Hindawi
Journal of Mathematics
Volume 2022, Article ID 2174192, 13 pages
https://doi.org/10.1155/2022/2174192

Research Article

Some Novel Approaches for Analyzing the Unforced and Forced Duffing–Van der Pol Oscillators

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Received 15 January 2022; Revised 27 May 2022; Accepted 21 June 2022; Published 31 July 2022

Academic Editor: Melike Kaplan

In the current investigation, both unforced and forced Duffing–Van der Pol oscillator (DVdPV) oscillators with a strong nonlinearity and external periodic excitations are analyzed and investigated analytically and numerically using some new and improved approaches. The new approach is constructed based on Krylov–Bogoliubov–Metroolsky method (KBMM). One of the most important features of this approach is that we do not need to solve a system of differential equations, but only solve a system of algebraic equations. Moreover, the ease and faster of applying this method gives high-accurate results and this approach is better than many approaches in the literature. This approach is applied for analyzing (un)forced DVdP oscillators. Also, some improvements are made to He's frequency-amplitude formulation in order to solve unforced DVdP oscillator to obtain high-accurate results. Furthermore, the He's homotopy perturbation method (He's HPM) is employed for analyzing unforced DVdP oscillator. The comparison between all mentioned approaches is carried out. The application of our approach is not limited to (un) forced DVdPV oscillators only but can be applied to analyze many higher-order nonlinearity oscillators for any odd power and it gives more accurate results than other approaches. Both used methods and obtained approximations will help many researchers in general and plasma physicists in particular in the interpretation of their results.

1. Introduction

The study of the dynamics of nonlinear oscillators is one of the topics of great importance for many researchers due to its many important applications in various areas of physics, applied mathematics, and practical engineering [1–17]. Differential equations are one of the best and most successful models in modeling many nonlinear dynamical systems. For instance, Duffing-type equation is one of the most famous and successful equations that has been used for modeling and interpreting many nonlinear oscillations in many different dynamical systems such as electrical circuit, optical stability, the buckled beam, and different oscillations in a plasma [16–19]. In plasma physics, there are many evolution equations that can be reduced to Duffing-type equation, Helmholtz-type equation, Duffing–Helmhlotz equation, and Mathieu equation in order to investigate the various oscillations that occur within complicated plasma systems [20–23]. There is another type of equation of motion that was used for modeling the nonlinear oscillations in biology, electronics, engineering, plasma physics, and chemistry which is called Van der Pol–Duffing (VdPD) (sometimes called Duffing–Van der Pol (DVdP)) equation and its family [24, 25]. For example, a forced modified VdPD (mVdPD) equation was adopted for investigating the strong nonlinear oscillations in plasma [26]. Also, a mVdPD equation with asymmetric potential was used for modeling the nonlinear chemical dynamics [27]. Many numerical and analytical approaches were applied for solving the second-order nonlinear oscillator equations. For example, both HBM and MTSs techniques were devoted for analyzing a forced Van der Pol (VdP) generalized oscillator to obtain the amplitudes of the forced harmonic, superharmonic, and subharmonic oscillatory states [26]. Melnikov's method was used for analyzing a mVdPD equation to derive analytical criteria for the appearance of horseshoe chaos in chemical oscillations [27]. He et al. [16] used the Poincare ´–Lindstedt technique (PLT) for solving and analyzing the Hybrid Rayleigh–van der pol–Duffing equation. Also, the homotopy analysis method (HAM) was used for analyzing DVdP oscillator [28]. Both methods of differentiable dynamics and Lie symmetry reduction method were devoted for analyzing the DVdP-type oscillator [29]. Moreover, DVdP oscillator was solved numerically via Adomian's decomposition method (ADM) [30]. Based on this approach, the equation of motion is converted to a system of first-order differential equations and then was solved to obtain a numerical approximation. Moreover, the authors made a comparison with Lindsted's method (LM) approximation. They found that the obtained approximation using ADM is better than LM. However, in the two approaches, the approximations become convergence and more accurate only in the short time domain but these approximations become dis-convergence and not accurate for long time domain. Most methods in the literature lead to complicated formulas for the obtained approximations and the analysis of such solutions are much difficult or not convergence for a long time. However, the Krylov–Bogoliubov–Mitropolsky method (KBMM) was adopted for deriving the periodic steady-state solutions to the following DVdP driven oscillator [6].

$$\begin{cases} \ddot{x} - \varepsilon\left(1 - x^2\right)\dot{x} + \omega_0^2 x + \beta x^3 = f\ \cos(\Omega t), \\ x(0) = x_0, \\ x'(0) = \dot{x}_0, \end{cases} \tag{1}$$

where the overdot indicates the derivative with respect to "t". Recently, Salas et al. [11] applied the ansatz method, HBM, PLT, and KBMM for analyzing the forced VdP oscillator and found that KBMM gives more accurate approximations. Motivated by the investigations in Ref. [6, 11], we proceed to analyze the DVdP oscillator using a new effective and simplification technique based on KBMM. In our approach, we will prove that the new approach does not demand to solve any ordinary differential equations (odes). Moreover, we will prove that the new suggested approach gives high-accurate and convergence approximations in the whole time domain. Note that for $\varepsilon = 0$, i.v.p. (1) reduces to the forced Duffing oscillator whose general solution is well known [11]. Moreover, in this investigation, we try to improve He's frequency-amplitude formulation to be suitable for analyzing the DVdP oscillator. Also, the He's homotopy perturbation method (He's HPM) will apply for analyzing and investigating the DVdP oscillator.

The rest of this paper is introduced in the following fashion: in Section 2, the new suggested approach is introduced. The analytical approximations to the unforced DVdP oscillator is reported in Section 3 using the new mentioned approach, the He's HPM, and improved He's frequency-amplitude formulation. Moreover, in Section 4, the new mentioned approach is devoted for getting an analytical approximation to the forced DVdP oscillator. The obtained results are summarized in Section 5.

2. New Approach Based on KBM for Solving Strongly Nonlinear Oscillators

Let us consider the following general form to the second-order i.v.p.:

$$\begin{cases} \ddot{x} + \omega_0^2 x + F(x, \dot{x}) = 0, \\ x(0) = x_0, \\ x'(0) = \dot{x}_0, \end{cases} \tag{2}$$

where the expression $F(x, 0)$ is an odd polynomial of x.

To introduce a p-parameter solution to i.v.p. (2), we rewrite this problem in the following new form:

$$\begin{cases} R_p(x) \equiv \ddot{x} + \omega_0^2 x + pF(x, \dot{x}) = 0, \\ x(0) = x_0, \\ x'(0) = \dot{x}_0, \end{cases} \tag{3}$$

where $x_p \equiv x_p(t) \equiv x(p, t)$ indicates the solution to i.v.p. (3), we can call a p-parameter solution. Then, the solution to the original i.v.p. (2) can be obtained for $p = 1$.

Assuming the solution to i.v.p. (3) is given by the following ansatz form:

$$x_p = a\ \cos(\psi) + \sum_{k=1}^{N} p^k \sum_{j=1}^{k} a^{2j+1} \sum_{i=1}^{j} \cdot \left\{ r_{2i+1,j,k} \cos[(2i+1)\psi] + s_{2i+1,j,k} \sin[(2i+1)\psi] \right\}, \tag{4}$$

where the functions $a \equiv a(t)$ and $\psi \equiv \psi(t)$ are assumed to vary with time according to the following ordinary differential equations (odes):

$$\dot{a} = \frac{\mathrm{d}a}{\mathrm{d}t} \equiv a\prime(t) = \sum_{k=1}^{N} p^k \sum_{j=k}^{k+1} B_{k,j} a^{2j-1} s, \tag{5}$$

$$\dot{\psi} = \frac{\mathrm{d}\psi}{\mathrm{d}t} \equiv \psi\prime(t) = \omega_0 + pC_{1,0}a^2 + \sum_{k=2}^{N} p^k \sum_{j=0}^{k} C_{k,j} a^{2j}. \tag{6}$$

We plug the ansatz (4) with the relations (5) and (6) into $R_p(x)$ given in equation (3) and equating the coefficients of p^n $(n = 1, 2, 3, \ldots,)$ $a^{2j+1}(t)$, cos $((2i+1)\psi(t))$, and sin $((2i+1)\psi(t))$ $(i, j = 1, 2, 3, \ldots, N)$ to zero, then we can get a system of algebraic equations. By solving this system, we can determine the unknown coefficients $r_{2i+1,j,k}$, $s_{2i+1,j,k}$, $B_{k,j}$, and $C_{k,j}$.

