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SOME NEW ZARISKI PAIRS OF SEXTIC CURVES

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Abstract. A topological invariant of reduced sextic curves with simple singularities is given. Twelve new Zariski pairs of sextic curves are determined. Each pair consists of two curves with non-isomorphic fundamental groups.

Introduction. Two plane curves C_1, C_2 of the same degree form a Zariski pair if C_1, C_2 have the same combinatorial data (cf. [1, 2]) and the pairs $(\boldsymbol{P}^2, C_1)$ and $(\boldsymbol{P}^2, C_2)$ are not homeomorphic. A brief account of the history of Zariski pairs can be found in [4]. It is remarkable that the degrees of all known Zariski pairs are at least six.

If the fundamental groups of the complements of C_1, C_2 in $\boldsymbol{P}^2$ are not isomorphic, then the Zariski pair C_1, C_2 is called a strong Zariski pair, otherwise it is a weak Zariski pair. The first strong Zariski pair was discovered in 1929 by Zariski in [16] which is the beginning of the long history of the study of this subject.

Let C be a reduced sextic curve with simple singularities only and let X be the K3 surface obtained from the double cover branched over C. Let N_C be the orthogonal complement in $H^2(X, \boldsymbol{Z})$ of the sublattice generated by all irreducible components of the inverse image of C in X. Shimada shows in [11] that N_C is a topological invariant of the pair $(\boldsymbol{P}^2, C)$. When C is a generic member of its equisingular deformation class, N_C is the transcendental lattice of the K3 surface X. Let γ_X be the discriminant form of the Picard lattice of X. For some special maximizing sextics there are two non-isomorphic positive definite lattices of rank two whose discriminant forms are isomorphic to $-\gamma_X$. By Shimada's theorem they are Zariski pairs, called arithmetic Zariski pairs. Shimada was able to enumerate all such pairs [10, 11].

For any reduced sextic with simple singularities, not necessarily maximizing, let M be the primitive hull of the sublattice generated by all irreducible components of the inverse image of C in X. By Shimada's theorem and Nikulin's lattice theory, the discriminant group A of M is a topological invariant of $(\boldsymbol{P}^2, C)$, which is weaker than N_C. In [12] and [15] this invariant was used to obtain a series of Zariski pairs and Zariski triplets of reduced sextics.

In this paper we show that there are Zariski pairs of sextic curves which cannot be detected by either invariants as mentioned before. We prove that the discriminant group of the primitive hull of the sublattice generated by the -2 curves arising from the simple singularities is a topological invariant of the sextic curve. Twelve new Zariski pairs are found by using this invariant.

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Then we compute the fundamental groups of the complements of the curves of one pair in details. It turns out that all these twelve Zariski pairs are strong Zariski pairs.

We are grateful to the referee for many useful comments and suggestions. He showed us a lattice theoretic technique by which Theorem 2.1 has been improved significantly. We have abandoned our tedious numerical computations of the fundamental groups of $D_7 + A_{11} + A_1$ after the referee suggested Degtyarev's method of dessin d'enfants as an alternative. The proof of Theorem 1.1 was rewritten upon referee's suggestions.

Notations and conventions:

$\boldsymbol{C}$ denotes the field of complex numbers.

$\mathbf{Z}_n = \mathbf{Z}/n\mathbf{Z}$ for a natural number n.

$\pi_1(X)$ denotes the fundamental group of a connected manifold X.

The even unimodular lattice of signature (3, 19) is denoted by Λ, called the K3 lattice. The cohomology group $H^i(X, \mathbf{Z})$ is abbreviated as $H^i(X)$.

1. A topological invariant of sextic curves with simple singularities. Let Y be a compact complex nonsingular algebraic surface. A reduced curve C on Y is called an *even* curve if there is a line bundle $\mathcal{L}$ on Y such that $\mathcal{O}(C) \cong \mathcal{L}^{\otimes 2}$. Such a line bundle $\mathcal{L}$ is uniquely determined by C if $H^1(Y; \mathbf{Z}/2\mathbf{Z}) = 0$. It is well known that there is a double cover $f : X \to Y$ of a compact surface X over Y branched over C.

THEOREM 1.1. *Let Y_1, Y_2 be two compact complex nonsingular surfaces such that $H^1(Y_1; \mathbf{Z}_2) = H^1(Y_1; \mathbf{Z}_2) = 0$. Let C_1, C_2 be reduced even curves in Y_1, Y_2, respectively. Let $\psi : (Y_1, C_1) \to (Y_2, C_2)$ be a homeomorphism. Let $f_i : X_i \to Y_i$ be the double cover branched over C_i for $i = 1, 2$. Then there is a homeomorphism $\phi : (X_1, f_1^{-1}(C_1)) \to (X_2, f_2^{-1}(C_2))$ such that $f_2\phi = \psi f_1$.*

PROOF. For $i = 1, 2$, the double cover $f_i : X_i \backslash f_i^{-1}(C_i) \to Y_i \backslash C_i$, as a $\mathbf{Z}_2$ bundle, is determined by its characteristic class $\omega_i \in H^1(Y_i \backslash C_i, \mathbf{Z}_2)$. Let $\sigma_i : H^1(Y_i \backslash C_i, \mathbf{Z}_2) \to H_3(Y_i, C_i; \mathbf{Z}_2)$ be the isomorphism from Poincaré-Lefschetz duality.

The pair (Y_i, C_i) yields an exact sequence

$$0 \to H_3(Y_i, C_i; \mathbf{Z}_2) \xrightarrow{\partial_i} H_2(C_i; \mathbf{Z}_2) \to H_2(Y_i; \mathbf{Z}_2),$$

due to $H_3(Y_i; \mathbf{Z}_2) \cong H^1(Y_i; \mathbf{Z}_2) = 0$. Since f_i is ramified at C_i, we have $\partial_i \sigma_i(\omega_i) = [C_i]$, where $[C_i]$ is the fundamental class in $H_2(C_i; \mathbf{Z}_2)$. Since the isomorphism from $H_2(C_1, \mathbf{Z}_2)$ to $H_2(C_2, \mathbf{Z}_2)$ induced by ψ carries $[C_1]$ to $[C_2]$, the class $\sigma_1(\omega_1)$ is carried to $\sigma_2(\omega_2)$. Let $\psi^* : H^1(Y_2 \backslash C_2; \mathbf{Z}_2) \to H^1(Y_1 \backslash C_1; \mathbf{Z}_2)$ be the isomorphism induced by ψ. Then $\psi^*(\omega_2) = \omega_1$. This implies that there is an isomorphism $\phi : X_1 \backslash f_1^{-1}(C_1) \to X_2 \backslash f_2^{-1}(C_2)$ such that $f_2\phi = \psi f_1$.

Extend ϕ to the whole X_1 by $\phi(q) = f_2^{-1}\psi f_1(q)$ for every $q \in f_1^{-1}(C_1)$. Then ϕ is the desired homeomorphism. □

Let C be a reduced sextic curve with simple singularities only. Let $\pi : X_0 \to \boldsymbol{P}^2$ be the double cover branched over C. Let $\rho : X \to X_0$ be the minimal resolution of singularities.

Then X is a K3 surface. Let $\mathcal{E}$ be the set of all exceptional -2 curves of ρ. Denote $E = \bigcup_{E_i \in \mathcal{E}} E_i$. Let M be the sublattice of $H^2(X, \boldsymbol{Z})$ generated by all members of $\mathcal{E}$. Denote the primitive hull of M in $H^2(X)$ by $\tilde{M}$.

THEOREM 1.2. *The orthogonal complement $M^\perp$ of M in $H^2(X)$ is a topological invariant of the pair $(\boldsymbol{P}^2, C)$.*

PROOF. (after [5]) Let $U = X \backslash E$. It follows from Theorem 1.1 that the homeomorphism class of U is determined by that of $(\boldsymbol{P}^2, C)$. Denote the inclusion map from U into X by j.

Let R be the kernel of the lattice $H_2(U)$ under the intersection pairing, i.e.,

$$R = \{u \in H_2(U); ux = 0 \text{ for all } x \in H_2(U)\}.$$

Then the lattice $H_2(U)/R$ is a topological invariant of $(\boldsymbol{P}^2, C)$.

