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James D. Stein

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SOME OBSERVATIONS ON LOCAL UNIFORM BOUNDEDNESS PRINCIPLES

J. D. STEIN Jr., Long Beach

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The Uniform Boundedness Principle for continuous linear maps from Banach spaces into normed spaces is one of the first major consequences of the Baire Category Theorem. One way to do this is to first prove Osgood's Theorem, sometimes known as the local uniform boundedness principle, which states that a pointwise-bounded family of continuous maps of a complete metric space into a metric space must be uniformly bounded on some open subset.

Uniform boundedness principles play an important role in automatic continuity. A basic variation introduced by Pták ([6]) and extended by others ([3], [7]) enables one to derive interesting results for systems theory.

This paper investigates several different aspects of local uniform boundedness principles. In the first section, we prove versions of the Gliding Hump Theorem from automatic continuity ([2]) for complete metric spaces, locally compact Hausdorff, and sequentially compact spaces; including versions based on variations of the Mittag-Leffler Inverse Limit Theorem. The conclusions of the theorems are weaker when the spaces are sequentially compact. In the second section, we show that the the weaker conclusions for sequentially compact spaces reflect the fact that Baire spaces can be characterized by the equivalence between the Baire Category Theorem and a specific version of the local uniform boundedness principle, and thus optimally strong uniform boundedness principles for sequentially compact spaces are unattainable.

SECTION 1. NON-LINEAR GLIDING HUMP THEOREMS

The Gliding Hump Theorem has appeared in many different versions (see [2] and [5]). It is an important result from automatic continuity which numbers among its consequences stronger and more useful versions of the Principle of Uniform Boundedness ([3], [7]). These versions have been shown to be valuable in proving results dealing with the automatic continuity of certain types of operators important in systems theory ([1]).

Although it has recently been shown that the Gliding Hump Theorem is equivalent to versions of a uniform boundedness theorem which trace back to a result of Pták ([6]), the proof of the Gliding Hump Theorem given in ([2]) uses Pták's idea in a very elegant fashion, and has led to speculation that the more elegant proof might lead to more powerful results.

Theorem 1. *Let $\{E_n: n = 1, 2, \ldots\}$ be a sequence of complete metric spaces, and let E_0 be a topological space. Let Y be a topological space covered by a sequence $\{Y_n: n = 1, 2, \ldots\}$ of closed subsets. For $n \geqq 1$,* let *$R_n: E_n \to E_{n-1}$ be continuous and onto. Let $\{T_a: a \in A\}$ be a collection of maps from E_0 into Y. Suppose further that*

[1] *for each $a \in A$, there is an integer n such that $T_a R_1 \ldots R_n$ is continuous;*

[2] *for each $x \in E_0$, there is an integer n such that $T_a x \in Y_n$ for all $a \in A$.*

Then there are integers M and N, and a non-empty open subset U of E_N, such that $T_a R_1 \ldots R_N x \in Y_M$ for all $a \in A$ and $x \in U$.

Proof. Assume the theorem is false, and let $n_1 = 1$. Choose $x_1 \in E_{n_1}$ and $a_1 \in A$ such that $T_{a_1} R_1 x_1 \notin Y_1$. Let W_1 be an open subset of Y such that $T_{a_1} R_1 x_1 \in W_1$, $W_1 \cap Y_1 = \emptyset$. Let V_1 be a neighborhood of x_1 such that diam $V_1 < 1$. Choose an open set U_{11} such that $x_1 \in U_{11} \subset \bar{U}_{11} \subset V_1$.

By [1], we can choose an integer $n_2 > n_1$ such that the map $T_{a_1} R_1 \ldots R_{n_2}$ is continuous. $(R_2 \ldots R_{n_2})^{-1} x_1$ is a nonempty subset of $(R_2 \ldots R_{n_2})^{-1} (U_{11})$, since each R_j is onto. If $x \in (R_2 \ldots R_{n_2})^{-1} x_1$, $R_2 \ldots R_{n_2} x = x_1$, and so we have $T_{a_1} R_1 \ldots R_{n_2} x = T_{a_1} R_1 x_1 \in W_1$. Since $T_{a_1} R_1 \ldots R_{n_2}$ is continuous, $(T_{a_1} R_1 \ldots R_{n_2})^{-1} (W_1)$ is open, and we can therefore choose an open set V_2 in E_{n_2} with diam $V_2 < 1/2$, and $V_2 \subset (T_{a_1} R_1 \ldots R_{n_2})^{-1} (W_1) \cap (R_2 \ldots R_{n_2})^{-1} (U_{11})$.