Remark 1. We can obtain another method by replacing (6) with

$$\dot{\psi} = \sqrt{\omega_0^2 + pC_{1,0}a^2 + \sum_{k=2}^{N} p^k \sum_{j=0}^{k} C_{k,j}a^{2j}}. \quad (7)$$

In the below section, we will use this method for analyzing both the unforced DVdP oscillator, i.e., i.v.p. (1) for $f = 0$ and the forced DVdP oscillator (1).

3. Analytical Approximations to the Unforced Duffing–Van Der Pol Oscillator

Here, we can analyze the unforced DVdP oscillator, i.e., i.v.p. (1) for $f = 0$ using three different approaches, namely:

(i) Our new approach mentioned in the above section
(ii) The He's HPM
(iii) Improved He's frequency-amplitude formulation (He's FAF)

3.1. Our New Approach. By using $N = 2$ in solution (4), then the solution to the following unforced problem (3)

$$\ddot{x} + \omega_0^2 x + p\left[-\varepsilon\left(1 - x^2\right)\dot{x} + \beta x^3\right] = 0, \quad (8)$$

can be introduced in following form:

$$x_p = a\cos(\psi) + \sum_{k=1}^{2} a^{2k+1} p^k \sum_{j=1}^{k} \cdot \left[r_{2j+1,k}\cos((2j+1)\psi) + s_{2j+1,k}\sin((2j+1)\psi)\right]. \quad (9)$$

Accordingly, we get

$$x_p = a\cos(\psi) + pa^3 A_1(t) + p^2\left(a^3 A_2(t) + a^5 A_3(t)\right), \quad (10)$$

with

$$\begin{aligned} A_1(t) &= \left(r_{3,1,1}\cos(3\psi) + s_{3,1,1}\sin(3\psi)\right), \\ A_2(t) &= \left(r_{3,1,2}\cos(3\psi) + s_{3,1,2}\sin(3\psi)\right), \\ A_3(t) &= r_{3,2,2}\cos(3\psi) + r_{5,2,2}\cos(5\psi) + s_{3,2,2}\sin(3\psi) \\ &\quad + s_{5,2,2}\sin(5\psi), \end{aligned} \quad (11)$$

$$\begin{aligned} \dot{a} &= p\left(B_{1,2}a(t)^3 + B_{1,1}a(t)\right) + p^2\left(B_{2,3}a(t)^5 + B_{2,2}a(t)^3\right), \\ \dot{\psi} &= \omega_0 + pC_{1,0}a(t)^2 + p^2\left[C_{2,2}a(t)^4 + C_{2,1}a(t)^2 + C_{2,0}\right]. \end{aligned} \quad (12)$$

On the other hand

$$R_p = D_1 p + D_2 p^2 + O\left(p^3\right), \quad (13)$$

with

$$\begin{aligned} D_1 &= S_1 a\sin(\psi) + a^3\left(S_2\sin(\psi) + S_3\cos(\psi) + S_4\cos(3\psi) + S_5\sin(3\psi)\right), \\ D_2 &= a^5\left(S_6\sin(\psi) + S_7\cos(\psi) + S_8\sin(3\psi) + S_9\cos(3\psi) + S_{10}\sin(5\psi) + S_{11}\cos(5\psi)\right) \\ &\quad + a^3\left(S_{12}\sin(\psi) + S_{13}\cos(\psi) + S_{14}\sin(3\psi) + S_{15}\cos(3\psi)\right) + aS_{16}\cos(\psi), \end{aligned} \quad (14)$$

where the values of coefficients $S_i\,(i = 1, 2, \ldots, 16)$ are defined in Appendix (i).

Equating the coefficients of p and p^2 to zero in

$$R_p = \ddot{x} + \omega_0^2 x + p\left[-\varepsilon\left(1 - x^2\right)\dot{x} + \beta x^3\right], \quad (15)$$

also, equating to zero the coefficients of a^{2j+1}, $\cos((2i+1)\psi)$, and $\sin((2i+1)\psi)$, where $(i,\ j = 1, 2, 3, \ldots, N)$, i.e., $S_i = 0$, we get an algebraic system. The solution of this system yields

$$
\begin{aligned}
B_{1,1} &= \frac{\varepsilon}{2},\\
B_{1,2} &= -\frac{\varepsilon}{8},\\
B_{2,2} &= -\frac{3\beta\varepsilon}{16\omega_0^2},\\
B_{2,3} &= \frac{\beta\varepsilon}{32\omega_0^2},\\
C_{1,0} &= \frac{3\beta}{8\omega_0},\\
C_{2,0} &= -\frac{\varepsilon^2}{8\omega_0},\\
C_{2,1} &= \frac{\varepsilon^2}{8\omega_0},\\
C_{2,2} &= -\frac{15\beta^2 + 7\varepsilon^2\omega_0^2}{256\omega_0^3}\\
r_{3,1,1} &= \frac{\beta}{32\omega_0^2},\\
r_{3,1,2} &= -\frac{\varepsilon^2}{128\omega_0^2},\\
r_{3,2,2} &= -\frac{21\beta^2 + \varepsilon^2\omega_0^2}{1024\omega_0^4},\\
r_{5,2,2} &= \frac{3\beta^2 - 5\varepsilon^2\omega_0^2}{3072\omega_0^4},\\
s_{3,1,1} &= -\frac{\varepsilon}{32\omega_0},\\
s_{3,1,2} &= -\frac{3\beta\varepsilon}{128\omega_0^3},\\
s_{3,2,2} &= \frac{3\beta\varepsilon}{256\omega_0^3},\\
s_{5,2,2} &= -\frac{\beta\varepsilon}{384\omega_0^3}.
\end{aligned}
\tag{16}
$$

From the above values in equation (11), we have

$$
\begin{aligned}
A_1(t) &= \frac{1}{32\omega_0^2}\left(\beta\cos(3\psi) - \varepsilon\omega_0\sin(3\psi)\right),\\
A_2(t) &= -\frac{\varepsilon}{128\omega_0^3}\left(3\beta\sin(3\psi) + \varepsilon\omega_0\cos(3\psi)\right),\\
A_3(t) &= -\frac{1}{3072\omega_0^4}\left[3\left(21\beta^2 + \varepsilon^2\omega_0^2\right)\cos(3\psi)\right.\\
&\quad + \left(5\varepsilon^2\omega_0^2 - 3\beta^2\right)\cos(5\psi)\\
&\quad \left. + 4\beta\varepsilon\omega_0\left(2\sin(5\psi) - 9\sin(3\psi)\right)\right].
\end{aligned}
\tag{17}
$$

Accordingly, the odes for determining the functions (a, ψ) read

$$
\begin{aligned}
\dot{a} &= p\left(\frac{1}{2}\varepsilon a(t) - \frac{1}{8}\varepsilon a(t)^3\right)\\
&\quad + p^2\left(-\frac{3\beta\varepsilon}{16\omega_0^2}a(t)^3 + \frac{\beta\varepsilon}{32\omega_0^2}a(t)^5\right),\\
\dot{\psi} &= \omega_0 + p\frac{3\beta}{8\omega_0}a(t)^2\\
&\quad + p^2\left(-\frac{\varepsilon^2}{8\omega_0} + \frac{\varepsilon^2}{8\omega_0}a(t)^2 - \frac{15\beta^2 + 7\varepsilon^2\omega_0^2}{256\omega_0^3}a(t)^4\right).
\end{aligned}
\tag{18}
$$