There is a commutative diagram

$$\begin{array}{ccccc} H_2(U) & \xrightarrow{j_*} & H_2(X) & & \\ \downarrow \cong & & \downarrow \cong & & \\ H^2(X, E) & \longrightarrow & H^2(X) & \xrightarrow{r} & H^2(E), \end{array}$$

where the vertical maps are isomorphisms from Poincaré-Lefschetz duality and the second row is exact. Hence

$$\mathrm{Im}(j_*) \cong \mathrm{Ker}(r) = M^\perp.$$

Since the homomorphism j_* preserves the intersection pairing and the cup product in $H^2(X)$ is nondegenerate on M, we obtain $\mathrm{Ker}(j_*) = R$. Therefore $M^\perp \cong H_2(U)/R$, which is a topological invariant of $(\boldsymbol{P}^2, C)$. □

For any lattice L, the group $\mathrm{disc}(L) = L^\vee/L$ is called the *discriminant group* of L, where $L^\vee$ is the dual lattice of L.

COROLLARY 1.3. *The discriminant group disc($\tilde{M}$) is a topological invariant of $(\boldsymbol{P}^2, C)$.*

PROOF. This is because $\mathrm{disc}(\tilde{M})$ is isomorphic to $\mathrm{disc}(M^\perp)$ by [7, 1.6.1] and the latter is a topological invariant by Theorem 1.2 □

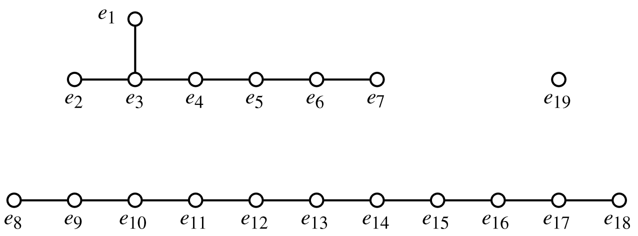

FIGURE 1. Dynkin graph of $D_7 + A_{11} + A_1$.

2. Zariski pairs of sextic curves. Using Corollary 1.3 we found twelve new Zariski pairs of sextic curves. Each of them consists of a conic and a quartic component. Among these pairs, one has Milnor number 19 and the other eleven are its perturbations.

THEOREM 2.1. *There are twelve Zariski pairs with singularity types*

$$D_7 + A_{11} + A_1,\ D_5 + A_{11} + 2A_1,\ D_7 + A_7 + A_3 + A_1,\ A_{11} + 2A_3 + A_1,$$

$$A_{11} + A_3 + 3A_1,\ D_5 + A_7 + A_3 + 2A_1,\ D_7 + 3A_3 + A_1,\ A_7 + 3A_3 + A_1,$$

$$A_7 + 2A_3 + 3A_1,\ D_5 + 3A_3 + 2A_1,\ 5A_3 + A_1,\ 4A_3 + 3A_1.$$

Every curve in these pairs is the union of an irreducible conic and an irreducible quartic curve.

PROOF. Let L denote the negative definite lattice of the Dynkin graph $D_7 + A_{11} + A_1$. The 19 generators of L are labeled according to Figure 1.

For any subset S of $\{e_4, e_6, e_{11}, e_{15}\}$, let D_S be the Dynkin subgraph obtained by deleting the vertices in S. Then D_S is one of the types as listed. Let L' be the root lattice corresponding to D_S. Let $V = \boldsymbol{Q} \otimes_{\boldsymbol{Z}} (\boldsymbol{Z}\lambda \oplus L')$, in which $\lambda^2 = 2$. Let

$$u_1 = \frac{e_1 + 3e_2 + 2e_3 + 2e_5 + 2e_7}{4} + \sum_{i=0}^{2} \frac{3e_{4i+8} + 2e_{4i+9} + e_{4i+10}}{4} \in V$$

and

$$u_2 = u_1 + \frac{\lambda + e_{19}}{2}.$$

It can be verified that $u_i^2 \in 2\boldsymbol{Z}$ and $u_i w \in \boldsymbol{Z}$ for any $w \in \boldsymbol{Z}\lambda \oplus L'$ for $i = 1, 2$. Let P_i be the lattice generated by $\boldsymbol{Z}\lambda \oplus L'$ and u_i. Then P_1 and P_2 are overlattices of $\boldsymbol{Z}\lambda \oplus L'$. Using Nikulin's criterion for lattice embeddings [7, 1.12.2], one verifies that there are primitive embeddings $\sigma_1 : P_1 \to \Lambda$ and $\sigma_2 : P_2 \to \Lambda$ from P_1 and P_2 into the K3 lattice Λ. Moreover, it is not hard to check that P_1 and P_2 satisfy the two conditions in Urabe's theorem [13]. It follows that there are reduced sextic curves C_1 and C_2 with D_S as its type of singularities such that the Picard lattices of the corresponding K3 surfaces are isomorphic to P_1 and P_2, respectively. We can use the algorithm in [14] or [12] to verify that the configurations of C_1 and C_2 are the same, i.e., the union of an irreducible quartic and a conic. The divisor classes representing the strict transforms of the conic components are shown in Table 1 and the configurations are shown in Table 2. Two local components of A_{2p-1} are labeled I and II. The smooth local component of D_{2p+1} is labeled I and the other local component II.

The primitive hull of $\sigma_1(L')$ is isomorphic to $L' + \boldsymbol{Z}u_1$ and that of $\sigma_2(L')$ is isomorphic to $L' + 2\boldsymbol{Z}u_1$. Hence

$$|\operatorname{disc}(\widetilde{\sigma_1(L')})| < |\operatorname{disc}(\widetilde{\sigma_2(L')})|.$$

It follows from Corollary 1.3 that $(\boldsymbol{P}^2, C_1)$ is not homeomorphic to $(\boldsymbol{P}^2, C_2)$. □

REMARK 2.2. In the case of $D_7 + A_{11} + A_1$, let X_1 and X_2 be the K3 surfaces corresponding to C_1 and C_2, respectively. It is verified that the transcendental lattices of both X_1

singularities	strict transform of the conic component
$D_7+A_{11}+A_1$	$\lambda-\frac{e_1+e_2+2e_3+2e_4+2e_5+2e_6+2e_7}{2}-\frac{\sum_{i=1}^{6} ie_{i+7}+\sum_{i=1}^{5} ie_{19-i}}{2}$
$D_5+A_{11}+2A_1$	$\lambda-\frac{e_1+e_2+2e_3+2e_4+2e_5}{2}-\frac{\sum_{i=1}^{6} ie_{i+7}+\sum_{i=1}^{5} ie_{19-i}}{2}$
$D_7+A_7+A_3+A_1$	$\lambda-\frac{e_1+e_2+2e_3+2e_4+2e_5+2e_6+2e_7}{2}-\frac{\sum_{i=1}^{4} ie_{i+7}+\sum_{i=1}^{3} ie_{15-i}}{2}-\frac{e_{16}+2e_{17}+e_{18}}{2}$
$A_{11}+2A_3+A_1$	$\lambda-\frac{e_1+2e_2+e_3}{2}-\frac{\sum_{i=1}^{6} ie_{i+7}+\sum_{i=1}^{5} ie_{19-i}}{2}$
$A_{11}+A_3+3A_1$	$\lambda-\frac{e_1+2e_2+e_3}{2}-\frac{\sum_{i=1}^{6} ie_{i+7}+\sum_{i=1}^{5} ie_{19-i}}{2}$
$D_5+A_7+A_3+2A_1$	$\lambda-\frac{e_1+e_2+2e_3+2e_4+2e_5}{2}-\frac{\sum_{i=1}^{4} ie_{i+7}+\sum_{i=1}^{3} ie_{15-i}}{2}-\frac{e_{16}+2e_{17}+e_{18}}{2}$
$D_7+3A_3+A_1$	$\lambda-\frac{e_1+e_2+2e_3+2e_4+2e_5+2e_6+2e_7}{2}-\frac{e_8+2e_9+e_{10}}{2}-\frac{e_{12}+2e_{13}+e_{14}}{2}-\frac{e_{16}+2e_{17}+e_{18}}{2}$
$A_7+3A_3+A_1$	$\lambda-\frac{e_1+2e_2+e_3}{2}-\frac{\sum_{i=1}^{4} ie_{i+7}+\sum_{i=1}^{3} ie_{15-i}}{2}-\frac{e_{16}+2e_{17}+e_{18}}{2}$
$A_7+2A_3+3A_1$	$\lambda-\frac{e_1+2e_2+e_3}{2}-\frac{\sum_{i=1}^{4} ie_{i+7}+\sum_{i=1}^{3} ie_{15-i}}{2}-\frac{e_{16}+2e_{17}+e_{18}}{2}$
$D_5+3A_3+2A_1$	$\lambda-\frac{e_1+e_2+2e_3+2e_4+2e_5}{2}-\frac{e_8+2e_9+e_{10}}{2}-\frac{e_{12}+2e_{13}+e_{14}}{2}-\frac{e_{16}+2e_{17}+e_{18}}{2}$
$5A_3+A_1$	$\lambda-\frac{e_1+2e_2+e_3}{2}-\frac{e_8+2e_9+e_{10}}{2}-\frac{e_{12}+2e_{13}+e_{14}}{2}-\frac{e_{16}+2e_{17}+e_{18}}{2}$
$4A_3+3A_1$	$\lambda-\frac{e_1+2e_2+e_3}{2}-\frac{e_8+2e_9+e_{10}}{2}-\frac{e_{12}+2e_{13}+e_{14}}{2}-\frac{e_{16}+2e_{17}+e_{18}}{2}$

