212. *Some Nonlinear Evolution Equations of Second Order*

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1. Introduction. Let H and W be two real separable Hilbert spaces and V be a real separable reflexive Banach space with $V\subset W\subset H$. Let V be dense in W and in H and the natural injections of V into W and of W into H be respectively continuous and compact. We identify H with its dual:

$$V\subset W\subset H\subset W^*\subset V^*$$

where W^* and V^* are the duals of W and V, respectively. The pairing between V and V^* is denoted by $(\ ,\ )$ and that of W and W^* by $\langle\ ,\ \rangle$.

We consider the following second order differential equation

$$u''+A(u)+Bu'=f \tag{1.1}$$

with initial conditions

$$u(0)=u_0, \qquad u'(0)=u_1, \tag{1.2}$$

where $u=u(t)$, $u'=du/dt$, $u''=d^2u/dt^2$ and data u_0, u_1, f are given.

Assume that the nonlinear operator $A: V\to V^*$ has the following properties:

1) A is hemicontinuous and $\|A(u)\|_{V^*}\leqq c\,\|u\|_V^{p-1}$, $p>1$, $c>0$.

2) A is monotone, i.e., $(A(u)-A(v),\ u-v)\geqq 0$, $\forall u, v\in V$.

3) $(A(u), u)=\|u\|_V^p$.

4) $A(u)$ is Fréchet differentiable at every $u\in V$.

5) $A(u)$ is strongly homogeneous of degree $p-1$ in the sense of Dubinskii [1], i.e., for every u, $\eta\in V$

$$(A'(u)\eta, u)=(A'(u)u, \eta)=(p-1)(A(u), \eta) \tag{1.3}$$

where $A'(u)$ is a Fréchet derivative.

Let $B: W\to W^*$ be a bounded linear operator associated with a bounded symmetric bilinear form $b(\cdot,\cdot)$ on W, i.e.,

$$|b(u, v)|\leqq\|u\|_W\,\|v\|_W, \qquad b(u, v)=b(v, u),$$
$$b(u, v)=\langle Bu, v\rangle, \qquad \forall u, v\in W,$$

such that

$$b(u, u)\geqq\alpha\,\|u\|_W^2-\beta\,\|u\|_H^2, \qquad \alpha, \beta>0, \tag{1.4}$$

and that if $u_n\to u$ weakly in W as $n\to\infty$,

$$\liminf_n b(u_n, u_n)\geqq b(u, u). \tag{1.5}$$

The main result of this note is the following theorem.

Theorem 1. *Suppose that $u_0\in V$, $u_1\in H$ and $f\in L^2(0, T; H)$. Then there exists at least one function u such that*

(1.6) $$u(t) \in L^\infty(0, T; V),$$

(1.7) $$u'(t) \in L^\infty(0, T; H) \cap L^2(0, T; W)$$

(1.8) $$u''(t) \in L^2(0, T; V^*)$$

and satisfies (1.1) *and* (1.2).

The proof of Theorem 1 is stated in Section 2. In Section 3, as applications, the existence of the weak solutions of the initial-Dirichlet boundary value problem for the equation of the form

(1.9) $$\frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right)^{2p-1} - \Delta \frac{\partial u}{\partial t} = f,$$

$$\Delta v = \sum_{i=1}^n \frac{\partial^2 v}{\partial x_i^2}, \qquad p > 1,$$

will be established. When $n=1$, the equation (1.9) was studied by Greenberg, MacCamy and Mizel [2] and Greenberg [3].

2. Proof of Theorem 1.

Lemma 1. *For* $u(t) \in C^1([0, T]; V)$, *we have*

(2.1) $$\begin{aligned}\int_0^t (A(u(s)), u'(s))ds &= \frac{1}{p}(A(u(t)), u(t)) - \frac{1}{p}(A(u(0)), u(0)) \\ &= \frac{1}{p}\|u(t)\|_V^p - \frac{1}{p}\|u(0)\|_V^p.\end{aligned}$$

Proof. By the chain rule, we have

$$\begin{aligned}\left(\frac{d}{dt}A(u(t)), u(t)\right) &= (A'(u(t))u'(t), u(t)) \\ &= (p-1)(A(u(t)), u'(t))\end{aligned}$$

since $A(u)$ is strongly homogeneous of degree $p-1$. Then we get

$$\frac{d}{dt}(A(u(t)), u(t)) = p(A(u(t)), u'(t))$$

which implies (2.1). q.e.d.