Since the theorem has been assumed false, choose $a_2 \in A$ and $x_2 \in V_2$ such that $T_{a_2} R_1 \ldots R_{n_2} x_2 \notin Y_2$. Choose an open set $W_2 \subset Y$ such that $T_{a_2} R_1 \ldots R_{n_2} x_2 \in W_2$ and $W_2 \cap Y_2 = \emptyset$.

Since $x_2 \in V_2 \subset (R_2 \ldots R_{n_2})^{-1} (U_{11})$; we see that $R_2 \ldots R_{n_2} x_2 \in U_{11}$. Choose an open set U_{12} such that diam $\bar{U}_{12} < 1/2$, $U_{12} \subset \bar{U}_{12} \subset U_{11}$, and $R_2 \ldots R_{n_2} x_2 \in U_{12}$. Choose an open neighborhood U_{22} of x_2 such that diam $\bar{U}_{22} < 1/2$, $\bar{U}_{22} \subset V_2 \cap (R_2 \ldots R_{n_2})^{-1} (U_{12})$. Then $R_2 \ldots R_{n_2}$ maps U_{22} into U_{12}. Note that $T_{a_1} R_1 \ldots R_{n_2}: U_{22} \to W_1$.

At the completion of step p of the induction, assume we have chosen integers $1 = n_1 < \ldots < n_p$; $a_1, \ldots, a_p \in A$; open subsets $W_1, \ldots, W_p$ of Y such that $W_k \cap Y_k = \emptyset$ for $1 \leqq k \leqq p$, and open subsets U_{jk} of E_{n_j}, for $j \leqq k \leqq p$, as well as points $x_1 \in E_{n_1}, \ldots, x_p \in E_{n_p}$, such that the following properties all hold

(1) $\quad$ diam $\bar{U}_{jk} < 1/k$ for $j \leqq k \leqq p$

(2) $\quad U_{jk} \subset \bar{U}_{jk} \subset U_{j,k-1}$ for $j < k \leqq p$

(3) $\quad R_{n_i+1} \ldots R_{n_k}(U_{kj}) \subset U_{ij}$ for $i < k \leqq j \leqq p$

(4) $\quad T_{a_{k-1}} R_1 \ldots R_{n_k}: U_{kk} \to W_{k-1}$ for $1 < k \leqq p$

(5) $\qquad T_{a_k}R_1 \dots R_{n_k+1}$ is continuous for $1 \leqq k < p$

(6) $\qquad T_{a_k}R_1 \dots R_{n_k}x_k \in W_k$ for $1 \leqq k \leqq p$

(7) $\qquad U_{kk}$ is a neighborhood of x_k for $1 \leqq k \leqq p$.

We show that we can proceed with the induction. Begin by choosing an integer $n_{p+1} > n_p$ such that $T_{a_p}R_1 \dots R_{n_{p+1}}$ is continuous. If $x \in (R_{n_p+1} \dots R_{n_{p+1}})^{-1} x_p$, and such an x must exist by the assumption that the R_j are onto, then if $j \leqq p$, we have $R_{n_j+1} \dots R_{n_{p+1}}x = R_{n_j+1} \dots R_{n_p}R_{n_p+1} \dots R_{n_{p+1}}x = R_{n_j+1} \dots R_{n_p}x_p \in R_{n_j+1} \dots \dots R_{n_p}(U_{pp}) \subset U_{jp}$ by (3). Also, $T_{a_p}R_1 \dots R_{n_{p+1}}x = T_{a_p}R_1 \dots R_{n_p}R_{n_p+1} \dots R_{n_{p+1}}x = T_{a_p}R_1 \dots R_{n_p}x_p \in W_p$ by (6). We can conclude that $Q_{p+1} = (T_{a_p}R_1 \dots R_{n_{p+1}})^{-1} . (W_p) \cap \bigcap_{j=1}^{p} (R_{n_j+1} \dots R_{n_{p+1}})^{-1} (U_{jp})$ is both open and non-empty.