TABLE 1. The strict transforms of the conic components.

and X_2 are isomorphic to the one represented by the matrix $\begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}$. Hence Shimada's criterion [11] cannot tell whether $\{C_1, C_2\}$ is a Zariski pair.

REMARK 2.3. For the eight Dynkin graphs

$$D_7+A_7+A_3+A_1,\ D_5+A_7+A_3+2A_1,\ D_7+3A_3+A_1,\ A_7+3A_3+A_1,$$
$$A_7+2A_3+3A_1,\ D_5+3A_3+2A_1,\ 5A_3+A_1,\ 4A_3+3A_1$$

appearing in Theorem 2.1, there are Zariski pairs which can be detected by Shimada's invariant [12], [15]. However, each curve in these pairs has an irreducible quartic and two lines as its components.

In order to compute the equations of these two curves, we need to determine the splitting curves of low degrees.

For a reduced sextic curve C with simple singularities, let $\pi : X_0 \to \boldsymbol{P}^2$ be the double cover branched over C and let $\rho : X \to X_0$ be the minimal resolution of singularities. Let $f = \pi\rho$. A curve E in X is called an exceptional -2 curve if $\rho(E)$ is a point. Let λ be the divisor class of the pull back of a line on $\boldsymbol{P}^2$. An irreducible curve D on $\boldsymbol{P}^2$ is called a splitting curve if it is not a component of C and there is an irreducible curve D_1 on X such that the restriction of f on D_1 is a birational morphism onto D. Such a curve D_1 is called a

singularities	configuration							
$D_7 + A_{11} + A_1$		D_7	A_{11}	A_1				
	conic	I	I					
	quartic	II	II	I,II				
$D_5 + A_{11} + 2A_1$		D_5	A_{11}	A_1	A_1			
	conic	I	I					
	quartic	II	II	I,II	I,II			
$D_7 + A_7 + A_3 + A_1$		D_7	A_7	A_3	A_1			
	conic	I	I	I				
	quartic	II	II	II	I,II			
$A_{11} + 2A_3 + A_1$		A_{11}	A_3	A_3	A_1			
	conic	I	I					
	quarticic	II	II	I,II	I,II			
$A_{11} + A_3 + 3A_1$		A_{11}	A_3	A_1	A_1	A_1		
	conic	I	I					
	quartic	II	II	I,II	I,II	I,II		
$D_5 + A_7 + A_3 + 2A_1$		D_5	A_7	A_3	A_1	A_1		
	conic	I	I	I				
	quartic	II	II	II	I,II	I,II		
$D_7 + 3A_3 + A_1$		D_7	A_3	A_3	A_3	A_1		
	conic	I	I	I	I			
	quartic	II	II	II	II	I,II		
$A_7 + 3A_3 + A_1$		A_7	A_3	A_3	A_3	A_1		
	conic	I	I	I				
	quartic	II	II	II	I,II	I,II		
$A_7 + 2A_3 + 3A_1$		A_7	A_3	A_3	A_1	A_1	A_1	
	conic	I	I	I				
	quartic	II	II	II	I,II	I,II	I,II	
$D_5 + 3A_3 + 2A_1$		D_5	A_3	A_3	A_3	A_1	A_1	
	conic	I	I	I	I			
	quartic	II	II	II	II	I,II	I,II	
$5A_3 + A_1$		A_3	A_3	A_3	A_3	A_3	A_1	
	conic	I	I	I	I			
	quartic	II	II	II	II	I,II	I,II	
$4A_3 + 3A_1$		A_3	A_3	A_3	A_3	A_1	A_1	A_1
	conic	I	I	I	I			
	quartic	II	II	II	II	I,II	I,II	I,II

TABLE 2. Configurations of Zariski pairs.

lift of D. A splitting curve D is called a Z-splitting curve if the divisor class of D_1 is in the primitive hull of the sublattice generated by λ and all exceptional −2 curves.

Assume that C is lattice-generic, i.e., the Picard number of X is equal to $\mu_C + 1$, where μ_C is the Milnor number of C.

Let Sing(C) be the set of all singularities of C. For each $p \in$ Sing(C), let $\mathcal{E}_p$ be the set of −2 curves in X mapped to p under f. Let $\mathcal{L}_C$ be the set of $x \in$ Pic(X) satisfying the following conditions:

(1) $x\lambda = 1, x^2 = -2$ and $xE \geq 0$ for every exceptional −2 curve in X;

(2) $\sum_{E \in \mathcal{E}_p} xE \leq 1$ for every $p \in$ Sing(C).

Let L be the subgroup of Pic(X) generated by all exceptional -2 curves and let L^+ be the subset of L consisting of effective divisors. The involution of X over $\boldsymbol{P}^2$ induces an involution ι_C on Pic(X).

Let $\mathcal{C}_C$ be the set of $x \in$ Pic(X) satisfying the following conditions:

(1) $x\lambda = 2$, $x^2 = -2$ and $xE \geq 0$ for every exceptional -2 curve in X;

(2) $\sum_{E \in \mathcal{E}_p} xE \leq 1$ for every $p \in$ Sing(C);

(3) $x \notin \boldsymbol{Z}\lambda \oplus L$ and $\iota_C(x) \neq x$;

(4) $x - l_1 - l_2 \notin L^+$ for any $l_1, l_2 \in \mathcal{L}_C$.

Shimada [12, 5.18] gave the following numerical criterion for splitting conics.

PROPOSITION 2.4. *A divisor class $x \in$ Pic(X) is in $\mathcal{C}_C$ if and only if $x = [D]$ such that D is a lift of a splitting conic.*

LEMMA 2.5. *Let (C_1, C_2) be a Zariski pair in Theorem* 2.1. The curve C_1 has a Z-splitting conic curve. The curve C_2 has no Z-splitting conic curve but has a family of Z-splitting cubic curves.