For $p = 1$, the value of $\dot{a}$ given in equation (18) reduces to

$$
\dot{a}|_{p=1} = \left(\frac{1}{2}\varepsilon a - \frac{1}{8}\varepsilon a^3\right) + \left(-\frac{3\beta\varepsilon}{16\omega_0^2}a^3 + \frac{\beta\varepsilon}{32\omega_0^2}a^5\right). \tag{19}
$$

The amplitude for the limit cycle is obtained from the condition $\dot{a}|_{p=1} = 0$:

$$
\left(\frac{1}{2}\varepsilon a - \frac{1}{8}\varepsilon a^3\right) + \left(-\frac{3\beta\varepsilon}{16\omega_0^2}a^3 + \frac{\beta\varepsilon}{32\omega_0^2}a^5\right) = 0. \tag{20}
$$

Solving equation (20) gives

$$
A = \sqrt{\frac{\omega_0^2\left(2 - \sqrt{\left(\beta\left(9\beta - 4\omega_0^2\right)/\omega_0^4\right) + 4}\right)}{\beta} + 3}. \tag{21}
$$

Observe that

$$
\lim_{\beta \longrightarrow 0} A = 2, \tag{22}
$$

this is called the cycle amplitude for the VdP oscillator.

We can use the following Chebyshev approximation in order to facilitate the solution to the ode system (18):

$$\dot{a}|_{p=1} = \left(\frac{1}{2}\varepsilon a - \frac{1}{8}\varepsilon a^3\right) + \left(-\frac{3\beta\varepsilon}{16\omega_0^2}a^3 + \frac{\beta\varepsilon}{32\omega_0^2}a^5\right)$$
$$\approx \frac{1}{16}\varepsilon\left(8 - \frac{\beta}{\omega_0^2}\right)a - \frac{1}{16}\varepsilon\left(2 + \frac{\beta}{\omega_0^2}\right)a^3, \quad (23)$$
$$= \kappa a - \mu a^3,$$

with

$$\kappa = \frac{1}{16}\varepsilon\left(8 - \frac{\beta}{\omega_0^2}\right), \mu = \frac{1}{16}\varepsilon\left(2 + \frac{\beta}{\omega_0^2}\right). \quad (24)$$

By solving equation (23), we get

$$a = \frac{c_0\sqrt{\kappa}e^{\kappa t}}{\sqrt{c_0^2\mu\left(e^{2\kappa t} - 1\right) + \kappa}}. \quad (25)$$

Also, the expression for determining ψ can be obtained from the second equation in (18) for $p = 1$ whose solution reads

$$\psi = \sum_{i=1}^{4} W_i + c_1, \quad (26)$$

where the values of W_i $(i = 1, 2, 3, 4)$ are defined in Appendix (ii). The constants c_0 and c_1 are determined from the initial conditions (ICs) $x(0) = x_0$ and $x'(0) = \dot{x}_0$.

In all above expressions, for $p = 1$, the approximation to the following i.v.p. can be obtained:

$$\begin{cases} \ddot{x} - \varepsilon\left(1 - x^2\right)\dot{x} + \omega_0^2 x + \beta x^3 = 0, \\ x(0) = x_0 \text{ and } x'(0) = \dot{x}_0, \end{cases} \quad (27)$$

3.2. He's Homotopy Perturbation Method. Moreover, the approximate solution to the DVdP i.v.p. (1) using He's HPM is obtained. Briefly, He's HPM can be used for a series of nonlinear oscillators differential equations which many classical perturbation methods failed to solve them or to give some accurate solutions. This method suggests the solution in the following ansatz:

$$x_{LP} = \sum_{i=0}^{\infty} p^i x_i(\omega t), \quad (28)$$

with

$$\omega^2 = \omega_0^2 + \sum_{i=1}^{\infty} p^i \omega_i. \quad (29)$$

Substitute equations (28) and (29) into i.v.p. (1), and by collecting the coefficients of same powers of p, we finally obtain some reduced equations. We have

$$\begin{aligned}\mathbb{R}_F &= \left(x_0(\tau) + x_0''(\tau)\right)\omega_0 \\ &+ \Bigg(\beta x_0(\tau)^3 + \omega_0^2 x_1(\tau) + \frac{1}{2}\left(-2\varepsilon\omega_0 + \tau\omega_1\right)x_0'(\tau) \\ &+ \varepsilon\omega_0 x_0(\tau)^2 x_0'(\tau) + \omega_1 x_0''(\tau) + \omega_0^2 x_1''(\tau) \\ &+ \frac{1}{2}\tau\omega_1 x_0^{(3)}(\tau) - F\cos\left(\frac{\tau\Omega}{\omega_0}\right)\Bigg)p + O\left(p^2\right),\end{aligned} \quad (30)$$

where $\tau = t\omega$.

Equating to zero the coefficients of p^j and solving the resulting odes gives

$$x_0(\tau) = A\cos(\tau),$$

$$\begin{aligned}x_1(\tau) &= \frac{A^3}{32\omega_0^2}\left(\beta\cos(3\tau) - \varepsilon\omega_0\sin(3\tau)\right) \\ &+ \frac{- - A^3\varepsilon\omega_0 + 16c_2\omega_0^2 + \left(8A\omega_1 - 6A^3\beta\right)\tau}{16\omega_0^2}\sin(\tau) \\ &+ \frac{8A\omega_1 + 16c_1\omega_0^2 - 5A^3\beta + \left(8A\varepsilon\omega_0 - 2A^3\varepsilon\omega_0\right)\tau}{16\omega_0^2}\cos(\tau) \\ &+ \frac{F}{\omega_0^2 - \Omega^2}\cos\left(\frac{\Omega}{\omega_0}\tau\right),\end{aligned} \quad (31)$$

where A represents the amplitude of the oscillator.

Secularity terms in the last expression are not allowed so that the coefficients of $\cos(t)$ and $\sin(t)$ must be equal to zero which lead to

$$\begin{cases} -5A^3\beta + 8A\omega_1 + 16c_1\omega_0^2 = 0, \\ -2A\left(A^2 - 4\right)\varepsilon\omega_0 = 0, \\ \omega_0\left(16c_2\omega_0 - A^3\varepsilon\right) = 0, \\ 8A\omega_1 - 6A^3\beta = 0. \end{cases} \quad (32)$$

Solving the last system gives

$$\begin{cases} A = 2, \\ c_1 = -\dfrac{\beta}{2\omega_0^2}, \\ c_2 = \dfrac{\varepsilon}{2\omega_0}, \\ \omega_1 = 3\beta. \end{cases} \quad (33)$$

Then, the following solution to $O(p^2)$ and for $p = 1$ is obtained:

$$x_{LP} = 2\cos(\omega t) + \frac{1}{4\omega_0^2}\left(\beta\cos(3\omega t) - \varepsilon\omega_0\sin(3\omega t)\right) + \frac{F}{\omega_0^2 - \Omega^2}\cos\left(\frac{\Omega\omega}{\omega_0}t\right), \tag{34}$$

where $\omega = \sqrt{(3\beta + \omega_0^2)}$ and $\omega_0^2 \neq \Omega^2$. This solution is obtained under the initial conditions (ICs)

$$\begin{cases} x_0 = 2 + \dfrac{\beta}{4\omega_0^2} + \dfrac{F}{\omega_0^2 - \Omega^2}, \\ \dot{x}_0 = -\dfrac{3\varepsilon\sqrt{\Theta}}{4\omega_0}. \end{cases} \tag{35}$$

Solution (34) recovers the unforced case for $F = 0$.

3.3. Improved He's Frequency-Amplitude Formulation. To demonstrate the general idea of He's frequency-amplitude formulation [31–35], let us consider the following oscillator:

$$\ddot{x} + f(x) = 0, \tag{36}$$

where $f(x)$ indicates the nonlinear restoring force. The following conditions are hold: $f(0) = 0$ and $f(x)/x > 0$.

