SOME NEW TEST CRITERIA IN MULTIVARIATE ANALYSIS

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1. Summary. Three new test criteria are proposed for overall tests of hypotheses in multivariate analysis. They are based on the characteristic roots of certain matrices obtained from the product moment matrices of samples drawn from multivariate normal populations. The approximate distributions of the statistics involved in the tests are found as Type I or Type II Beta distributions.

2. Introduction. In multivariate analysis, we generally wish to test three hypotheses, namely

(I) that of equality of the dispersion matrices of two p-variate normal populations,

(II) that of equality of the p-dimensional mean vectors for l p-variate normal populations (which is mathematically identical with the general problem of multivariate analysis of variance of means); and

(III) that of independence between a p-set and a q-set of variates in a $(p + q)$-variate normal population, with $p \leqq q$.

All tests proposed so far for these hypotheses have been shown to depend, when the hypotheses to be tested are true, only on the characteristic roots of matrices based on sample observations. For example, in case (I), all the tests proposed so far are based on the characteristic roots of the matrix $S_1(S_1 + S_2)^{-1}$, where S_1 and S_2 denote the usual sum of product (S.P.) matrices and where both are almost everywhere positive definite (a.e. p.d.). Thus $S_1(S_1 + S_2)^{-1}$ is a.e., p.d., whence it follows that all the p characteristic roots are greater than zero and less than unity. In case (II), the matrix is $S^*(S^* + S)^{-1}$, where S^* denotes the "between" S.P. matrix of means weighted by the sample sizes and S denotes the "within" S.P. matrix (pooled from the S.P. matrices of l samples). Then S is a.e. p.d., and S^* is at least positive semidefinite of rank $s = \min (p, \ l - 1)$. Thus, a.e., s of the characteristic roots are greater than zero and less than unity and the $p - s$ remaining roots are zero. In case (III), the matrix is $S_{11}^{-1}S_{12}S_{22}^{-1}S_{12}'$, where S_{11} is the S.P. matrix of the sample of observations on the p-set of variates, S_{22} that on the q-set, and S_{12}, the S.P. matrix between the observations on the p-set and those on the q-set. If $p \leqq q$ and $p + q < k$, where k is the sample size, then a.e. the p characteristic roots of this matrix are greater than zero and less than unity.

In each case, if the hypothesis to be tested is true, the $s \leqq p$ nonzero roots θ_i, where $0 < \theta_1 \leqq \theta_2 \leqq \cdots \leqq \theta_s < 1$, have the same joint distribution, the form of which was given by Roy [7], Hsu [3], and Fisher [1]. The distribution

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can be written in the form

$$(1)\quad p(\theta_1, \cdots, \theta_s) = C(s, m, n,) \prod_{i=1}^{s} \theta_i^m (1 - \theta_i)^n \prod_{i>j} (\theta_i - \theta_j),$$

$$0 < \theta_1 \leqq \cdots \leqq \theta_s < 1,$$

where

$$C(s, m, n) = \frac{\pi^{s/2} \prod_{i=1}^{s} \Gamma\frac{1}{2}(2m + 2n + s + i + 2)}{\prod_{i=1}^{s} \Gamma\frac{1}{2}(2m + i + 1)\, \Gamma\frac{1}{2}(2n + i + 1)\, \Gamma\frac{1}{2}i}.$$

Here m and n are to be interpreted differently for the different situations. For example, in case (I), with n_1 and n_2 as the sample sizes,

$$(2)\quad m = \tfrac{1}{2}(n_1 - p - 2), \qquad n = \tfrac{1}{2}(n_2 - p - 2).$$

In case (II), with N the total of the sizes of l samples,

$$(3)\quad m = \tfrac{1}{2}(|l - p - 1| - 1), \qquad n = \tfrac{1}{2}(N - l - p - 1).$$

In case (III),

$$(4)\quad m = \tfrac{1}{2}(q - p - 1), \qquad n = \tfrac{1}{2}(k - p - q - 2).$$

3. Current test criteria. The test criteria previously proposed for tests of hypotheses in multivariate analysis are mainly

Hotelling's $T_0^2 = c \sum_1^s \lambda_i$, where $\lambda_i = \theta_i / (1 - \theta_i)$ and c is a constant depending on the degrees of freedom of a matrix from which the characteristic roots were obtained [2];

Roy's [8], [9], [10] criteria of the largest root $\theta_s(\lambda_s)$ and the smallest root $\theta_1(\lambda_1)$;

Wilks' [11] criterion, $\Lambda = \prod_1^s (1 - \theta_i)$; and

$V^{(s)} = \sum_1^s \theta_i$, [4], [5].