PROOF. For each pair in Theorem 2.1, let v be the element of $\boldsymbol{Q} \otimes (\boldsymbol{Z}\lambda \oplus L')$ shown in Table 3. It follows from $v - u_1 \in L'$ that $v \in P_1$. An easy calculation shows that $v \in \mathcal{C}_{C_1}$. Hence v is the class of a lift of a splitting conic by Proposition 2.4. The intersection numbers of

singularities	-2 divisor class v
$D_7 + A_{11} + A_1$	$\lambda - \frac{7e_1+5e_2+10e_3+8e_4+6e_5+4e_6+2e_7}{4} - \frac{\sum_{i=1}^{9} ie_{i+7}+6e_{17}+3e_{18}}{4}$
$D_5 + A_{11} + 2A_1$	$\lambda - \frac{3e_1+5e_2+6e_3+4e_4+2e_5}{4} - \frac{e_7}{2} - \frac{\sum_{i=1}^{9} ie_{i+7}+6e_{17}+3e_{18}}{4}$
$D_7 + A_7 + A_3 + A_1$	$\lambda - \frac{7e_1+5e_2+10e_3+8e_4+6e_5+4e_6+2e_7}{4} - \frac{\sum_{i=1}^{6} ie_{i+7}+3e_{14}}{4} - \frac{e_{16}+2e_{17}+3e_{18}}{4}$
$A_{11} + 2A_3 + A_1$	$\lambda - \frac{3e_1+e_2+2e_3}{4} - \frac{2e_5+4e_6+2e_7}{4} - \frac{\sum_{i=1}^{9} ie_{i+7}+6e_{17}+3e_{18}}{4}$
$A_{11} + A_3 + 3A_1$	$\lambda - \frac{3e_1+e_2+2e_3}{4} - \frac{e_5+e_7}{2} - \frac{\sum_{i=1}^{9} ie_{i+7}+6e_{17}+3e_{18}}{4}$
$D_5 + A_7 + A_3 + 2A_1$	$\lambda - \frac{3e_1+5e_2+6e_3+4e_4+2e_5}{4} - \frac{e_7}{2} - \frac{\sum_{i=1}^{6} ie_{i+7}+3e_{14}}{4} - \frac{e_{16}+2e_{17}+3e_{18}}{4}$
$D_7 + 3A_3 + A_1$	$\lambda - \frac{7e_1+5e_2+10e_3+8e_4+6e_5+4e_6+2e_7}{4} - \frac{\sum_{i=1}^{3} ie_{i+7}}{4} - \frac{\sum_{i=1}^{3} ie_{i+11}}{4} - \frac{\sum_{i=1}^{3} ie_{i+15}}{4}$
$A_7 + 3A_3 + A_1$	$\lambda - \frac{3e_1+e_2+2e_3}{4} - \frac{2e_5+4e_6+2e_7}{4} - \frac{\sum_{i=1}^{6} ie_{i+7}+3e_{14}}{4} - \frac{e_{16}+2e_{17}+3e_{18}}{4}$
$A_7 + 2A_3 + 3A_1$	$\lambda - \frac{3e_1+e_2+2e_3}{4} - \frac{e_5+e_7}{2} - \frac{\sum_{i=1}^{6} ie_{i+7}+3e_{14}}{4} - \frac{e_{16}+2e_{17}+3e_{18}}{4}$
$D_5 + 3A_3 + 2A_1$	$\lambda - \frac{3e_1+5e_2+6e_3+4e_4+2e_5}{4} - \frac{e_7}{2} - \frac{\sum_{i=1}^{3} ie_{i+7}}{4} - \frac{\sum_{i=1}^{3} ie_{i+11}}{4} - \frac{\sum_{i=1}^{3} ie_{i+15}}{4}$
$5A_3 + A_1$	$\lambda - \frac{3e_1+e_2+2e_3}{4} - \frac{2e_5+4e_6+2e_7}{4} - \frac{\sum_{i=1}^{3} ie_{i+7}}{4} - \frac{\sum_{i=1}^{3} ie_{i+11}}{4} - \frac{\sum_{i=1}^{3} ie_{i+15}}{4}$
$4A_3 + 3A_1$	$\lambda - \frac{3e_1+e_2+2e_3}{4} - \frac{e_5+e_7}{2} - \frac{\sum_{i=1}^{3} ie_{i+7}}{4} - \frac{\sum_{i=1}^{3} ie_{i+11}}{4} - \frac{\sum_{i=1}^{3} ie_{i+15}}{4}$

TABLE 3. Special divisor classes of splitting conics.

the corresponding splitting conic with the conic and quartic components of C_1 are determined by the intersection numbers of v with $e_i' s$. They are shown in Table 4.

Assume that there is some $v \in \mathcal{C}_{C_2}$. The condition $v\lambda = 2$ implies that v is in the lattice M generated by $2u_1$ and $\mathbf{Z}\lambda \oplus L'$. It can be verified that the only divisor class $v \in M$ satisfying the conditions $v^2 = -2$, $xE \geq 0$ for every exceptional -2 curve E and $\sum_{E \in \mathcal{E}_p} xE \leq 1$ for every $p \in \mathrm{Sing(C_2)}$ is the strict transform of the conic component of C_2, as displayed in Table 1. This would imply that $\iota_{C_2}(v) = v$. Hence $\mathcal{C}_{C_2}$ is empty. By Proposition 2.4 C_2 has no splitting conic.

singularities	intersection indices							
$D_7 + A_{11} + A_1$		D_7	A_{11}	A_1				
	conic	1	3	0				
	quartic	5	3	0				
$D_5 + A_{11} + 2A_1$		D_5	A_{11}	A_1	A_1			
	conic	1	3	0	0			
	quartic	3	3	2	0			
$D_7 + A_7 + A_3 + A_1$		D_7	A_7	A_3	A_1			
	conic	1	2	1	0			
	quartic	5	2	1	0			
$A_{11} + 2A_3 + A_1$		A_{11}	A_3	A_3	A_1			
	conic	3	1	0	0			
	quartic	3	1	4	0			
$A_{11} + A_3 + 3A_1$		A_{11}	A_3	A_1	A_1	A_1		
	conic	3	1	0	0	0		
	quartic	3	1	2	2	0		
$D_5 + A_7 + A_3 + 2A_1$		D_5	A_7	A_3	A_1	A_1		
	conic	1	2	1	0	0		
	quartic	3	2	1	2	0		
$D_7 + 3A_3 + A_1$		D_7	A_3	A_3	A_3	A_1		
	conic	1	1	1	1	0		
	quartic	5	1	1	1	0		
$A_7 + 3A_3 + A_1$		A_7	A_3	A_3	A_3	A_1		
	conic	2	1	1	0	0		
	quartic	2	1	1	4	0		
$A_7 + 2A_3 + 3A_1$		A_7	A_3	A_3	A_1	A_1	A_1	
	conic	2	1	1	0	0	0	
	quartic	2	1	1	2	2	0	
$D_5 + 3A_3 + 2A_1$		D_5	A_3	A_3	A_3	A_1	A_1	
	conic	1	1	1	1	0	0	
	quartic	3	1	1	1	2	0	
$5A_3 + A_1$		A_3	A_3	A_3	A_3	A_3	A_1	
	conic	1	1	1	1	0	0	
	quartic	1	1	1	1	4	0	
$4A_3 + 3A_1$		A_3	A_3	A_3	A_3	A_1	A_1	A_1
	conic	1	1	1	1	0	0	0
	quartic	1	1	1	1	2	2	0

TABLE 4. Intersection indices of the splitting conic.

Let $w = v + \lambda/2 - e_{19}/2$. Then $w \in P_2, w^2 = 0$ and $we_i \geq 0$ for each $e_i \notin S$. The divisor class w determines a pencil of elliptic curves on the K3 surface X_2, whose image in $\boldsymbol{P}^2$ is a pencil of splitting cubic curves (compare [12, 5.20]). □

REMARK 2.6. Every member of the cubic pencil has even intersection number with C_2 at every point. This phenomenon is refered to as "porism".

Once we have an equation of sextic curve with its configuration to be one of those in Theorem 2.1, we can use Lemma 2.5 to tell if it is C_1 or C_2.

To conclude this section, we compute the equations of C_1 and C_2 for $D_7 + A_{11} + A_1$.

