=0}^n \frac{(-1)^i i!}{m^i} S(j, i) s_{\bar{\alpha}}(n, n-i). \quad (16)$$

Proof. The  $j$ th moments can be obtained from the moment generating function,  $M_X(t)$  where

$$E[X^j] = \left[ \frac{d^n}{dt^n} M_X(t) \right]_{t=0}.$$

This completes the proof.

Then from (16), we can calculate the mean and variance for the random variable  $X$  as follows.

$$\mu = E(X) = \sum_{i=0}^n \frac{(-1)^i i!}{m^i} S(1, i) s_{\bar{a}}(n, n-i) = -\frac{s_{\bar{a}}(n, n-1)}{m}. \quad (17)$$

$$\begin{aligned} E(X^2) &= \sum_{i=0}^n \frac{(-1)^i i!}{m^i} S(2, i) s_{\bar{a}}(n, n-i) \\ &= -\frac{S(2,1) s_{\bar{a}}(n, n-1)}{m} + \frac{2! S(2,2) s_{\bar{a}}(n, n-2)}{m^2} \\ &= -\frac{s_{\bar{a}}(n, n-1)}{m} + \frac{2s_{\bar{a}}(n, n-2)}{m^2}, \end{aligned} \quad (18)$$

hence the variance is given by

$$\begin{aligned} \text{Var}(X) &= E[X^2] - \mu^2 = -\frac{s_{\bar{a}}(n, n-1)}{m} + \frac{2s_{\bar{a}}(n, n-2)}{m^2} - \left( -\frac{s_{\bar{a}}(n, n-1)}{m} \right)^2 \\ &= -\frac{s_{\bar{a}}(n, n-1)}{m^2} [s_{\bar{a}}(n, n-1) + m] + \frac{2s_{\bar{a}}(n, n-2)}{m^2}. \end{aligned} \quad (19)$$

**Corollary 2.** The Skewness and kurtosis for the random variable  $X$  are given by

$$\text{Skewness} = \frac{\mu_3}{\mu_2^{3/2}}, \text{ kurtosis} = \frac{\mu_4}{\mu_2^2} \quad (20)$$

where

$$\begin{aligned} \mu_2 &= -\frac{s_{\bar{a}}(n, n-1)}{m} + \frac{2s_{\bar{a}}(n, n-2)}{m^2} - \frac{s_{\bar{a}}^2(n, n-1)}{m^2}, \\ \mu_3 &= -\frac{s_{\bar{a}}(n, n-1)}{m} - \frac{3s_{\bar{a}}^2(n, n-1)}{m^2} - \frac{2s_{\bar{a}}^3(n, n-1)}{m^3} + \frac{6s_{\bar{a}}(n, n-2)}{m^2} \\ &\quad + \frac{6s_{\bar{a}}(n, n-2)s_{\bar{a}}(n, n-1)}{m^3} - \frac{6s_{\bar{a}}(n, n-3)}{m^3}, \\ \mu_4 &= -\frac{s_{\bar{a}}(n, n-1)}{m} - \frac{4s_{\bar{a}}^2(n, n-1)}{m^2} - \frac{6s_{\bar{a}}^3(n, n-1)}{m^3} - \frac{3s_{\bar{a}}^4(n, n-1)}{m^4} + \frac{14s_{\bar{a}}(n, n-2)}{m^2} \\ &\quad + \frac{24s_{\bar{a}}(n, n-2)s_{\bar{a}}(n, n-1)}{m^3} + \frac{12s_{\bar{a}}(n, n-2)s_{\bar{a}}^2(n, n-1)}{m^4} - \frac{36s_{\bar{a}}(n, n-3)}{m^3} \\ &\quad - \frac{24s_{\bar{a}}(n, n-3)s_{\bar{a}}(n, n-1)}{m^4} + \frac{24s_{\bar{a}}(n, n-4)}{m^4}. \end{aligned}$$

Proof. We can find the  $j$ th moments about the mean by using

$$\mu_j = E[(X - E[X])^j] = \sum_{k=0}^j \binom{j}{k} (-1)^k (E[X])^k E[X^{j-k}]. \quad (21)$$

From (16) and (21), we can find the moments  $\mu_2, \mu_3, \mu_4$  about mean which can be used to calculate the skewness and kurtosis.

**Special Case:**

If  $a_i = i, m = n$ , from (14), we have

$$P(X = k) = (-1)^k \sum_{i=k}^n \binom{i}{k} \frac{s(n+1, n+1-i)}{n^i},$$

and from (16) the  $j$ th moments has the form

$$E[X^j] = \sum_{i=0}^n \frac{(-1)^i i! S(j, i) s(n+1, n+1-i)}{n^i},$$

and the mean is given by

$$\mu = -\frac{s(n+1, n)}{n} = \frac{n+1}{2}.$$

$$E[X^2] = -\frac{s(n+1, n)}{n} + \frac{2s(n+1, n-1)}{n^2},$$

the variance can be obtained as follows.

$$\begin{aligned} \text{Var}(X) &= E[X^2] - \mu^2 = -\frac{s(n+1, n)}{n} + \frac{2s(n+1, n-1)}{n^2} - \left(-\frac{s(n+1, n)}{n}\right)^2 \\ &= -\frac{s(n+1, n)}{n^2} [s(n+1, n) + n] + \frac{2s(n+1, n-1)}{n^2} \\ &= \frac{(n-1)(n+1)}{6n}. \end{aligned}$$

where we used  $s(n, n-1) = -\binom{n}{2}$ ,  $s(n, n-2) = \frac{1}{4}(3n-1)\binom{n}{3}$ , see [3].

#### 4. Numerical Results

Setting  $n = 5, m = 6, \alpha_i = i, i = 1, 2, \dots, n$ . Therefore the numerical values of  $s_{\bar{\alpha}}(n, k)$ , are reduced to  $s(n, k)$ , see [4] [5].

From Equation (14), we can find the probability distribution of the number of paths  $X$  along  $A_{5 \times 6}$  as follows

$X$	0	1	2	3	4	5	Total
$p(x)$	0.09259	0.34259	0.38889	0.15741	0.01852	0	1

From (16), we can compute the 4th moments as follows.

$j$	1	2	3	4
$E[X^j]$	1.66667	3.61111	8.88889	24.0556

The 4<sup>th</sup> moments about mean can be obtained as

$j$	1	2	3	4
$\mu_j$	0	0.83333	0.09259	1.83333

The values of mean and variance can be obtained from (17) and (19) as follows.

$$\mu = 1.66667, \text{Var}(X) = 0.83333.$$

The skewness and kurtosis, respectively can be obtained from (20) as follows.

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skewness = 0.12172, kurtosis = 2.64.

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