Therefore, choose an open subset V_{p+1} of Q_{p+1} defined above such that diam $V_{p+1} < 1/(p+1)$. Since the theorem has been assumed false, choose $x_{p+1} \in V_{p+1}$ and $a_{p+1} \in A$ such that $T_{a_{p+1}}R_1 \dots R_{n_{p+1}}x_{p+1} \notin Y_{p+1}$. Now choose an open subset W_{p+1} of Y such that $T_{a_{p+1}}R_1 \dots R_{n_{p+1}}x_{p+1} \in W_{p+1}$, and such that $W_{p+1} \cap Y_{p+1} = \emptyset$.

Since $x_{p+1} \in V_{p+1}$, $R_2 \dots R_{n_{p+1}}x_{p+1} \in U_{1p}$. Choose a neighborhood $U_{1,p+1}$ in $E_1 = E_{n_1}$ such that $R_2 \dots R_{n_{p+1}}x_{p+1}$ is a member of $U_{1,p+1} \subset \bar{U}_{1,p+1} \subset U_{1p}$, and with diam $\bar{U}_{1,p+1} < 1/(p+1)$. Again, since $x_{p+1} \in V_{p+1}$, $R_{n_2+1} \dots R_{n_{p+1}}x_{p+1} \in U_{2p}$. But $R_2 \dots R_{n_2}R_{n_2+1} \dots R_{n_{p+1}}x_{p+1} = R_2 \dots R_{n_{p+1}}x_{p+1} \in U_{1,p+1}$, so $R_{n_2+1} \dots \dots R_{n_{p+1}}x_{p+1} \in U_{2p} \cap (R_2 \dots R_{n_2})^{-1} (U_{1,p+1})$. Choose a neighborhood $U_{2,p+1}$ in E_{n_2} with diam $\bar{U}_{2,p+1} < 1/(p+1)$, $R_{n_2+1} \dots R_{n_{p+1}}x_{p+1} \in U_{2,p+1}$, $U_{2,p+1} \subset \bar{U}_{2,p+1} \subset U_{2p}$, and $R_2 \dots R_{n_2}(U_{2,p+1}) \subset U_{1,p+1}$. Continuing, since $x_{p+1} \in V_{p+1}$, $R_{n_3+1} \dots \dots R_{n_{p+1}}x_{p+1} \in U_{3p}$. As before, since we also have $R_{n_2+1} \dots R_{n_3}R_{n_3+1} \dots R_{n_{p+1}}x_{p+1} = R_{n_2+1} \dots R_{n_{p+1}}x_{p+1}$, and this latter element belongs to $U_{2,p+1}$, we can continue this process of backtracking to construct $U_{j,p+1}$ for $1 \leqq j \leqq p$ which satisfy properties (1)–(3). We must still construct $U_{p+1,p+1}$.

The backtracking process outlined above finishes with the element $R_{n_p+1} \dots \dots R_{n_{p+1}}x_{p+1} \in U_{p,p+1}$. Since $x_{p+1} \in V_{p+1}$, choose a neighborhood $U_{p+1,p+1}$ of x_{p+1} with $\bar{U}_{p+1,p+1} \subset V_{p+1}$, $R_{n_p+1} \dots R_{n_{p+1}} : U_{p+1,p+1} \to U_{p,p+1}$, and also such that diam $\bar{U}_{p+1,p+1} < 1/(p+1)$. It will be seen that properties (1)–(7) above still hold, and the induction is complete.

The restriction on the diameters of the closed sets $\bar{U}_{pj}$ insures that there is a unique element $z_p \in \bigcap_{j=p+1}^{\infty} \bar{U}_{pj}$. If $k < p$, $R_{n_k+1} \dots R_{n_p}z_p \in \bigcap_{j=p+1}^{\infty} R_{n_k+1} \dots R_{n_p}(\bar{U}_{pj})$. Property (2) shows that this is a subset of $\bigcap_{j=p+1}^{\infty} R_{n_k+1} \dots R_{n_p}(U_{p,j-1})$, and by property (3) this is contained in $\bigcap_{j=p+1}^{\infty} U_{k,j-1} = \{z_k\}$. Therefore, $k < p \Rightarrow R_{n_k+1} \dots R_n z_p = z_k$.