For this purpose we compute the moduli space $\mathfrak{M}$ of sextics of $D_7 + A_{11}$ with a conic and a quartic as the irreducible components. Let $X = C + Q$ be such a curve with $\deg(C) = 2, \deg(Q) = 4$. Then $C \cap Q = \{p, q\}$, where p and q are the D_7 point and A_{11} point of X, respectively. Under the homogeneous coordinates $(x_0 : x_1 : x_2)$ assign $(1 : 0 : 0)$ and $(0 : 1 : 0)$ to p and q, respectively. Choose the tangent line of C at q to be the line $x_0 = 0$ and the line with the maximal intersection number with Q at the point p as the line $x_1 = 0$. Then the three points $(1 : 0 : 0)$, $(0 : 1 : 0)$, $(0 : 0 : 1)$ are fixed. Under the affine coordinates $x = x_1/x_0$, $y = x_2/x_0$, the equation of C and Q are

$$x + \lambda y + my^2 = 0$$

and

$$x^2 - 2\mu xy^2 + \mu^2 y^4 + ax^3 + bx^2 y + cx^2 y^2 + dxy^3 = 0\,,$$

respectively, in which the coefficients $\lambda, \mu, m, a, b, c, d$ are to be determined.

It is easy to see that both λ and μ are nonzero. After a linear change $x_1 \mapsto sx_1, x_2 \mapsto tx_2$ for suitable $s, t \in \boldsymbol{C}^*$ of the homogeneous coordinates, we may assume that $\lambda = \mu = 1$.

Let C' and Q' be the conic and quartic curves defined by

$$x + y + my^2 = 0 \tag{1}$$

and

$$x^2 - 2xy^2 + y^4 + ax^3 + bx^2 y + cx^2 y^2 + dxy^3 = 0\,, \tag{2}$$

respectively. For generic values of m, a, b, c, d, the point $(1 : 0 : 0)$ is a D_7 point of $C' \cup Q'$ and the point $(0 : 1 : 0)$ is an A_3 point. We need to find four conditions on m, a, b, c, d to make $(0 : 1 : 0)$ an A_{11} point.

Under the affine coordinates $w = x_0/x_1$, $y = x_2/x_1$, the equations (1) and (2) become

$$w + wy + my^2 = 0 \tag{3}$$

and

$$w^2 - 2y^2 w + y^4 + aw + byw + cy^2 + dy^3 = 0\,, \tag{4}$$

respectively. Solving (3) yields

$$w = -\frac{my^2}{1 + y}\,.$$

Plug this into (4) and we obtain $y^2(\lambda_0 + \lambda_1 y + \lambda_2 y^2 + \lambda_3 y^3 + \lambda_4 y^4) = 0$ where

$$\begin{aligned}\lambda_0 &= am - c\,,\\ \lambda_1 &= am + bm - 2c - d\,,\\ \lambda_2 &= -m^2 - 2m - 1 + bm - c - 2d\,,\\ \lambda_3 &= -2m - 2 - d\,,\\ \lambda_4 &= -1\,.\end{aligned}$$

In order that $(C', Q')_q = 6$, all $\lambda_i (0 \leq i \leq 3)$ must be zero. The system of equations $\lambda_0 = \lambda_1 = \lambda_2 = 0$ in terms of b, c, d has a unique solution

$$\left\{ b = \frac{-m^2 - 2m - 1 + am}{m}, \quad c = am, d = -(m+1)^2 \right\}.$$

Thus λ_3 becomes $m^2 - 1$, which has two zeros 1 and -1.

Hence the moduli space $\mathfrak{M}$ has two connected components corresponding to the values -1 and 1 of m. The equations are

$$f(x, y) = (x + y - y^2)(x^2 - 2xy^2 + y^4 + ax^3 + ax^2y - ax^2y^2) = 0 \tag{5}$$

and

$$f(x, y) = (x + y + y^2)(x^2 - 2xy^2 + y^4 + ax^3 - 4x^2y + ax^2y + ax^2y^2 - 4xy^3)\,, \tag{6}$$

respectively.

REMARK 2.7. The equations (5) and (6) can be used for the explicit computation of the fundamental groups (see Remark 3.5) of lattice-generic sextics of type $D_7 + A_{11}$.

In order to obtain an extra A_1 point, the parameter a must take special values, which are found by solving the system of equations $f(x, y) = 0, \partial f(x, y)/\partial x = 0, \partial f(x, y)/\partial y = 0$ in three variables a, x, y. The nontrivial solutions are $a = -27, x = -1/18, y = 1/6$ for (5) and $a = -1, x = -9/2, y = 3/2$ for (6). Thus the equations for $D_7 + A_{11} + A_1$ are

$$(x + y - y^2)(-x^2 + 2xy^2 - y^4 + 27x^3 + 27x^2y - 27x^2y^2) = 0 \tag{7}$$

and

$$(x + y + y^2)(-x^2 + 2xy^2 - y^4 + x^3 + 5x^2y + x^2y^2 + 4xy^3) = 0\,. \tag{8}$$

By Lemma 2.5 exactly one of these two sextic curves has a Z-splitting conic D. Since $(C, D)_q = 3$ and $(D, Q)_p > 2$, the equation of D is $x - y^2 = 0$ for (7) and $x + y^2 = 0$ for (8), respectively. Since $(D, Q)_p = 5$ for (7) and $(D, Q)_p = 4$ for (8), the curve defined by (7) has a Z-splitting conic. Hence (7) and (8) are the equations of C_1 and C_2, respectively.

3. Fundamental groups.

In this section we show in detail that the Zariski pair of sextic curves with singularities $D_7 + A_{11} + A_1$ is a strong Zariski pair. As a matter of fact, all other Zariski pairs in Theorem 2.1 are strong Zariski pairs, whose fundamental groups are the same as those of $D_7 + A_{11} + A_1$.

The computation of the fundamental group of the complement of a plane curve needs Zariski-Van Kampen theorem more or less. One can do the computation directly on $\boldsymbol{P}^2$, like in [6] or [8, 9] and many other papers. We have used this classical method to compute our curves. It is tedious. For sextic curves with at least one singularity of triple point, Degtyarev developed an elegant method by using Grothendieck's dessin d'enfants [3, 4], which makes the computation almost combinatorial in nature. We will use this method to compute the fundamental groups of the Zariski pair $D_7 + A_{11} + A_1$.

Let C be a simple sextic curve in $\boldsymbol{P}^2$ with one singularity p of type D_7 and some double points. Let $X = \boldsymbol{P}^2 \backslash C$. Denote the fundamental group of X by $\pi_1(X)$. Let $\sigma_0 : Y_0 \to \boldsymbol{P}^2$ be the blowing up of $\boldsymbol{P}^2$ at the D_7 point p and let $E_0 = \sigma_0^{-1}(p)$. Let C_0 be the proper transform of C in Y_0. Then X is isomorphic to $Y_0 \backslash (E_0 \cup C_0)$. There are two intersection points p_1 and p_2 of E_0 and C_0 such that $(C_0, E_0)_{p_i} = i$ for $i = 1, 2$ and p_2 is a singularity of type A_2. For simplicity we assume that $\pi_0^{-1}(p_1) \cap C_0$ contains three distinct points, in which π_0 is the projection from Y_0 to E_0.

Let $\bar{Y}$ be the surface obtained by performing two elementary transformations with centers at p_1 and p_2. Let $\bar{E}$ and $\bar{C}$ be the proper transforms of E_0 and C_0 in $\bar{Y}$ respectively. Then $\bar{Y}$ is the Hirzebruch surface Σ_3 with $\bar{E}$ as its minimal section. The projection from $\bar{Y}$ to $\bar{E}$ is denoted by π. Let F_i be the inverse image of the point p_i in $\bar{Y}$ for $i = 1, 2$. By abuse of nation, the points $\pi(F_i)$ in $\bar{E}$ is also denoted by p_i for $i = 1, 2$.

The curve $\bar{C}$ does not meet $\bar{E}$ and $(\bar{C}, F) = 3$ for any fiber F. This curve is called a trigonal curve. Let $X' = \bar{Y} \backslash (\bar{E} \cup \bar{C})$. Then X' is no longer homeomorphic to X. In Degtyarev's theory, the fundamental group of X can still be computed by braid monodromies in $(\bar{Y}, \bar{C} \cup \bar{E})$. Here we briefly explain a restrictive version of the algorithm without proof. For details and the full version, see [3].