The following lemma can be found in [4].

Lemma 2. *Let X be a reflexive separable Banach space. Then there exists a separable Hilbert space Y, being dense in X, such that the injection of Y into X is continuous.*

Hence, we can construct a separable Hilbert space $\tilde{H} \subset W$, being dense in V, such that the injection of $\tilde{H}$ into V is continuous. Then the injection of $\tilde{H}$ into H is compact. Therefore we have

Lemma 3. *The spectral problem*:

(2.2) $$(w, v)_{\tilde{H}} = \lambda (w, v)_H, \qquad \forall v \in \tilde{H},$$

has the sequence of non zero solutions w_j *corresponding to the sequence of eigenvalues* λ_j:

(2.3) $$(w_j, v)_{\tilde{H}} = \lambda_j (w_j, v)_H, \quad \forall v \in \tilde{H}, \quad \lambda_j > 0,$$

where $(\, , \,)_H$ *and* $(\, , \,)_{\tilde{H}}$ *are the scalar products in* H *and* $\tilde{H}$, *respectively.*

In order to prove Theorem 1, we shall employ the Galerkin's method. We use the sequence of the functions w_j as the basis of $\tilde{H}$.

We look for an approximate solution $u_m(t)$ in the form:

$$u_m(t)=\sum_{i=1}^{m} g_{im}(t)w_i, \qquad g_{im}(t)\in C^\infty[0,T],$$

where the unknown functions g_{im} are determined by the following system of ordinary differential equations:

(2.4) $(u_m''(t), w_j)+(A(u_m(t)), w_j)+b(u_m'(t), w_j)=(f(t), w_j) \qquad 1\leqq j\leqq m,$

with initial conditions:

$$u_m(0)=u_{0m}, \quad u_{0m}=\sum_{i=1}^{m}\alpha_{im}w_i\to u_0 \text{ in } V \text{ strongly as } m\to\infty, \tag{2.5}$$

$$u_m'(0)=u_{1m}, \quad u_{1m}=\sum_{i=1}^{m}\beta_{im}w_i\to u_1 \text{ in } H \text{ strongly as } m\to\infty. \tag{2.6}$$

Then we have

Lemma 4. *There exists a constant c independent of m, such that*

$$\|u_m\|_{L^\infty(0,T;V)}\leqq c, \tag{2.7}$$

$$\|u_m'\|_{L^\infty(0,T;H)\cap L^2(0,\pi;W)}\leqq c, \tag{2.8}$$

$$\|A(u_m)\|_{L^\infty(0,T;V^*)}\leqq c, \tag{2.9}$$

$$\|Bu_m'\|_{L^2(0,T;W^*)}\leqq c, \tag{2.10}$$

and

$$\|u_m''\|_{L^2(0,T;\tilde H^*)}\leqq c. \tag{2.11}$$

Proof. Multiplication of the i-th equation in (2.4) by g_{im}', summation over i from 1 to m, integration with respect to t and Lemma 1 give

$$\frac{1}{2}\|u_m'(t)\|_H^2+\frac{1}{p}\|u_m(t)\|_V^p+\int_0^t b(u_m'(s), u_m'(s))ds = \frac{1}{2}\|u_m'(0)\|_H^2+\frac{1}{p}\|u_m(0)\|_V^p+\int_0^t (f(s), u_m'(s))ds \tag{2.12}$$

from which it follows that

$$\frac{1}{2}\|u_m'(t)\|_H^2+\frac{1}{p}\|u_m(t)\|_V^p+\alpha\int_0^t \|u_m'(s)\|_W^2\,ds \leqq c\left(1+\int_0^t\|u_m'(s)\|_H^2\,ds\right). \tag{2.13}$$

The inequality (2.13) and our hypotheses on A and B yield (2.7)–(2.10).

Let P_m be the projection of $H\to[w_1,\cdots,w_m]$ (=the space spanned by $w_1,\cdots,w_m$): $P_m h=\sum_{i=1}^{m}(h, w_i)_H w_i$.