Let $z_0 = R_1 z_1$; then since $z_1 = R_2 \dots R_{n_p}z_p$, we have $z_0 = R_1 \dots R_{n_p}z_p$. Since $z_p \in U_{pp}$, if $p > 1$ we have $T_{a_{p-1}}z_0 = T_{a_{p-1}}R_1 \dots R_{n_p}z_p \in W_{p-1}$, by property (4).

But W_{p-1} and Y_{p-1} are disjoint, and so $T_{a_{p-1}} z_0 \notin Y_{p-1}$ for $p > 1$, and this contradicts hypothesis [2]. ■

The surjectivity of the maps $\{R_j: j = 1, 2, \ldots\}$ can be replaced by a somewhat artificial-appearing combination of conditions on the maps $\{R_j: j = 1, 2, \ldots\}$ and $\{T_a: a \in A\}$. Given an open subset U of E and an open subset W of Y, suppose that $x \in U$ and $T_a R_1 \ldots R_j x \in W$. Then, for infinitely many k, $(R_{j+1} \ldots R_k)^{-1}(U) \cap \cap (T_a R_1 \ldots R_k)^{-1}(W)$ is non-empty. The existence of an element in this set was necessary to assure that the induction could continue, whereas the requirement that there exist infinitely many k such that this condition is satisfied is made to ensure that one can find a k so large that the second set in the intersection above will be open.

The above condition is satisfied if all the $\{R_j: j = 1, 2, \ldots\}$ are onto. If $k > j$ and $z \in (R_{j+1} \ldots R_k)^{-1} x$, then $T_a R_1 \ldots R_k z = T_a R_1 \ldots R_j R_{j+1} \ldots R_k z = T_a R_1 \ldots \ldots R_k x$, and so $z \in (R_{j+1} \ldots R_k)^{-1}(U) \cap (T_a R_1 \ldots R_k)^{-1}(W)$.

From the standpoint of systems theory, interest is focused not so much on the fact that an operator is continuous, but that it is bounded, as a signal processor which acts as a bounded operator exhibits amplitude-dependent response to input signals. Recent scientific developments have heightened the interest in non-linear phenomena, and uniform boundedness principles for nonlinear operators might well have useful applications.

It is possible to prove a Gliding Hump Theorem similar to the one above for sequentially compact spaces. However, this theorem differs notably from Theorem 1 in one very important respect; the range space Y is required to be a metric space. This obviously prompts the question: is this restriction necessary? In Section 2 we shall show that this restriction, or some other similar restriction, is indeed necessary.

Theorem 2. *Let $\{E_n: n = 1, 2, \ldots\}$ be sequentially compact spaces, and let E_0 be a topological space. Let Y be a metric space. For $n \geqq 1$, let $R_n: E_n \to E_{n-1}$ be continuous and onto. Let $\{T_a: a \in A\}$ be a collection of maps from E_0 into Y. Let $y_0 \in Y$. Suppose further that*

[1] *for each $a \in A$, there is an integer n such that $T_a R_1 \ldots R_n$ is continuous*

[2] *for each $x \in E_0$, $\sup\{d(T_a x, y_0): a \in A\} < \infty$.*

Then there is an integer N and a non-empty open subset U of E_N such that $\sup\{d(T_a R_1 \ldots R_N x, y_0): a \in A, x \in U\} < \infty$.

Proof. Not unsurprisingly, the proof is similar to the proof of Theorem 1, but is mercifully shorter. Assume the result is false. Let $n_1 = 1$, and choose $x_1 \in E_{n_1}$, $a_1 \in A$ such that $d(T_{a_1} R_1 x_1, y_0) > 2$.

Suppose that $n_1 < \ldots < n_p$; $a_1, \ldots, a_p \in A$, and elements $x_1 \in E_{n_1}, \ldots, x_p \in E_{n_p}$ have been chosen. Choose $n_{p+1} > n_p$ such that $T_{a_p} R_1 \ldots R_{n_{p+1}}$ is continuous. We assume that, for $1 \leqq j < p$, $T_{a_j} R_1 \ldots R_{n_{j+1}}$ is continuous.