A general fiber F of $\bar{Y}$ over a point $z \in \bar{E}$ meets $\bar{C} \cup \bar{E}$ at four distinct points. Let $j(z)$ be the j-invariant of these four points. Then j gives a rational map from $\boldsymbol{P}^1 \cong \bar{E}$ to $\boldsymbol{A}^1$, which determines a morphism $j : \boldsymbol{P}^1 \to \boldsymbol{P}^1 = \boldsymbol{C} \cup \{\infty\}$. We always assume that j is not a constant map, i.e., the trigonal curve $\bar{C}$ is not isotrivial. Let [0, 1] denote the closed interval from 0 to 1 on the complex plane. The graph $Sk = j^{-1}([0, 1])$ on $\boldsymbol{P}^1$ is called the skeleton of the trigonal curve $\bar{C}$. The points in $j^{-1}(0)$ are called $\bullet$-vertices while those in $j^{-1}(1)$ are called $\circ$-vertices.

For our purpose we restrict the discussion to the case that $\bar{C}$ satisfies the following conditions:

(1) all singularities of $\bar{C}$ are of type A_n and the intersection number $(\bar{C}, F)_q$ is at most 2 for any fiber F and any point $q \in \bar{Y}$;

(2) the map j has no critical values other than $0, 1, \infty$;

(3) the pull-back $j^*(0)$ of 0 as a divisor on $\boldsymbol{P}^1$ is $3(D_1 + \cdots + D_m)$, where $D_1, \ldots, D_m$ are distinct points;

(4) the pull-back $j^*(1)$ of 1 is $2(E_1 + \cdots + E_r)$, where $E_1, \ldots, E_r$ are distinct points.

Such a trigonal curve is a maximal trigonal curve in the sense of [3].

Under this assumption, the skeleton Sk (called generic skeleton in [3]) satisfies the following conditions:

(1) every edge of Sk is smooth;

(2) every $\bullet$-vertex has valency 3, i.e., there are exactly three edges issuing from it;

(3) every $\circ$-vertex has valency 2.

Due to the last condition, in the drawing of skeleton the $\circ$-vertices can be omitted. It is understood that such a vertex is hidden in the middle of every edge connecting two $\bullet$-vertices. This makes Sk look like an ordinary graph.

The skeleton Sk is a connected closed graph in $\boldsymbol{P}^1$, of which each open region contains exactly one point z such that $j(z) = \infty$ and the intersection number of $\pi^{-1}(z)$ with $\bar{C}$ at one point q is equal to two. The type of this point is A_r and $r + 1$ happens to be the number of corners of the corresponding closed region. It should be understood that q is a smooth point of $\bar{C}$ if $r = 0$. The type of this region is defined to be A_r too. A region containing p_1 or p_2 is called a distinguished region.

For any $\bullet$-vertex u, label the three edges issuing from it by u_1, u_2, u_3 in counterclockwise orientation. An edge connecting vertices u and v is denoted by $[u_i, v_j]$ where u_i and v_j are the labels of this edge at u and v, respectively.

Assume that all vertices have been labeled in such a way that each region has two adjacent edges on its boundary whose labels at their common vertex u are u_i, u_{i+1} with $u = 1$ or 2. We use $\langle u_i, u_{i+1}\rangle$ to denote this region. Since there may be several vertices on the boundary of a region, there are many choices of the representation $\langle u_i, u_{i+1}\rangle$ of the region. Since a change of the choice does not change the result, the choice of $\langle u_i, u_{i+1}\rangle$ can be made arbitrarily.

Choose a nerve $\mathcal{N}$ of Sk, which is a subtree of the graph Sk containing all $\bullet$-vertices. Let G be the free group generated by all labels u_1, u_2, u_3, where u runs over all $\bullet$-vertices of Sk.

The fundamental group of the complement of the original sextic curve is isomorphic to G modulo the following sets of relations:

(1) (translation relations) For every edge $[u_i, v_j]$ of the nerve $\mathcal{N}$, as elements in G, u_1, u_2, u_3 are related to v_1, v_2, v_3 by fixed rules determined by i and j. For example

$v_1 = u_2, v_2 = u_2^{-1}u_1u_2, v_3 = u_3, \quad \text{if} \quad i = 1, j = 2.$

$v_1 = u_1u_2u_1^{-1}, v_2 = u_1, v_3 = u_3, \quad \text{if} \quad i = 2, j = 1.$

$v_1 = u_1u_2u_3u_2^{-1}u_1^{-1}, v_2 = u_2, v_3 = u_2^{-1}u_1u_2, \quad \text{if} \quad i = 1, j = 3.$

(2) (monodromy relations) For each non-distinguished region $\langle u_i, u_{i+1}\rangle$ of type A_m, the monodromy relation is

$$(u_iu_{i+1})^r = (u_{i+1}u_i)^r \quad \text{if} \quad m = 2r - 1$$

and

$$(u_iu_{i+1})^r u_i = u_{i+1}(u_iu_{i+1})^r \quad \text{if} \quad m = 2r\,.$$

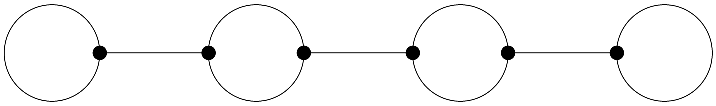

FIGURE 2. Skeleton of the sextic curves with $D_7 + A_{11} + A_1$.

For the distinguished region $\langle u_i, u_{i+1}\rangle$ containing p_1, the monodromy relation is

$$u_1u_2u_3 = u_3u_1u_2 \quad \text{if} \quad i = 1$$

and

$$u_1u_2u_3 = u_2u_3u_1 \quad \text{if} \quad i = 2\,.$$

For the distinguished region $\langle u_i, u_{i+1}\rangle$ containing p_2, the monodromy relations are

$$u_1 = u_3u_2u_3^{-1}, u_1u_2u_1^{-1} = u_3u_1u_3^{-1} \quad \text{if} \quad i = 1$$

and

$$u_2 = u_1u_3u_1^{-1}, u_2u_3u_2^{-1} = u_1u_2u_1^{-1} \quad \text{if} \quad i = 2\,.$$

(3) (relation at infinity) Assume that the distinguished regions containing p_1 and p_2 are $\langle u_i, u_{i+1}\rangle$ and $\langle v_j, v_{j+1}\rangle$, respectively. The relation is

$$(w_1w_2w_3)^3 = u_iu_{i+1}v_3 \quad \text{if} \quad j = 1$$

and

$$(w_1w_2w_3)^3 = u_iu_{i+1}v_1 \quad \text{if} \quad j = 2$$

where w can be any $\bullet$-vertex of Sk.

These relations are not independent. Any one of the monodromy relations corresponding to a non-distinguished region can be omitted.

Now we use this algorithm to compute the fundamental groups of the curves C_1 and C_2 in Theorem 2.1. After the blowing-up of the D_7 point and elementary transformations, we obtain trigonal curves satisfying all conditions of the above algorithm. The skeletons of trigonal curves for both C_1 and C_2 are the same, which is illustrated in Figure 2. The difference is the positions of the distinguished regions. For C_2 the two distinguished regions are adjacent, in the sense that they are connected by a single edge, but not for C_1.

Assign letters $\alpha, \beta, \gamma, \delta, \varepsilon, \eta$ to the vertices of Sk of C_1 and label all edges as shown in Figure 3. There are five regions I through V. The distinguished regions are II and IV. The chosen nerve consists of the edges $[\eta_3, \varepsilon_1]$, $[\varepsilon_2, \delta_1]$, $[\delta_2, \alpha_1]$, $[\alpha_2, \beta_1]$, $[\beta_2, \gamma_1]$, as shown by the thickened path.