He considered Duffing oscillator

$$\begin{cases} \ddot{x} + x + \varepsilon x^3 = 0, \\ x(0) = A, \\ \dot{x}(0) = 0, \end{cases} \tag{37}$$

where A represents the amplitude of the oscillator. Based on He's principle, we have

$$\begin{cases} f(x) = x + \varepsilon x^3 \\ \dot{\psi}^2 = \omega^2 = \dfrac{f(x)}{x}\Big|_{x=(\sqrt{3}/2)A} = 1 + 3\left(\dfrac{A}{2}\right)^2, \end{cases} \tag{38}$$

where ω denotes the frequency of oscillator.

Now, by considering the following DVdP oscillator:

$$\ddot{x} - \varepsilon\left(1 - x^2\right)\dot{x} + \omega_0^2 x + \beta x^3 = 0, \tag{39}$$

in this case, the function $f(x)$ reads

$$f(x) = \omega_0^2 x + \beta x^3, \tag{40}$$

which leads to

$$\dot{\psi}^2 = \omega^2 = \frac{f(x)}{x}\Big|_{x=(\sqrt{3}/2)A} = \omega_0^2 + \frac{3}{4}\beta A^2. \tag{41}$$

Since the amplitude now depends on time, we will reason heuristically to determine it

$$\begin{cases} A = a(t), \\ \dot{a} = \dfrac{\varepsilon\kappa}{2}a - \dfrac{\varepsilon}{8}a^3, \end{cases} \tag{42}$$

with

$$\kappa = \frac{\omega_0^2\left(2 - \sqrt{\left(\beta\left(9\beta - 4\omega_0^2\right)/\omega_0^4\right) + 4}\right) + 3\beta}{4\beta}. \tag{43}$$

Note that $\kappa \longrightarrow 1$ as $\beta \longrightarrow 0$.

Then, the improved He's solution becomes

$$x(t) = a(t)\cos(\omega(t)), \tag{44}$$

with

$$a(t) = \frac{2\sqrt{\kappa}e^{(\varepsilon\kappa t/2)}}{\sqrt{\left(4\kappa/c_0^2\right) + e^{\varepsilon\kappa t} - 1}},$$
$$\omega(t) = c_1 + \int_0^t \sqrt{\omega_0^2 + \frac{3}{4}\beta a^2(\tau)\mathrm{d}\tau} = W(t) - W(0) + c_1, \tag{45}$$

where

$$W(t) = \frac{1}{\varepsilon\kappa}\left[2\sqrt{3\beta\kappa + \omega_0^2}\tanh^{-1}\cdot\left(\frac{\sqrt{\left(3\beta\kappa e^{\varepsilon\kappa t}/\left(4\kappa/c_0^2\right) + e^{\varepsilon\kappa t} - 1\right) + \omega_0^2}}{\sqrt{3\beta\kappa + \omega_0^2}}\right) - 2\omega_0\coth^{-1}\left(\frac{\omega_0}{\sqrt{\left(3\beta\kappa e^{\varepsilon\kappa t}/\left(4\kappa/c_0^2\right) + e^{\varepsilon\kappa t} - 1\right) + \omega_0^2}}\right)\right]. \tag{46}$$

The constants c_0 and c_1 are determined from the ICs $x(0) = x_0$ and $x'(0) = \dot{x}_0$. The amplitude for the limit cycle reads

$$r_\beta = \lim_{t\longrightarrow+\infty} a(t) = \frac{\sqrt{\sqrt{9\beta^2 - 4\beta\omega_0^2 + 4\omega_0^4} + 3\beta + 2\omega_0^2}}{\omega_0} \longrightarrow 2\ \text{as}\ \beta \longrightarrow 0. \tag{47}$$

As a numerical example, we can use the same model and data that were given in Ref. [36], which lead to the following unforced DVdP i.v.p. (27):

$$\begin{cases} \ddot{x} - 0.1\left(1 - x^2\right)\dot{x} + x + 0.01x^3 = 0, \\ x(0) = 2, \\ x'(0) = 0. \end{cases} \tag{48}$$

Solution (10) and RK numerical approximation to i.v.p. (48) are graphically mapped as shown in Figure 1. Moreover, the approximation (34) using He's HMP and the approximation (44) using the improved He's FAF are compared with the obtained analytical approximation (10) and RK numerical approximation as illustrated in Figure 1. In addition, the maximum distance error in the whole time domain $(0 \leq t \leq 50)$ with respect to RK numerical approximation is estimated

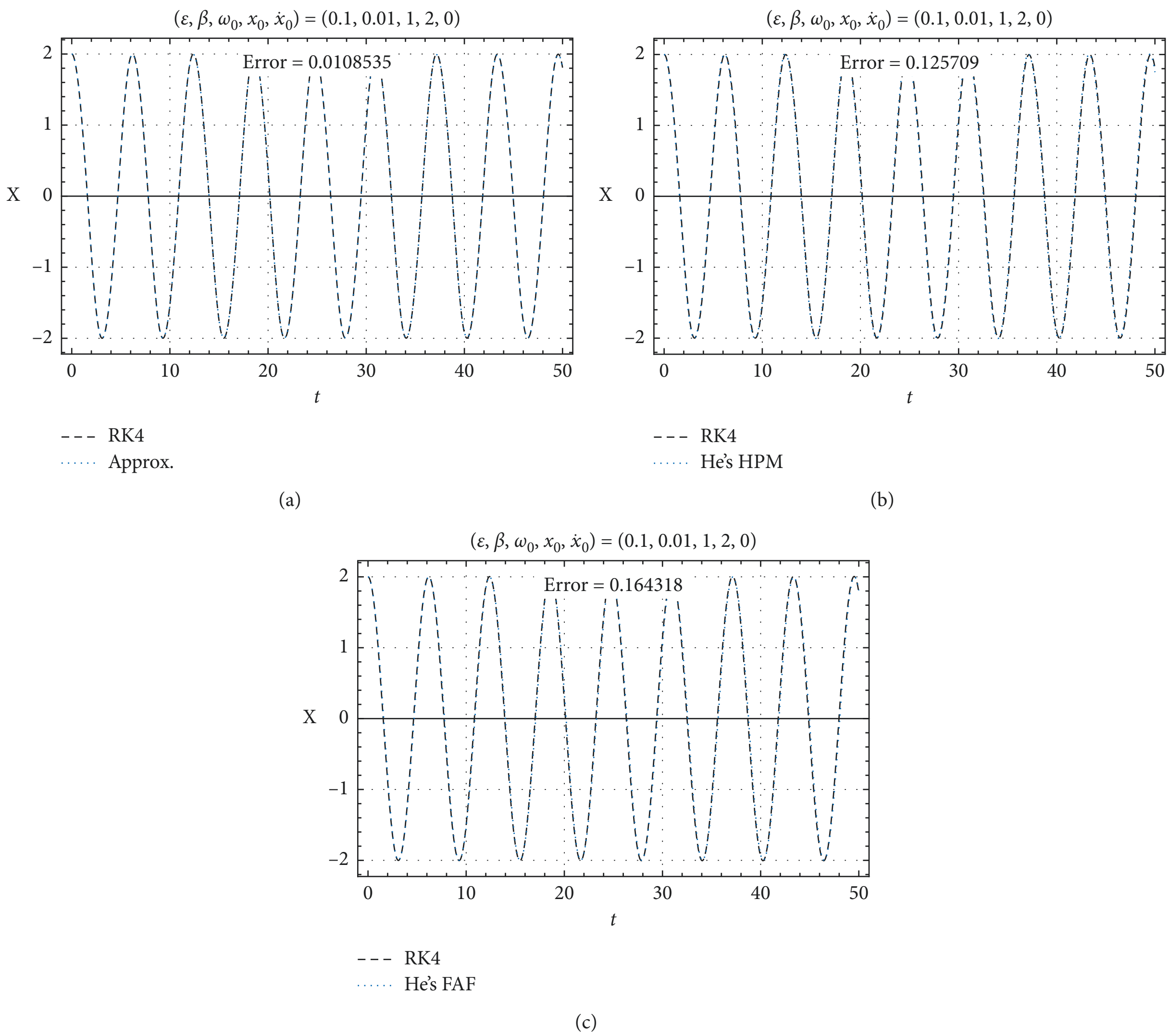

Figure 1: The approximate solutions to the unforced i.v.p. (48) using the analytical approximation (10) and RK numerical approximation as well as He's FAF, He's HPM, and PLT approximations are plotted in the $(x, t)-$ plane.