Then we have $P_m\in\mathcal{L}(\tilde H,\tilde H)$; $\|P_m\|_{\mathcal{L}(\tilde H,\tilde H)}\leqq c$.

Since $\tilde H\subset V\subset W$, we get

$$\|P_m\|_{\mathcal{L}(\tilde H,V)}\leqq c \quad\text{and}\quad \|P_m\|_{\mathcal{L}(\tilde H,W)}\leqq c$$

which imply

$$\|P_m^*\|_{\mathcal{L}(V^*,\tilde H^*)}\leqq c \quad\text{and}\quad \|P_m^*\|_{\mathcal{L}(W^*,\tilde H^*)}\leqq c.$$

The equation (2.4) may be written as

$$u_m''=-P_m^*A(u_m)-P_m^*Bu_m'+P_m^*f,$$

which assures (2.11). q.e.d.

From Lemma 4 we see that there exist a function u and a subsequence u_μ of u_m such that

(2.14) $u_\mu \to u$ in $L^\infty(0, T; V)$ weakly star,

(2.15) $u'_\mu \to u'$ in $L^\infty(0, T; H)$ weakly star and in $L^2(0, T; W)$ weakly,

(2.16) $u''_\mu \to u''$ in $L^2(0, T; \tilde{H}^*)$ weakly,

(2.17) $u_\mu(T) \to u(T)$ in W weakly,

(2.18) $u'_\mu(T) \to u'(T)$ in H weakly,

(2.19) $A(u_\mu) \to \chi$ in $L^\infty(0, T; V^*)$ weakly star,

and

(2.20) $Bu'_\mu \to Bu'$ in $L^2(0, T; W^*)$ weakly.

Since the injections of V into H and of W into H are compact, we can furthermore assume that

(2.21) $u_\mu \to u$ in $L^2(0, T; H)$ strongly,

(2.22) $u_\mu(T) \to u(T)$ in H strongly

and

(2.23) $u'_\mu \to u'$ in $L^2(0, T; H)$ strongly.

To show that the function $u(t)$ is a solution of (1.1), (1.2), it is sufficient to prove that

$$\chi = A(u).$$

Multiplying (2.4) by an arbitrary smooth function $\alpha(t)$, integrating over $[0, T]$ and integrating the first term by parts, we have

$$\begin{aligned} -\int_0^T (u'_m(t), \alpha'(t)w_j)dt + \int_0^T (A(u_m(t)), \alpha(t)w_j)dt \\ + \int_0^T b(u'_m(t), \alpha(t)w_j)dt \\ = \int_0^T (f(t), \alpha(t)w_j)dt + (u'_m(0), \alpha(0)w_j) - (u'_m(T), \alpha(T)w_j). \end{aligned} \tag{2.24}$$

Taking the limit of both sides with $m = \mu$, j fixed, we get

$$\begin{aligned} -\int_0^T (u'(t), \alpha'(t)w_j)dt + \int_0^T (\chi, \alpha(t)w_j)dt + \int_0^T b(u'(t), \alpha(t)w_j)dt \\ = \int_0^T (f(t), \alpha(t)w_j)dt + (u_1, \alpha(0)w_j) - (u'(T), \alpha(T)w_j), \ \forall j, \end{aligned}$$

which implies

$$\begin{aligned} -\int_0^T (u'(t), \psi'(t))dt + \int_0^T (\chi, \psi(t))dt + \int_0^T b(u'(t), \psi(t))dt \\ = \int_0^T (f(t), \psi(t))dt + (u_1, \psi(0)) - (u'(T), \psi(T)) \end{aligned} \tag{2.25}$$

for any $\psi \in G$, where G denotes a family of functions defined by

$$G = \{\psi \mid \psi \in L^2(0, T; V),\ \psi' \in L^2(0, T; H)\}.$$

In particular, setting $\psi = u$, we have

$$\begin{aligned} -\int_0^T \|u'\|_H^2\, dt + \int_0^T (\chi, u)dt + \frac{1}{2}b(u(T), u(T)) - \frac{1}{2}b(u_0, u_0) \\ = \int_0^T (f, u)dt + (u_1, u_0) - (u'(T), u(T)). \end{aligned} \tag{2.26}$$