The edges $[\alpha_2, \beta_1]$ and $[\beta_2, \gamma_1]$ yield the translation relations

$$\beta_1 = \alpha_1\alpha_2\alpha_1^{-1},\ \beta_2 = \alpha_1,\ \beta_3 = \alpha_3$$

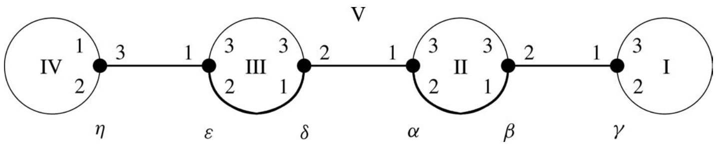

FIGURE 3. Labeling of the skeleton of C_1.

and

$$\gamma_1 = \alpha_1\alpha_2\alpha_1\alpha_2^{-1}\alpha_1^{-1},\ \gamma_2 = \alpha_1\alpha_2\alpha_1^{-1},\ \gamma_3 = \alpha_3\,.$$

The monodromy relation of the region I is

(9) $$\alpha_3 = \alpha_1\alpha_2\alpha_1^{-1}\,.$$

The monodromy relation of the distinguished region II is $\alpha_1\alpha_2\alpha_3 = \alpha_2\alpha_3\alpha_1$, which is equivalent to

(10) $$\alpha_1\alpha_2\alpha_1\alpha_2 = \alpha_2\alpha_1\alpha_2\alpha_1$$

by (9).

The edges $[\delta_2, \alpha_1]$ and $[\varepsilon_2, \delta_1]$ yield the translation relations

$$\delta_1 = \alpha_2,\ \delta_2 = \alpha_2^{-1}\alpha_1\alpha_2,\ \delta_3 = \alpha_3$$

and

$$\varepsilon_1 = \alpha_2^{-1}\alpha_1\alpha_2,\ \varepsilon_2 = \alpha_2^{-1}\alpha_1^{-1}\alpha_2\alpha_1\alpha_2,\ \varepsilon_3 = \alpha_3\,.$$

The region III yields the relation $\varepsilon_2\varepsilon_3 = \varepsilon_3\varepsilon_2$, which is redundant.

The edge $[\eta_3, \varepsilon_1]$ yields the translation relations

$$\eta_1 = \alpha_1\alpha_2\alpha_3\alpha_2^{-1}\alpha_1^{-1} = \alpha_2,\ \eta_2 = \alpha_1\alpha_2\alpha_1^{-1},\ \eta_3 = \alpha_1\alpha_2^{-1}\alpha_1^{-1}\alpha_2^{-1}\alpha_1\alpha_2\alpha_1\alpha_2\alpha_1^{-1} = \alpha_1\,.$$

The distinguished region IV yields two monodromy relations

$$\eta_1 = \eta_3\eta_2\eta_3^{-1}$$

and

$$\eta_1\eta_2\eta_1^{-1} = \eta_3\eta_1\eta_3^{-1}\,,$$

of which both are equivalent to

(11) $$\alpha_1^2\alpha_2 = \alpha_2\alpha_1^2\,.$$

Finally, the relation at infinity is

$$(\alpha_1\alpha_2\alpha_3)^3 = \alpha_1\alpha_2\alpha_3\,,$$

which is equivalent to

(12) $$(\alpha_2^2\alpha_1)^2 = 1\,.$$

In summary, the fundamental group of the complement of C_1 is isomorphic to

$$\langle \alpha_1, \alpha_2;\ \alpha_1\alpha_2\alpha_1\alpha_2 = \alpha_2\alpha_1\alpha_2\alpha_1, \alpha_1^2\alpha_2 = \alpha_2\alpha_1^2, (\alpha_2^2\alpha_1)^2 = 1\rangle.$$

Thus we obtain

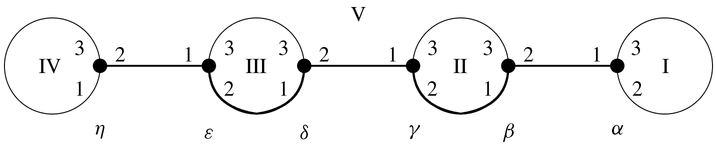

FIGURE 4. Labeling of the skeleton of C_2.

$$\pi_1(\boldsymbol{P}\setminus C_1) \cong \langle a, b; abab = baba, b^2 = 1\rangle \tag{13}$$

due to the following lemma.

LEMMA 3.1. *There are isomorphisms*

$$\langle a, b; abab = baba, b^2a = ab^2, a^2ba^2b = 1\rangle \cong \langle a, b; abab = baba, a^2ba^2b = 1\rangle$$
$$\cong \langle a, b; abab = baba, b^2 = 1\rangle.$$

PROOF. Assume that $abab = baba, a^2ba^2b = 1$. It follows from

$$a^2ba^2b = 1 \Rightarrow aba^2ba = 1 \Rightarrow aba^2bab = b \Rightarrow abababa = b$$
$$\Rightarrow a^2bab^2a = a^2ba^2b^2 \Rightarrow b^2a = ab^2$$

that

$$\langle a, b; abab = baba, b^2a = ab^2, a^2ba^2b = 1\rangle \cong \langle a, b; abab = baba, a^2ba^2b = 1\rangle.$$

Let $\alpha = a^{-1}$, $\beta = a^2b$. Then $a = \alpha^{-1}$, $b = \alpha^2\beta$. Hence $\langle a, b; abab = baba, a^2ba^2b = 1\rangle$ is generated by α, β. Since

$$abab = baba \Leftrightarrow \alpha\beta\alpha\beta = \alpha^2\beta\alpha\beta\alpha^{-1} \Leftrightarrow \beta\alpha\beta\alpha = \alpha\beta\alpha\beta$$

and

$$a^2ba^2b = 1 \Leftrightarrow \beta^2 = 1\,,$$

we have $\langle a, b; abab = baba, a^2ba^2b = 1\rangle \cong \langle a, b; abab = baba, b^2 = 1\rangle$. □

For the second curve C_2, we change the labels as shown in Figure 3 and take the same nerve as before. The distinguished regions are I and II.

The region I yields two relations

$$\alpha_3 = \alpha_1^{-1}\alpha_2\alpha_1$$

and

$$\alpha_2\alpha_1^{-1}\alpha_2\alpha_1\alpha_2^{-1} = \alpha_1\alpha_2\alpha_1^{-1}\,. \tag{14}$$

The translation relations are

$$\beta_1 = \alpha_2,\ \ \beta_2 = \alpha_2^{-1}\alpha_1\alpha_2\,,$$
$$\gamma_1 = \beta_2,\ \ \gamma_2 = \beta_2^{-1}\beta_1\beta_2\,,$$

$$\delta_1 = \gamma_2,\ \delta_2 = \gamma_2^{-1}\gamma_1\gamma_2\,,$$
$$\varepsilon_1 = \delta_2,\ \varepsilon_2 = \delta_2^{-1}\delta_1\delta_2\,,$$
$$\eta_1 = \varepsilon_2,\ \eta_2 = \varepsilon_2^{-1}\varepsilon_1\varepsilon_2$$

and

$$\eta_3 = \varepsilon_3 = \delta_3 = \gamma_3 = \beta_3 = \alpha_3\,.$$

The monodromy relation of the distinguished region II are $\gamma_1\gamma_2\gamma_3 = \gamma_2\gamma_3\gamma_1$, which by (14) is equivalent to

$$\alpha_1\alpha_2\alpha_1^2\alpha_2 = \alpha_2\alpha_1^2\alpha_2\alpha_1\,. \tag{15}$$

The monodromy relation of the region III is $\varepsilon_2\varepsilon_3 = \varepsilon_3\varepsilon_2$, which is reduced to the relation

$$\alpha_2\alpha_1^2 = \alpha_1^2\alpha_2 \tag{16}$$

by using (14) and (15). Then (15) is equivalent to

$$\alpha_1\alpha_2^2 = \alpha_2^2\alpha_1\,. \tag{17}$$

The relation at infinity is

$$(\alpha_1\alpha_2\alpha_3)^3 = \alpha_1\alpha_2^{-1}\alpha_1^{-1}\alpha_2\alpha_1\alpha_2\alpha_1^{-1}\alpha_2\alpha_1\,,$$

which is equivalent to

$$\alpha_2^3\alpha_1\alpha_2\alpha_1 = 1\,. \tag{18}$$

Obviously all relations are generated by (16), (17) and (18). Therefore the fundamental group of the complement of C_2 is isomorphic to

$$\langle\alpha_1, \alpha_2; \alpha_2\alpha_1^2 = \alpha_1^2\alpha_2, \alpha_1\alpha_2^2 = \alpha_2^2\alpha_1, \alpha_2^3\alpha_1\alpha_2\alpha_1 = 1\rangle.$$

Thus we obtain

$$\pi_1(\boldsymbol{P}\backslash C_2) \cong \langle a, b; ab^2 = b^2a, a^2baba = 1\rangle \tag{19}$$

due to the following lemma.