Since each R_j is onto, let $u \in (R_{n_p+1} \dots R_{n_{p+1}})^{-1} x_p$. Then, for $1 \leqq j \leqq p$,

$$T_{a_j} R_1 \dots R_{n_{p+1}} u = T_{a_j} R_1 \dots R_{n_p} R_{n_p+1} \dots R_{n_{p+1}} u = T_{a_j} R_1 \dots R_{n_p} x_p .$$

Note that, if $1 \leqq j \leqq p$, $T_{a_j} R_1 \dots R_{n_{j+1}}$ is continuous, and so $T_{a_j} R_1 \dots R_{n_{p+1}}$ is continuous. Therefore, for $1 \leqq j \leqq p$, choose a neighborhood $V_{p+1,j}$ of u such that for each $x \in V_{p+1,j}$, we have $d(T_{a_j} R_1 \dots R_{n_{p+1}} x,\ T_{a_j} R_1 \dots R_{n_p} x_p) < 1/2^{p+1}$. Let $U_{p+1} = \bigcap_{j=1}^{p} V_{p+1,j}$; note that $u \in U_{p+1}$ and U_{p+1} is open. By assumption, we can choose $a_{p+1} \in A$ and $x_{p+1} \in U_{p+1}$ such that $d(T_{a_{p+1}} R_1 \dots R_{n_{p+1}} x_{p+1}, y_0) > p + 2$.

Now observe that, for any integer p,

$$\begin{aligned} d(T_{a_p} R_1 \dots R_{n_p} x_p, y_0) &< d(T_{a_p} R_1 \dots R_{n_{p+1}} x_{p+1}, y_0) + \\ &+ d(T_{a_p} R_1 \dots R_{n_p} x_p,\ T_{a_p} R_1 \dots R_{n_{p+1}} x_{p+1}) \leqq \dots \\ \dots &\leqq \sum_{j=p}^{k-1} d(T_{a_p} R_1 \dots R_{n_j} x_j,\ T_{a_p} R_1 \dots R_{n_{j+1}} x_{j+1}) + \\ &+ d(T_{a_p} R_1 \dots R_{n_k} x_k, y_0) < \sum_{j=p}^{k-1} 1/2^{j+1} + d(T_{a_p} R_1 \dots R_{n_k} x_k, y_0) < \\ &< 1 + d(T_{a_p} R_1 \dots R_{n_k} x_k, y_0) . \end{aligned}$$

Therefore, if $k > p$, then $d(T_{a_p} R_1 \dots R_{n_k} x_k, y_0) > p$.

Since E_1 is sequentially compact, choose a sequence $\{k_j : j = 1, 2, \dots\}$ and $z_1 \in E_1$ such that $R_2 \dots R_{n_{k_j}} x_{k_j} \to z_1$. Let $x_0 = R_1 z_1$; then $R_1 \dots R_{n_{k_j}} x_{k_j} \to R_1 z_1 = x_0$. For any integer p, since $E_{n_{p+1}}$ is sequentially compact, we can choose a subsequence $\{k_{j_l} : l = 1, 2, \dots\}$ such that

$$R_{n_{p+1}+1} \dots R_{n_{k_{j_l}}} x_{k_{j_l}} \to z_{p+1} \in E_{n_{p+1}} .$$

Applying the continuous map $R_1 \dots R_{n_{p+1}}$ to both sides of the limit above, we obtain $R_1 \dots R_{n_{p+1}} z_{p+1} = x_0$. Additionally, $T_{a_p} R_1 \dots R_{n_{k_{j_l}}} x_{k_{j_l}} \to T_{a_p} R_1 \dots R_{n_{p+1}} z_{p+1} = T_{a_p} x_0$. But $d(T_{a_p} R_1 \dots R_{n_{k_{j_l}}} x_{k_{j_l}}, y_0) > p$ if $k_{j_l} > p$. Therefore $d(T_{a_p} x_0, y_0) > p$ for any integer p, a contradiction. ■

Notice that, if $E_0 = E_1 = \dots = E_n = \dots$ and each R_j is the identity map, then both Theorems 1 and 2 yield Osgood's Theorem.

Although one would like to prove Theorems 1 and 2 without assuming that the maps $\{R_j : j = 1, 2, \dots\}$ are onto, such a result is too strong to hold. In fact, one cannot even prove Theorem 1 or Theorem 2 under the assumption that each of the continuous maps R_n has closed range.