$$\begin{cases} L_R = \max_{0\le t\le 50}\left|RK - x_{\text{Approx}}\right| = 0.0108535, \\ L_R = \max_{0\le t\le 50}\left|RK - x_{\text{He}'\text{sHPM}}\right| = 0.125709, \\ L_R = \max_{0\le t\le 50}\left|RK - x_{\text{He}'\text{sFAF}}\right| = 0.164318. \end{cases} \tag{49}$$

It is clear that the analytical approximation (10) and RK numerical approximation are very compatible with each other. Also, they are more accurate than He's FAF and He's HPM approximations.

4. Analytical Approximation to the Forced Duffing–Van Der Pol Oscillator

Let us consider the following forced DVdP i.v.p.:

$$\begin{cases} \mathbb{R}_F = \ddot{x} - \varepsilon\left(1 - x^2\right)\dot{x} + \omega_0^2 x + \beta x^3 - F\cos(\Omega t) = 0, \\ x(0) = x_0, \\ x'(0) = \dot{x}_0. \end{cases} \tag{50}$$

Assume that the solution to i.v.p. (50) is given by the following ansatz:

$$x(t) = y(t) + d_0 \cos(\Omega t) + d_1 \sin(\Omega t), \tag{51}$$

where $y \equiv y(t)$ is a solution to the unforced DVdP oscillator

$$\begin{cases} \ddot{y} - \varepsilon\left(1 - x^2\right)\dot{y} + \omega_0^2 y + \beta y^3 = 0, \\ y(0) = \left(x_0 - d_0\right), \\ y\prime(0) = \left(\dot{x}_0 - d_1\Omega\right). \end{cases} \tag{52}$$

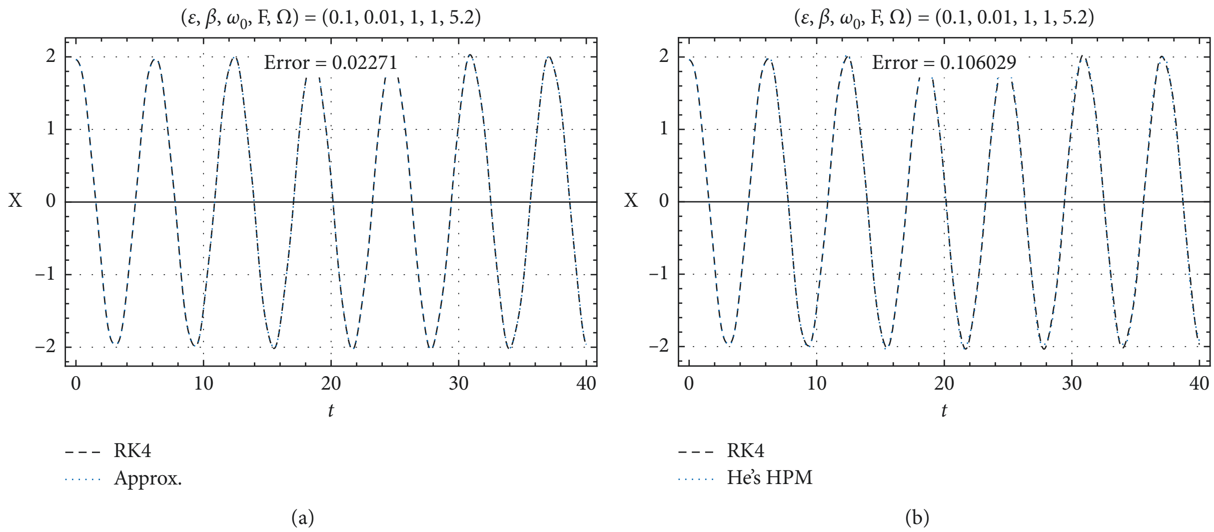

FIGURE 2: The two approximations (51) and (34) according to ICs (35) for the forced i.v.p. (50) are plotted in the $(x, t)-$ plane.

Putting solution (51) into (50), we have

$$\mathbb{R}_F = R\cos(\Omega t) + S\sin(\Omega t) + h.o.t, \tag{53}$$

with

$$\begin{cases} R = \dfrac{3\beta d_0^3}{4} + \dfrac{3}{4}\beta d_1^2 d_0 + \dfrac{1}{4}d_1 d_0^2 \varepsilon\Omega + \dfrac{1}{4}d_1^3\varepsilon\Omega - d_1\varepsilon\Omega + d_0\omega_0^2 - d_0\Omega^2 - F + y(t)^2\left(3\beta d_0 + d_1\varepsilon\Omega\right) + 2d_0\varepsilon y\prime(t)y(t), \\ S = \dfrac{3}{4}\beta d_1 d_0^2 + \dfrac{3\beta d_1^3}{4} - \dfrac{1}{4}d_0^3\varepsilon\Omega - \dfrac{1}{4}d_1^2 d_0\varepsilon\Omega + d_0\varepsilon\Omega + d_1\omega_0^2 - d_1\Omega^2 - y(t)^2\left(3\beta d_1 + d_0\varepsilon\Omega\right) + 2d_1\varepsilon y\prime(t)y(t), \end{cases} \tag{54}$$

where h.o.t. represents higher-order terms. By neglecting $y(t)$ and $y\prime(t)$ from system (54) at $(R, S) = (0, 0)$, then the constants d_0 and d_1 can be determined from the system

$$\begin{cases} \dfrac{3\beta d_0^3}{4} + \dfrac{3}{4}\beta d_1^2 d_0 + \dfrac{1}{4}d_1 d_0^2\varepsilon\Omega + \dfrac{1}{4}d_1^3\varepsilon\Omega - d_1\varepsilon\Omega + d_0\omega_0^2 - d_0\Omega^2 - F = 0, \\ \dfrac{3}{4}\beta d_1 d_0^2 + \dfrac{3\beta d_1^3}{4} - \dfrac{1}{4}d_0^3\varepsilon\Omega - \dfrac{1}{4}d_1^2 d_0\varepsilon\Omega + d_0\varepsilon\Omega + d_1\omega_0^2 - d_1\Omega^2 = 0. \end{cases} \tag{55}$$

From this system, we get

$$Y_0 + Y_1 d_0 + Y_2 d_0^2 + Y_3 d_0^3 = 0, \tag{56}$$

$$Z_0 + Z_1 d_1 + Z_2 d_1^2 + Z_3 d_1^3 = 0, \tag{57}$$

where the values of $Y_i\,(i = 0, 1, 2, 3)$ and $Z_i\,(i = 0, 1, 2, 3)$ are defined in Appendix (iii). We choose the least in magnitude real roots to cubics (56) and (57).