LEMMA 3.2. *Let a, b be two elements in a group G such that $a^2baba = 1$. Then $abab = baba$ and $a^2b = ba^2$ hold.*

PROOF. The relation $a^2baba = 1$ implies $ababa = a^{-1}$. So $ababa^2 = 1$. Therefore $baba = a^{-2} = abab$.

The relation $a^2baba = 1$ implies $a^2b = (aba)^{-1}$. Since $ababa^2 = 1$, we have $ba^2 = (aba)^{-1}$. Hence $a^2b = ba^2$. □

THEOREM 3.3. *The pair C_1, C_2 in Theorem 2.1 is a strong Zariski pair.*

PROOF. Let $G_1 = \langle a, b; abab = baba, b^2 = 1\rangle$ and let Γ_1 be the subgroup of $\mathrm{GL}_2(\boldsymbol{Q})$ generated by

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since

$$ABAB = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = BABA, B^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

there is a unique homomorphism $f_1 : G_1 \to \Gamma_1$ such that $f_1(a) = A$ and $f_1(b) = B$.

It is easy to see that

$$\Gamma_1 = \left\{ \begin{pmatrix} 2^n & 0 \\ 0 & 2^m \end{pmatrix}; m, n \in \boldsymbol{Z} \right\} \cup \left\{ \begin{pmatrix} 0 & 2^n \\ 2^m & 0 \end{pmatrix}; m, n \in \boldsymbol{Z} \right\}.$$

Define the map $g_1 : \Gamma_1 \to G_1$ by

$$g_1 \begin{pmatrix} 2^n & 0 \\ 0 & 2^m \end{pmatrix} = a^n b a^m b, \; g_1 \begin{pmatrix} 0 & 2^n \\ 2^m & 0 \end{pmatrix} = a^n b a^m.$$

By using the equality $(bab)^r a = a(bab)^r$, it can be verified that g_1 is a homomorphism. It is obvious that $f_1 \circ g_1 = 1$ and $g_1 \circ f_1 = 1$. Hence G_1 is isomorphic to Γ_1.

Since

$$\left\{ \begin{pmatrix} 2^n & 0 \\ 0 & 2^n \end{pmatrix}; n \in \boldsymbol{Z} \right\}$$

is the center of Γ_1, the center of $\pi(\boldsymbol{P}^2 \backslash C_1)$ is isomorphic to $\boldsymbol{Z}$ by (13).

Let $G_2 = \langle a, b; a^2 b = b a^2, b^2 abab = 1\rangle$ and let Γ_2 be the subgroup of $\mathrm{GL}_2(\boldsymbol{Q})$ generated by

$$A = \begin{pmatrix} 0 & -1/4 \\ 1/4 & 0 \end{pmatrix}, B = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.$$

Since

$$A^2 B = \begin{pmatrix} -1/8 & 0 \\ 0 & 1/8 \end{pmatrix} = BA^2, B^2 ABAB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

there is a unique homomorphism $f_2 : G_2 \to \Gamma_2$ such that $f_2(a) = A$ and $f_2(b) = B$.

Since A^2, B^2 are in the center and $BA = -AB$, every element in Γ_2 can be written uniquely as

$$\pm \begin{pmatrix} 4^n & 0 \\ 0 & 4^n \end{pmatrix} A^p B^q,$$

where $p, q \in \{0, 1\}$. Define the map $g_2 : \Gamma_2 \to G_2$ by

$$g_2 \left(\begin{pmatrix} 4^n & 0 \\ 0 & 4^n \end{pmatrix} A^p B^q \right) = a^p b^{q+2n}, \; g_2 \left(- \begin{pmatrix} 4^n & 0 \\ 0 & 4^n \end{pmatrix} A^p B^q \right) = a^{p+2} b^{q+2n+4}.$$

It can be verified that g_2 is a homomorphism. It is obvious that $f_2 \circ g_2 = 1$ and $g_2 \circ f_2 = 1$. Hence G_2 is isomorphic to Γ_2. Since

$$\left\{ \pm \begin{pmatrix} 4^n & 0 \\ 0 & 4^n \end{pmatrix}; n \in \boldsymbol{Z} \right\}$$

is the center of Γ_2, the center of $\pi(\boldsymbol{P}^2\backslash C_2)$ is isomorphic to $\boldsymbol{Z}\oplus\boldsymbol{Z}_2$ by (19). Therefore the centers of $\pi(\boldsymbol{P}^2\backslash C_1)$ and $\pi(\boldsymbol{P}^2\backslash C_2)$ are not isomorphic. □

REMARK 3.4. The fundamental groups can also be distinguished by their subgroups of commutators, as observed by the referee. In fact, $[G_1, G_1]\cong\boldsymbol{Z}$ and $[G_2, G_2]\cong\boldsymbol{Z}_2$.

REMARK 3.5. The fundamental groups of the curves of D_7+A_{11} defined by the equations (5) and (6) for generic value of a are also isomorphic to (13) and (19), respectively, as we have computed by numerical method.

REFERENCES

[1] E. ARTAL-BARTOLO, Sur les couples de Zariski, J. Algebraic Geom. 3 (1994), 223–247.

[2] E. ARTAL-BARTOLO AND H. TOKUNAGA, Zariski pairs of index 19 and Mordell-Weil groups of K3 surfaces, Proc. London Math. Soc. (3) 80 (2000), 127–144.

[3] A. DEGTYAREV, The fundamental group of a generalized trigonal curve, Osaka J. Math. 48 (2011), 749–782.

[4] A. DEGTYAREV, Zariski k-plets via dessins d'enfants, Comment. Math. Helv. 84 (2009), 639–671

[5] A. KATANAGA AND I. SHIMADA, Remarks on Zariski pairs of surfaces in the projective space, preprint.

[6] M. NAMBA AND H. TSUCHIHASHI, On the fundamental groups of Galois covering spaces of the projective plane, Geom. Dedicata 105 (2004), 85–105.

[7] V. V. NIKULIN, Integral symmetric bilinear forms and some of their applications, Math. USSR Izvestiya 14 (1980), 103–167.

[8] M. OKA, Geometry of cuspidal sextics and their dual curves, Singularities-Sapporo 1998, 245–277, Adv. Stud. Pure Math. 29, Kinokuniya, Tokyo, 2000.

[9] M. OKA AND D. T. PHO, Fundamental group of sextics of torus type, Trends in singularities, 151–180, Trends Math., Birkhäuser, Basel, 2002.

[10] I. SHIMADA, Non-homeomorphic conjugate complex varieties, Singularities-Niigata-Toyama 2007, 285–301, Adv. Stud. Pure Math., 56, Math. Soc. Japan, Tokyo, 2009.

[11] I. SHIMADA, On Arithmetic Zariski Pairs in degree 6, Adv. Geom. 8 (2008), 205–225.

[12] I. SHIMADA, Lattice Zariski k-plets of plane sextic curves and Z-splitting curves for double plane sextics, Michigan J. Math. 59 (2010), 621–665.

[13] T. URABE, Combinations of rational singularities on plane sextic curves with the sum of Milnor numbers less than sixteen, Banach Center Publ. 20 (1988), 429–456.

[14] J. YANG, Sextic curves with simple singularities, Tohoku Math. J. 48 (2) (1996), 203–227.

[15] J. YANG AND J. XIE, Discrminantal groups and Zariski pairs of sextic curves, Geom. Dedicata (2011), 1–25.

[16] O. ZARISKI, On the problem of existence of algebraic functions of two variables possessing a given branch curve, Amer. J. Math. 51 (2) (1929), 305–328.

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