Suppose that such a result holds, and that $\{T_a : a \in A\}$ is a pointwise-bounded family of maps of a complete metric space X into a space Y such that each T_a is continuous on a closed subset S_a. We assert that $\{T_a : a \in A\}$ is uniformly bounded on a relatively open subset of a finite intersection of the $\{S_a : a \in A\}$. If not, we can find a sequence $\{T_{a_n} : n = 1, 2, \dots\}$ which is not uniformly bounded on any relatively open subset of a finite intersection of the $\{S_a\colon a \in A\}$. Let $E_0 = X$, and let $E_n = \bigcap_{k=1}^{n} S_{a_k}$. Let R_n denote the injection map; obviously, this map has closed range. It is easy to check that all the hypotheses of the conjectured version of Theorem 1 hold, using the family $\{T_{a_n}\colon n = 1, 2, \ldots\}$ yields a contradiction.

However, the following counterexample to the above result shows that the hypothesized version of Theorem 1 cannot hold.

Let $X = [0, 1]$, and let $\{r_n\colon n = 1, 2, \ldots\}$ denote an ordering of the rationals in X. For each integer n, let T_n map X into the reals as follows:

$$T_n x = 0 \quad \text{if} \quad 0 \leqq x \leqq 1/n \quad \text{or } x \text{ is irrational},$$

$$T_n x = 0 \quad \text{if} \quad x = r_k, \quad x > 1/n, \quad \text{and} \quad k < n,$$

$$T_n x = k \quad \text{if} \quad x = r_k, \quad x > 1/n, \quad \text{and} \quad k \geqq n.$$

Let $S_n = [0, 1/n]$; $T_n \mid S_n = 0$, and so is continuous. $\{S_n\colon n = 1, 2, \ldots\}$ is closed under finite intersections. If x is irrational, $T_n x = 0$ for all n, and if x is rational, $T_n x = 0$ for all but finitely many n, and so $\{T_n\colon n = 1, 2, \ldots\}$ is pointwise-bounded. Given any N and any open subset U of S_N, choose p such that $U \setminus [0, 1/p]$ contains infinitely many rationals; if r_k is such a rational and $k > p$, then $T_k r_k = k$, and consequently $\{T_n\colon n = 1, 2, \ldots\}$ cannot be uniformly bounded on S_N. This suggests that linearity of the maps and spaces, or some analogous condition, is not just sufficient to prove the Gliding Hump Theorem, but might be necessary as well.

Another interesting situation arises when the ranges of the maps R_n are dense. We give a short proof of the Mittag-Leffler Inverse Limit Theorem for sequences of complete metric spaces or locally compact Hausdorff spaces; the theorem for complete metric spaces appears in Esterle ([4], Theorem 2.14). We then give an application of this theorem to local uniform boundedness.

Theorem 3. (Mittag-Leffler Inverse Limit Theorem) *Let $\{E_n\colon n = 0, 1, \ldots\}$ be a sequence of* (a) *complete metric spaces, or* (b) *locally compact Hausdorff spaces, and let $R_n : E_n \to E_{n-1}$ be a continuous map with dense range. Then there exists a dense subset X_0 of E_0 such that, for each $x \in X_0$, there is a sequence of points $\{x_n\colon n = 1, 2, \ldots, x_n \in E_n\}$, such that $R_n(x_n) = x_{n-1}$ for $n = 1, 2, \ldots$.*

Proof. (a) We give a proof which uses the same basic idea as the one given in ([4]), but incorporates ideas used in Theorems 1 and 2; it is included so that the paper may be self-contained.

Let V be an open subset of E_0. Since R_1 has dense range, choose $\{U_{k1}\colon k = 0, 1\}$ such that $U_{k1} \subset E_k$, $U_{01} \subset V$, and also diam $\overline{U}_{k1} < 1$ for $k = 0, 1$, and $R_1(U_{11}) \subset U_{01}$.