As a numerical example, the two approximations (51) and (34) according to ICs (35) for i.v.p. (50) are displayed in Figure 2 for $(\varepsilon, \omega_0, \beta, F, \Omega) = (0.1, 1, 0.01, 1, 5.2)$. Also, the maximum distance error for the two approximations is estimated as follows:

$$\begin{cases} L_R = \max_{0\le t\le 50}\left|RK - x_{\text{Approx}}\right| = 0.0227101, \\ L_R = \max_{0\le t\le 50}\left|RK - x_{\text{He}'\text{sHMP}}\right| = 0.106029. \end{cases} \tag{58}$$

On the other side, for arbitrary ICs, the analytical approximation (51) versus the RK numerical approximation is presented in Figure 3 for $(\omega_0, \beta, F, \Omega, x_0, \dot{x}_0) = (1, 0.01, 1, 5.2, 0, 0.183)$ and different values to ε. Also, the maximum distance error at the same values of the physical

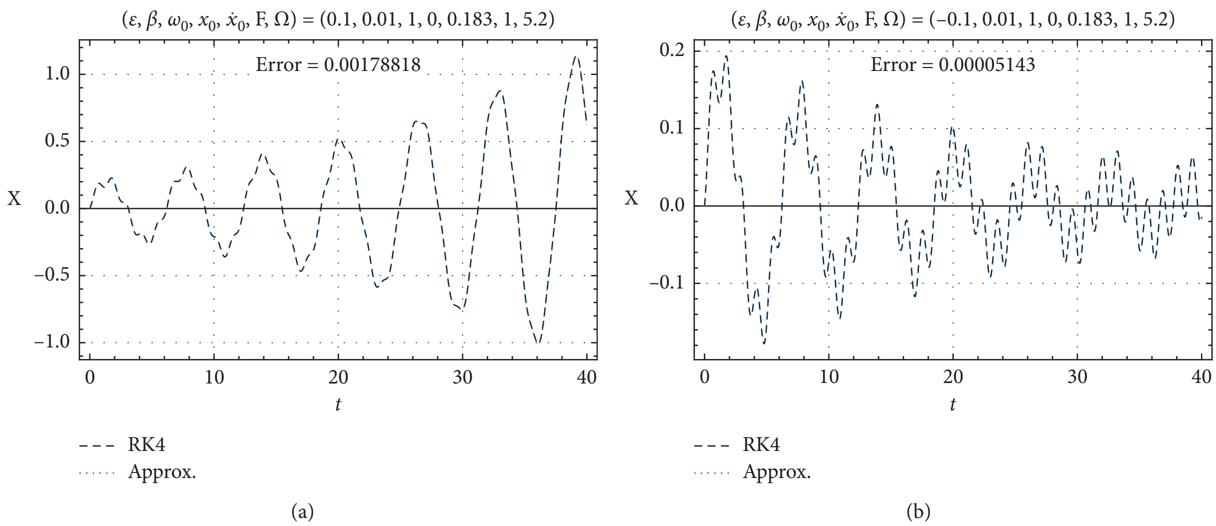

Figure 3: Both analytical approximation (51) and RK numerical approximation to the forced i.v.p. (50) for different values of ε are plotted in the $(x, t)-$ plane.

parameters and ICs mentioned in Figure 3 is estimated as follows:

$$\begin{cases} L_R|_{\varepsilon=0.1} = \max_{0\leq t\leq 40}\left|RK - x_{\text{Approx}}\right| = 0.00178818, \\ L_R|_{\varepsilon,=-0.1} = \max_{0\leq t\leq 40}\left|RK - x_{\text{Approx}}\right| = 0.00005143. \end{cases} \quad (59)$$

We can conclude that in all cases, both two analytical approximations (10) and (51) for the unforced and forced DVdP oscillators are more accurate and convergence as compared to He's FAF and He's HPM.

5. Conclusion

Both higher-order nonlinearity unforced Duffing–Van der Pol (DVdP) oscillator and forced DVdP oscillator having linear and cubic nonlinear terms have been analyzed using some effectiveness and more accurate approaches. The new approach was constructed based on the Krylov–Bogoliubov–Metroolsky method (KBMM). The new approach was discussed in detail for the two issues. In our analysis, we only stopped at the first approximation because it is sufficient in all cases. Also, this new approach can be used for analyzing many strongly nonlinear oscillators. Moreover, the new approach does not demand to solve any ordinary differential equations (odes) because we only simply equate the coefficients of the trigonometric functions $a(t)$ and p to zero in order to get a simple system of algebraic equations. Then, this system becomes very easy to solve it to determine the undetermined coefficients. Also, this new method is also characterized by being direct and fast. Moreover, it is characterized by high-accuracy if it is compared with other methods in the literature. Also, we improved He's frequency-amplitude formulation technique in order to solve unforced DVdP oscillator to obtain high-accurate results. Furthermore, the unforced DVdP oscillator was analyzed via He's homotopy perturbation method. The maximum distance error in the whole time domain with respect to Runge–Kutta numerical approach has been estimated. It was found that our new approach is better than all method approaches. Moreover, the new approach can be devoted for analyzing many strong nonlinearity oscillators with any odd power. Also, the new approach can be applied for arbitrary initial conditions.

Future work: the ansatz that has been used in this paper is called the KBM first-Variant. This approach cannot recover He's amplitude formula. On the other side, we will use another new ansatz which maybe called the KBM second-Variant, in this case, He's amplitude formula can be recovered.

Appendix

(i) The coefficients $S_1 - S_{16}$ of solution (13)

$$
\begin{aligned}
S_1 &= \frac{1}{4}\left(4\varepsilon\omega_0 - 8\omega_0 B_{1,1}\right),\\
S_2 &= \frac{1}{4}\left(-\varepsilon\omega_0 - 8\omega_0 B_{1,2}\right),\\
S_3 &= \frac{1}{4}\left(3\beta - 8\omega_0 C_{1,0}\right),\\
S_4 &= \frac{1}{4}\left(\beta - 32\omega_0^2 r_{3,1,1}\right),\\
S_5 &= \frac{1}{4}\left(-\varepsilon\omega_0 - 32\omega_0^2 s_{3,1,1}\right),\\
S_6 &= \left(-4B_{1,2}C_{1,0} - 2\omega_0 B_{2,3} - \frac{1}{4}\varepsilon C_{1,0} - \frac{1}{4}\varepsilon\omega_0 r_{3,1,1} + \frac{3}{4}\beta s_{3,1,1}\right),\\
S_7 &= \left(\frac{3}{4}\varepsilon B_{1,2} + 3B_{1,2}^2 - 2\omega_0 C_{2,2} - C_{1,0}^2 + \frac{3}{4}\beta r_{3,1,1} + \frac{1}{4}\varepsilon\omega_0 s_{3,1,1}\right),\\
S_8 &= \left(-18\omega_0 B_{1,2} r_{3,1,1} - \frac{1}{4}\varepsilon C_{1,0} - 18\omega_0 C_{1,0} s_{3,1,1} - \frac{3}{2}\varepsilon\omega_0 r_{3,1,1} + \frac{3}{2}\beta s_{3,1,1} - 8\omega_0^2 s_{3,2,2}\right),\\
S_9 &= \left(\frac{1}{4}\varepsilon B_{1,2} + 18\omega_0 B_{1,2} s_{3,1,1} - 18\omega_0 C_{1,0} r_{3,1,1} + \frac{3}{2}\beta r_{3,1,1} - 8\omega_0^2 r_{3,2,2} + \frac{3}{2}\varepsilon\omega_0 s_{3,1,1}\right),\\
S_{10} &= \left(-\frac{5}{4}\varepsilon\omega_0 r_{3,1,1} + \frac{3}{4}\beta s_{3,1,1} - 24\omega_0^2 s_{5,2,2}\right),\\
S_{11} &= \left(\frac{3}{4}\beta r_{3,1,1} - 24\omega_0^2 r_{5,2,2} + \frac{5}{4}\varepsilon\omega_0 s_{3,1,1}\right),\\
S_{12} &= \left(-4B_{1,1}C_{1,0} - 2\omega_0 B_{2,2} + \varepsilon C_{1,0}\right),\\
S_{13} &= \left(\frac{3}{4}\varepsilon B_{1,1} - \varepsilon B_{1,2} + 4B_{1,2}B_{1,1} - 2\omega_0 C_{2,1}\right),\\
S_{14} &= \left(-18\omega_0 B_{1,1} r_{3,1,1} + 3\varepsilon\omega_0 r_{3,1,1} - 8\omega_0^2 s_{3,1,2}\right),\\
S_{15} &= \left(\frac{1}{4}\varepsilon B_{1,1} + 18\omega_0 B_{1,1} s_{3,1,1} - 8\omega_0^2 r_{3,1,2} - 3\varepsilon\omega_0 s_{3,1,1}\right),\\
S_{16} &= \left(-\varepsilon B_{1,1} + B_{1,1}^2 - 2\omega_0 C_{2,0}\right).
\end{aligned}
\tag{A.1}
$$