It is well known (see e.g., [9], [10]) that (i) the sample roots are invariant under certain classes of linear transformations, different for the different situations discussed in the Introduction, and (ii) the joint distribution of these roots, when the hypothesis to be tested is true, involves as parameters only the so-called degrees of freedom, and, in addition, only a set of population roots (with a strong physical import) when the hypothesis to be tested is not true.

It is likely that any statistic (and a test based on it), in order to possess these desirable properties, must be a function of these sample roots. All the tests proposed so far, including the ones proposed in this paper, are, in fact, based on functions of these roots and therefore have all the above properties.

The choice of any specific function of the roots, as a basis for test criteria, has so far been made on additional considerations which are heuristic. The ultimate justification is found in properties like (iii) usability in the sense of availability of large or small sample distribution of the statistic involved (on the null hypothesis) and (iv) some good operating characteristics of the test. Such characteristics are, for example, (iva) unbiassedness and (ivb) even something stronger, namely a power which is a monotonically increasing function of each of the deviation parameters, (ivc) some good and convenient lower bound to the power, (ivd) possibility of getting, by inversion, suitable confidence interval statements, and (ive) admissibility.

Hotelling's T_0^2 has so far been studied and shown to be good under (i)–(iii). Wilks' criterion has, as of moment, been shown to be good under (i)–(iii) and also under (iva)–(ivb), in so far as the criterion is to be used for situations (II) and (III) discussed in the Introduction. Roy's criteria have so far been shown to be good under (i)–(iii) and (iva)–(ivd). The tests proposed in the present paper are also based on heuristic arguments, and are shown to be good under (i)–(iii). The further study of these tests in terms of (iva)–(ive) remains to be made.

4. Three new test criteria. Wilks' criterion is the sth power of a geometric mean, while T_0^2 and $V^{(s)}$ are s times the arithmetic means. This suggests the harmonic mean as another possibility. The following three criteria based on this statistic are offered.

$$H^{(s)} = s\left\{\sum_{i=1}^{s} (1-\theta_i)^{-1}\right\}^{-1}, \quad R^{(s)} = s\left\{\sum_{i=1}^{s} \theta_i^{-1}\right\}^{-1}, \quad T^{(s)} = s\left\{\sum_{i=1}^{s} \lambda_i^{-1}\right\}^{-1}.$$

Before proceeding to derive the approximate distributions of these criteria, we may derive the approximate distribution of the criterion defined by $W^{(s)} = 1 - V^{(s)}/s$, the tests based on which are exactly identical with those of $V^{(s)}$. It has been shown [4], [5] that the distribution of $V^{(s)}$ can be approximated by a Type I Beta distribution given by

$$(5) \qquad p(V^{(s)}) = C\{V^{(s)}\}^{s(2m+s+1)/2-1}(1 - V^{(s)}/s)^{s(2n+s+1)/2-1}, \qquad 0 < V^{(s)} < s,$$

where $C = 1 / s^{s(2m+s+1)/2}\beta\{\frac{1}{2}s(2m+s+1), \ \frac{1}{2}s(2n+s+1)\}$. Hence it is easily seen that $W^{(s)}$ has the approximate distribution

$$(6) \qquad p(W^{(s)}) = C'\{W^{(s)}\}^{s(2n+s+1)/2-1}(1 - W^{(s)})^{s(2m+s+1)/2-1}, \qquad 0 < W^{(s)} < 1,$$

where $C' = 1/\beta\{\frac{1}{2}s(2n+s+1), \ \frac{1}{2}s(2m+s+1)\}$. The approximate distribution (5) has been shown, [5], to be satisfactory for practical use for $m + n \geqq 30$ when $s = 2$; when s increases by one unit, $m + n$ must increase by 10 to give satisfactory results. Hence under these conditions (6) is also valid.

Now we consider the criterion $H^{(s)}$. If we put $U^{(s)} = T_0^2/c$, the approximate distribution of $U^{(s)}$ has been shown [5], [6] to be a Type II Beta distribution of the form

$$(7) \qquad p(U^{(s)}) = K\{U^{(s)}\}^{s(2m+s+1)/2-1} / [1 + U^{(s)}/s]^{s(2m+2n+s+1)/2+1}, \qquad 0 < U^{(s)} < \infty,$$

where $K = 1/s^{s(2m+s+1)/2}\beta\{\frac{1}{2}s(2m + s + 1), \ sn + 1\}$. It may be seen that

$$U^{(s)} = \sum_{i=1}^{s} \lambda_i = \sum_{i=1}^{s} \theta_i/(1 - \theta_i) = \sum_{i=1}^{s} (1 - \theta_i)^{-1} - s. \tag{8}$$

Hence $H^{(s)} = (1 + U^{(s)}/s)^{-1}$, and use of this transformation in (7) gives

$$p(H^{(s)}) = \{H^{(s)}\}^{sn}(1 - H^{(s)})^{s(2m+s+1)/2-1} / \beta\{sn + 1, \ \tfrac{1}{2}s(2m + s + 1)\}, \quad 0 < H^{(s)} < 1. \tag{9}$$

For this approximation to be valid it has been shown [5] that $n + s$ must satisfy the conditions stated for $m + n$ in the case of $V^{(s)}$.