The monotonicity of A gives

$$X_\mu=\int_0^T (A(u_\mu)-A(v), u_\mu-v)dt \geqq 0, \qquad \forall v \in L^\infty(0\cdot T; V). \tag{2.27}$$

From (2.4) we have

$$\int_0^T (A(u_\mu), u_\mu)dt = \int_0^T \|u'\|_H^2\, dt - \frac{1}{2} b(u_\mu(T), u_\mu(T)) + \frac{1}{2} b(u_\mu(0), u_\mu(0)) + \int_0^T (f, u_\mu)dt + (u'_\mu(0), u(0)) - (u'_\mu(T), u_\mu(T))$$

from which it follows that

$$X_\mu = \int_0^T \|u'_\mu\|_H^2\, dt - \frac{1}{2} b(u_\mu(T), u_\mu(T)) + \frac{1}{2} b(u_\mu(0), u_\mu(0)) + \int_0^T (f, u_\mu)dt + (u'_\mu(0), u_\mu(0)) - (u'_\mu(T), u_\mu(T)) - \int_0^T (A(v), u_\mu - v)dt - \int_0^T (A(u_\mu), v)dt.$$

Hence, in virtue of (1.5) and (2.17) we get

$$\liminf_\mu X_\mu \leqq \int_0^T \|u'\|_H^2\, dt - \frac{1}{2} b(u(T), u(T)) + \frac{1}{2} b(u_0, u_0) + \int_0^T (f, u)dt + (u_1, u_0) - (u'(T), u(T)) - \int_0^T (A(v), u-v)dt - \int_0^T (\chi, v)dt. \tag{2.28}$$

Combining (2.26) with (2.28), we have

$$\int_0^T (\chi - A(v), u-v)dt \geqq 0.$$

Then, a well-known argument of the theory of monotone operators gives

$$\chi = A(u).$$

From (1.1), we have

$$u'' = -A(u) - Bu' + f \in L^2(0, T; V^*).$$

This completes the proof of Theorem 1.

3. Some Examples. Let Ω be a bounded domain in R^n with a sufficient smooth boundary $\partial\Omega$. Points in Ω are denoted by $x=(x_1, \cdots, x_n)$ and the time variable is denoted by t. We consider the following initial boundary value problem

$$\frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^n \frac{\partial}{\partial x_j}\left(\frac{\partial u}{\partial x_i}\right)^{2p-1} - \Delta \frac{\partial u}{\partial t} = f, \tag{3.1}$$

$$u(x, 0) = u_0(x), \qquad \partial u(x, 0)/\partial t = u_1(x), \tag{3.2}$$

$$u(x, t) \equiv 0 \qquad \text{on } \partial\Omega \times [0, T], \tag{3.3}$$

where $f(x, t)$, $u_0(x)$ and $u_1(x)$ are given functions and T is an arbitrary positive number.

Put

(3.4) $$A(u)=-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right)^{2p-1}$$

and

(3.5) $$b(u, v)=\sum_{i=1}^{n}\int_{\Omega}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dx.$$

If we take $H=L^2(\Omega)$, $W=W_0^{1,2}(\Omega)$ and $V=W_0^{1,2p}(\Omega)$, we easily see that our hypotheses on A and B are satisfied. Furthermore the well-known theorem of Sobolev tells us that if

$$r>1+\frac{n}{2}-\frac{n}{2p}$$

then

$$\tilde{H}=W_0^{r,2}(\Omega)\subset W_0^{1,2p}(\Omega)$$

and the injection of $W_0^{r,2}(\Omega)$ into $W_0^{1,2p}(\Omega)$ is continuous. Hence, we have

Theorem 2. *For each* $f\in L^2(0, T; L^2(\Omega))$, $u_0\in W^{1,2p}(\Omega)$, $u_1\in L^2(\Omega)$, *the initial boundary value problem* (3.1)–(3.3) *has a solution* $u(x, t)\in L^\infty(0, T; W_0^{1,2p}(\Omega))$ *with*

$$\partial u(x, t)/\partial t\in L^\infty(0, T; L^2(\Omega))\cap L^2(0, T; W_0^{1,2}(\Omega))$$

and

$$\partial^2 u(x, t)/\partial t^2\in L^2(0, T; W^{-1,2p/2p-1}(\Omega)).$$

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