(ii) The values of W_1 $(i = 1, 2, 3, 4)$ for equation (26)

$$
\begin{aligned}
W_1 &= \left(\omega_0 - \frac{\varepsilon^2}{8\omega_0}\right)t,\\
W_2 &= \frac{-1}{512\mu^2\omega_0^3\left(c_0^2\mu\left(e^{2\kappa t}-1\right)+\kappa\right)}c_0^4\mu^2\left(e^{2\kappa t}-1\right)\left(15\beta^2+7\varepsilon^2\omega_0^2\right),\\
W_3 &= \frac{-c_0^2\mu\left(e^{2\kappa t}-1\right)}{512\mu^2\omega_0^3\left(c_0^2\mu\left(e^{2\kappa t}-1\right)+\kappa\right)}\left(\omega_0^2\left(\left(7\varepsilon^2\kappa-32\mu\left(3\beta+\varepsilon^2\right)\right)\log\left(\frac{c_0^2\mu\left(e^{2\kappa t}-1\right)}{\kappa}+1\right)-7\varepsilon^2\kappa\right)\right.\\
&\quad\left.-15\beta^2\kappa\left(1-\log\left(\frac{c_0^2\mu\left(e^{2\kappa t}-1\right)}{\kappa}+1\right)\right)\right),\\
W_4 &= -\frac{\kappa\left(15\beta^2\kappa+\omega_0^2\left(7\varepsilon^2\kappa-32\mu\left(3\beta+\varepsilon^2\right)\right)\right)\log\left(\left(c_0^2\mu\left(e^{2\kappa t}-1\right)/\kappa\right)+1\right)}{512\mu^2\omega_0^3\left(c_0^2\mu\left(e^{2\kappa t}-1\right)+\kappa\right)}.
\end{aligned}
\tag{B.1}
$$

(iii) The values of Y_i $(i = 0, 1, 2, 3)$ and Z_i $(i = 0, 1, 2, 3)$ for equations (56) and (57)

$$
\begin{aligned}
Y_0 &= 4F\begin{pmatrix} 27F^2\beta^3 - 36\beta^2\varepsilon^2\Omega^4 + 24\beta\varepsilon^2\Omega^6 - 4\varepsilon^2\Omega^8 + 36\beta^2\varepsilon^2\Omega^2\omega_0^2 - 48\beta\varepsilon^2\Omega^4\omega_0^2 \\ +12\varepsilon^2\Omega^6\omega_0^2 + 24\beta\varepsilon^2\Omega^2\omega_0^4 - 12\varepsilon^2\Omega^4\omega_0^4 + 4\varepsilon^2\Omega^2\omega_0^6 \end{pmatrix},\\
Y_1 &= 4\begin{pmatrix} 27F^2\beta^3\Omega^2 + 36F^2\beta^2\varepsilon^2\Omega^2 - 9F^2\beta\varepsilon^2\Omega^4 - 36\beta^2\varepsilon^4\Omega^4 - 36\beta^2\varepsilon^2\Omega^6 + 24\beta\varepsilon^4\Omega^6 \\ +24\beta\varepsilon^2\Omega^8 - 4\varepsilon^4\Omega^8 - 4\varepsilon^2\Omega^{10} - 27F^2\beta^3\omega_0^2 + 9F^2\beta\varepsilon^2\Omega^2\omega_0^2 + 72\beta^2\varepsilon^2\Omega^4\omega_0^2 - 24\beta\varepsilon^4\Omega^4\omega_0^2 \\ -72\beta\varepsilon^2\Omega^6\omega_0^2 + 8\varepsilon^4\Omega^6\omega_0^2 + 16\varepsilon^2\Omega^8\omega_0^2 - 36\beta^2\varepsilon^2\Omega^2\omega_0^4 + 72\beta\varepsilon^2\Omega^4\omega_0^4 - 4\varepsilon^4\Omega^4\omega_0^4 \\ -24\varepsilon^2\Omega^6\omega_0^4 - 24\beta\varepsilon^2\Omega^2\omega_0^6 + 16\varepsilon^2\Omega^4\omega_0^6 - 4\varepsilon^2\Omega^2\omega_0^8 \end{pmatrix},\\
Y_2 &= -8F\varepsilon^2\Omega^2\left(3\beta - \Omega^2 + \omega_0^2\right)\left(9\beta^2 - 6\beta\Omega^2 - \varepsilon^2\Omega^2 + 6\beta\omega_0^2\right),\\
Y_3 &= -F^2\left(9\beta^2 + \varepsilon^2\Omega^2\right)^2,\\
Z_0 &= -4F\varepsilon^3\Omega^3\left(F - 6\beta + 2\Omega^2 - 2\omega_0^2\right)\left(F + 6\beta - 2\Omega^2 + 2\omega_0^2\right),\\
Z_1 &= 4\varepsilon^2\Omega^2\begin{pmatrix} 27F^2\beta^2 - 12F^2\beta\Omega^2 - F^2\varepsilon^2\Omega^2 + 36\beta^2\varepsilon^2\Omega^2 + 36\beta^2\Omega^4 - 24\beta\varepsilon^2\Omega^4 - 24\beta\Omega^6 + 4\varepsilon^2\Omega^6 + 4\Omega^8 + 12F^2\beta\omega_0^2 \\ -72\beta^2\Omega^2\omega_0^2 + 24\beta\varepsilon^2\Omega^2\omega_0^2 + 72\beta\Omega^4\omega_0^2 - 8\varepsilon^2\Omega^4\omega_0^2 - 16\Omega^6\omega_0^2 + 36\beta^2\omega_0^4 - 72\beta\Omega^2\omega_0^4 + 4\varepsilon^2\Omega^2\omega_0^4 \\ +24\Omega^4\omega_0^4 + 24\beta\omega_0^6 - 16\Omega^2\omega_0^6 + 4\omega_0^8 \end{pmatrix},\\
Z_2 &= 8F\varepsilon\Omega\left(3\beta - \Omega^2 + \omega_0^2\right)\left(9\beta^2\Omega^2 + 6\beta\varepsilon^2\Omega^2 - \varepsilon^2\Omega^4 - 9\beta^2\omega_0^2 + \varepsilon^2\Omega^2\omega_0^2\right),\\
Z_3 &= F^2\left(9\beta^2 + \varepsilon^2\Omega^2\right)^2.
\end{aligned}
\tag{C.1}
$$

Data Availability

All data generated or analyzed during this study are included in this published article (more details can be requested from El-Tantawy).

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to this study and approved the final manuscript.