Again, to obtain the distribution of $R^{(s)}$, if we make a transformation $\theta_i' = 1 - \theta_i$, for $i = 1, 2, \cdots, s$, the joint distribution of θ_i''s resulting from (1) will have the same form as the joint distribution of θ_i's with m and n interchanged. Also we have $0 < \theta_s' \leqq \cdots \leqq \theta_1' < 1$. Hence if we start with the distribution of θ_i''s, we obtain for

$$U'^{(s)} = \sum_{i=1}^{s} \theta_i'/(1 - \theta_i')$$

the same form of approximate distribution as the one given by (7) with m and n interchanged. Now we consider

$$U'^{(s)} = \sum_{i=1}^{s} \theta_i^{-1} - s. \tag{10}$$

Hence $R^{(s)} = (1 + U'^{(s)}/s)^{-1}$, and with this transformation we have, as in (9) the approximate distribution of $R^{(s)}$ in the form

$$p(R^{(s)}) = \{R^{(s)}\}^{sm}(1 - R^{(s)})^{s(2n+s+1)/2-1} / \beta\{sm + 1, \ \tfrac{1}{2}s(2n + s + 1)\}, \quad 0 < R^{(s)} < 1. \tag{11}$$

Here $m + s$ must satisfy the conditions stated for $m + n$ in the case of $V^{(s)}$, for the validity of the approximation.

It is interesting to note the parallelisms between the approximate distributions of $V^{(s)}$ and $W^{(s)}$, and between those of $H^{(s)}$ and $R^{(s)}$. One is obtainable from the other by an interchange of m and n. It may be further noted that

$$U'^{(s)} = \sum_{i=1}^{s} \lambda_i^{-1} = \sum_{i=1}^{s} \theta_i^{-1} - s \tag{12}$$

Hence $T^{(s)} = R^{(s)} / (1 - R^{(s)})$, and from (11), the distribution of $T^{(s)}$ can be obtained in the form

$$p(T^{(s)}) = \frac{\{T^{(s)}\}^{sm}}{(1 + T^{(s)})^{s(2m+2n+s+1)/2+1}\beta\{sm + 1, \tfrac{1}{2}s(2n + s + 1)\}}, \quad 0 < T^{(s)} < \infty. \tag{13}$$

The approximate distribution $p(T^{(s)})$ also must satisfy the conditions stated for $p(R^{(s)})$ in (11). The condition that m must be large is more often suited to case (I) of the Introduction than to cases (II) and (III).

REFERENCES

[1] R. A. FISHER, "The sampling distribution of some statistics obtained from non-linear equations," *Ann. Eugenics*, Vol. 9 (1939), pp. 238–249.

[2] H. HOTELLING, "A generalized T test and measure of multivariate dispersion," *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, University of California Press, 1951, pp. 23–41.

[3] P. L. HSU, "On the distribution of roots of certain determinantal equations," *Ann. Eugenics*, Vol. 9 (1939), pp. 250–258.

[4] K. C. S. PILLAI, "On the distribution of the sum of the roots of a determinantal equation," (Abstract) *Ann. Math. Stat.*, Vol. 24 (1953), p. 495.

[5] K. C. S. PILLAI, "On some distribution problems in multivariate analysis," Mimeograph Series No. 88, Institute of Statistics, University of North Carolina 1954.

[6] K. C. S. PILLAI, "On the distribution of Hotelling's generalized T test," (Abstract) *Ann. Math. Stat.*, Vol. 25 (1954), p. 412.

[7] S. N. ROY, "p-statistics and some generalizations in analysis of variance appropriate to multivariate problems," *Sankhyā*, Vol. 4 (1939), pp. 381–396.

[8] S. N. ROY, "The individual sampling distributions of the maximum, the minimum and any intermediate of the p-statistics on the null hypothesis," *Sankhyā*, Vol. 7 (1945), pp. 133–158.

[9] S. N. ROY, "On a heuristic method of test construction and its use in multivariate analysis," *Ann. Math. Stat.*, Vol. 24 (1953), pp. 220–228.

[10] S. N. ROY AND R. C. BOSE, "Simultaneous confidence interval estimation," *Ann. Math. Stat.*, Vol. 24 (1953), pp. 513–536.

[11] S. S. WILKS, "Certain generalizations in analysis of variance," *Biometrika*, **Vol. 24** (1932), pp. 471–494.