Acknowledgments

The authors expressed their gratitude to Princess Nourah bint Abdulrahman University Researchers Supporting

Project number (PNURSP2022R157), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

References

[1] J. Soukup and J. Volek, *Vibration of Mechanic Systems-Vehicles*, UJEP, Usti nad Labem, Czechia, 2008.
[2] J. R. Ellis, *Road Vehicle Dynamics*, SAE International, Warrendale, PA, USA, 1989.
[3] A. H. Nayfeh and D. T. Mook, *Nonlinear Oscillations*, John Wiley, New York, NY, USA, 1973.
[4] W. Albalawi, A. H. Salas, S. A. El-Tantawy, and A. A. A. R. Youssef, "Approximate analytical and numerical solutions to the damped pendulum oscillator: Newton–Raphson and moving boundary methods," *Journal of Taibah University for Science*, vol. 15, no. 1, pp. 479–485, 2021.
[5] G. M. Moatimid, "Stability analysis of a parametric duffing oscillator," *Journal of Engineering Mechanics*, vol. 146, no. 5, Article ID 05020001, 2020.
[6] J. Kyzioł and A. Okniński, "The duffing–van der Pol equation: metamorphoses of resonance curves," *Nonlinear Dynamics and Systems Theory*, vol. 15, no. 1, pp. 25–31, 2015.
[7] A. Pikovsky, M. Rosenblum, and J. S. Kurths, *A Universal Concept in Nonlinear Sciences*, Cambridge University Press, Cambridge, UK, 2001.
[8] A. H. Salas and S. A. El-Tantawy, "On the approximate solutions to a damped harmonic oscillator with higher-order nonlinearities and its application to plasma physics: semi-analytical solution and moving boundary method," *European Physical Journal A: Hadrons and Nuclei*, vol. 135, no. 10, p. 833, 2020.
[9] A. H. Salas and S. A. El-Tantawy, *Analytical Solutions of Some Strong Nonlinear Oscillators*, IntechOpen, London, UK, 2021.
[10] Y. O. El-Dib, "Multiple scales homotopy perturbation method for nonlinear oscillators," *Nonlinear Sci Lett A*, vol. 8, pp. 352–364, 2017.
[11] A. Salas, J. M. H. Lorenzo, and L. O. R. David, "Analytical and Numerical Study to a Forced Van der," *Pol Oscillator" Mathematical Problems in Engineering*, vol. 2022, Article ID 9736427, 9 pages, 2022.
[12] Ji-H. He, N. Anjum, and P. Skrzypacz, "A variational principle for a nonlinear oscillator arising in the microelectromechanical system," *Journal of Applied and Computational Mechanics*, vol. 7, no. 1, pp. 78–83, 2021.
[13] Ji-H. He, Q. Yang, C.-H. He, and Y. Khan, "A simple frequency formulation for the tangent oscillator," *Axioms*, vol. 10, no. 4, p. 320, 2021.
[14] Ji-H. He and Y. O. El-Dib, "Homotopy perturbation method with three expansions for Helmholtz-Fangzhu oscillator," *International Journal of Modern Physics B*, vol. 35, no. 24, Article ID 2150244, 2021.
[15] Ji-H. He, Y. O. El-Dib, and A. A. Mady, "Homotopy perturbation method for the fractal toda oscillator," *Fractal Fract*, vol. 5, no. 3, p. 93, 2021.
[16] C.-H. He, D. Tian, G. M. Moatimid, H. F. Salman, and M. H. Zekry, "Hybrid Rayleigh–van der pol–duffing oscillator: stability analysis and controller," *Journal of Low Frequency Noise, Vibration and Active Control*, vol. 41, no. 1, pp. 244–268, 2022.
[17] Y. Ueda, "Randomly transitional phenomena in the system governed by Duffing's equation," *Journal of Statistical Physics*, vol. 20, no. 2, pp. 181–196, 1979.
[18] H. G. Enjieu Kadji, J. B. Chabi Orou, and P. Woafo, "Regular and chaotic behaviors of plasma oscillations modeled by a modified Duffing equation," *Physica Scripta*, vol. 77, no. 2, Article ID 025503, 2008.
[19] S. A. El-Tantawy, A. H. Salas, and M. R. Alharthi, "On the analytical solutions of the forced damping duffing equation in the form of weierstrass elliptic function and its applications," *Mathematical Problems in Engineering*, vol. 2021, pp. 1–9, 2021.
[20] N. H. Aljahdaly and S. A. El-Tantawy, "On the multistage differential transformation method for analyzing damping Duffing oscillator and its applications to plasma physics," *Mathematics*, vol. 9, no. 4, p. 432, 2021.
[21] H. S. Alvaro, S. A. El-Tantawy, and H. J. E. Castillo, "On the approximate and analytical solutions to the fifth-order duffing oscillator and its physical applications," *Waves Random Complex Media*, 2021, in Press.
[22] A. H. Salas S, S. A. El-Tantawy, M. R. Alharthi, and M. R. Alharthi, "Novel solutions to the (un)damped Helmholtz-Duffing oscillator and its application to plasma physics: moving boundary method," *Physica Scripta*, vol. 96, no. 10, Article ID 104003, 2021.
[23] S. A. El-Tantawy, A. H. Salas, and M. R. Alharthi, "A new approach for modelling the damped Helmholtz oscillator: applications to plasma physics and electronic circuits," *Communications in Theoretical Physics*, vol. 73, no. 3, Article ID 035501, 2021.
[24] W. Szemplińska-Stupnicka and J. Rudowski, "The coexistence of periodic, almost-periodic and chaotic attractors in the Van der Pol – duffing oscillator," *Journal of Sound and Vibration*, vol. 199, no. 2, pp. 165–175, 1997.
[25] A. Chudzik, P. Perlikowski, A. Stefański, and T. Kapitaniak, "Multistability and rare attractors in van der Pol-Duffing oscillator," *International Journal of Bifurcation and Chaos*, vol. 21, no. 7, pp. 1907–1912, 2011.
[26] C. H. Miwadinou, L. A. Hinvi, A. V. Monwanou, and J. B. Chabi Orou, "Nonlinear dynamics of plasma oscillations modeled by a forced modified Van der Pol-Duffing oscillator," 2013, https://arxiv.org/abs/1308.6132.
[27] C. H. Miwadinou, A. V. Monwanou, J. Yovogan, L. A. Hinvi, P. R. Nwagoum Tuwa, and J. B. Chabi Orou, "Modeling nonlinear dissipative chemical dynamics by a forced modified Van der Pol-Duffing oscillator with asymmetric potential: chaotic behaviors predictions," *Chinese Journal of Physics*, vol. 56, no. 3, pp. 1089–1104, 2018.
[28] A. Kimiaeifar, A. R. Saidi, G. H. Bagheri, M. Rahimpour, and D Domairry, "Analytical solution for van der Pol–duffing oscillators," *Chaos, Solitons & Fractals*, vol. 42, no. 5, pp. 2660–2666, 2009.
[29] Z. Feng, "Duffing-van der Pol-type oscillator systems," *Discrete & Continuous Dynamical Systems—S*, vol. 7, no. 6, pp. 1231–1257, 2014.
[30] Gh. Asadi Cordshooli1 and A. R. Vahidi, "Solutions of duffing-van der Pol equation using decomposition method," *Advanced Studies in Theoretical Physics*, vol. 5, no. 3, pp. 121–129, 2011.
[31] J. H. He, "Some asymptotic methods for strongly nonlinear equations," *International Journal of Modern Physics B*, vol. 20, no. 10, pp. 1141–1199, 2006.

[32] J. H. He, "An improved amplitude-frequency formulation for nonlinear oscillators," *International Journal of Nonlinear Sciences and Numerical Stimulation*, vol. 9, no. 2, pp. 211-212, 2008.

[33] J. H. He, "Comment on He's frequency formulation for nonlinear oscillators," *European Journal of Physics*, vol. 29, no. 4, pp. 19–22, 2008.

[34] J. H. He, "An approximate amplitude-frequency relationship for a nonlinear oscillator with discontinuity," *Nonlinear Science Letters A Mathematics*, vol. 7, pp. 77–85, 2016.

[35] J. H. He, "Amplitude-frequency relationship for conservative nonlinear oscillators with odd nonlinearities," *International Journal of Applied and Computational Mathematics*, vol. 3, no. 2, pp. 1557–1560, 2017.

[36] A. F. N. Rasedee, M. H. A. Sathar, H. M. Ijam, K. I. Othman, N. Ishak, and S. R. Hamzah, "A numerical solution for Duffing-Van Der Pol oscillators using a backward difference formulation," *AIP Conference Proceedings*, vol. 2016, Article ID 020120, 2018.