Assume that, after p steps, we have constructed nested open subsets $\{U_{nk}\colon n \leqq k \leqq p\}$ in E_n such that $\overline{U}_{n,k+1} \subset U_{nk}$ for $n \leqq k < p$, diam $\overline{U}_{nk} < 1/k$, and $R_n(U_{nk}) \subset U_{n-1,k}$ as long as $n \leqq k \leqq p$. Since the range of R_{p+1} is dense in E_p, choose an element $u_{p+1} \in E_{p+1}$ such that $u_p = R_{p+1} u_{p+1} \in U_{pp}$. Then $u_{p-1} = R_p u_p \in$

$\in U_{p-1,p}, \ldots, u_0 = R_1 u_1 \in U_{0p}$. Backtracking through this chain of elements, we can find neighborhoods $U_{k,p+1}$ of u_k for $0 \leqq k \leqq p + 1$ such that diam $\bar{U}_{k,p+1} < 1/(p + 1)$, $\bar{U}_{k,p+1} \subset U_{kp}$ for $1 \leqq k \leqq p$, and $R_k(U_{k,p+1}) \subset U_{k-1,p+1}$ for $1 \leqq k \leqq p + 1$.

Let $z_p = \bigcap_{k=p}^{\infty} \bar{U}_{pk}$. Then $R_p z_p \in \bigcap_{k=p}^{\infty} R_p(U_{pk}) \subset \bigcap_{k=p}^{\infty} U_{p-1,k} = z_{p-1}$ if $p \geqq 1$. Since $z_0 \in V$, the theorem is proved.

(b) Let U be an open subset of E_0. Choose an open subset V_0 of E_0 such that $\bar{V}_0$ is compact and $\bar{V}_0 \subset U$. Choose a point $x_0 \in V_0 \cap R_1(E_1)$, and a point $x_1 \in E_1$ such that $R_1 x_1 = x_0$. Choose an open neighborhood U_1 of x_1 such that $R_1(U_1)$ is a subset of V_0. Now choose an open neighborhood V_1 of x_1 with compact closure such that $\bar{V}_1 \subset U_1$. Continus inductively to obtain a sequence $\{\bar{V}_n : n = 0, 1, \ldots\}$ of non-empty compact sets such that $\bar{V}_0 \subset U$, $\bar{V}_n \subset E_n$, and $R_n(\bar{V}_n) \subset \bar{V}_{n-1}$ for $n = 0, 1, \ldots$.

Order the family of sequences $\{K_n : n = 0, 1, \ldots\}$ of nonempty compact sets having the properties stated in the preceding paragraph by $\{K_n : n = 1, 2, \ldots\} \leqq \{J_n : n = 1, 2, \ldots\}$ if and only if $K_n \supset J_n$ for all n (equality of the sequences occurs precisely when equality holds for all n). The preceding paragraph shows that this collection is non-empty.

Let A be a set indexing a linearly-ordered subset of this family; for each $a \in A$, the sequence of compact sets is given by $\{K_{an} : n = 0, 1, \ldots\}$. For each n, let $F_n = \bigcap_{a \in A} K_{an}$. The sequence $\{F_n : n = 0, 1, \ldots\}$ is clearly an upper bound for the linearly-ordered subset. If $x \in F_n$ for $n > 0$, then $x \in K_{an}$ for each $a \in A$; consequently $R_n x \in K_{a,n-1}$ for each $a \in A$, and so $R_n x \in F_{n-1}$. The finite intersection property shows that each F_n is non-empty; in a Hausdorff space the intersection of compact sets is compact. Therefore, $\{F_n : n = 1, 2, \ldots\}$ is a member of the family, and we can apply Zorn's Lemma to obtain a maximal sequence $\{M_n : n = 0, 1, \ldots\}$ of non-empty compact sets.

The maximality of the sequence insures that R_n maps M_n onto M_{n-1}, for if some R_N is not onto, let $Q_n = M_n$ for $n \geqq N$, and let $Q_{N-1} = R_N(Q_N), \ldots, Q_0 = R_1(Q_1)$. The continuity of each R_n guarantees that each Q_n is compact, and if M_N is not onto, the sequence $\{Q_n : n = 0, 1, \ldots\}$ contradicts the maximality of the sequence $\{M_n : n = 0, 1, \ldots\}$.

If we can show that each M_n is a singleton, the proof will be complete, so suppose that M_N is not a singleton. Let $u \in M_N$, and let $Q_N = \{u\}$. Let $Q_{N+1} = M_{N+1} \cap R_{N+1}^{-1}(Q_N)$; since R_{N+1} maps M_{N+1} onto M_N, Q_{N+1} is a non-empty subset of M_{N+1}. Since we are now effectively working with continuous maps from and to compact Hausdorff spaces, Q_{N+1} must also be compact. Having now defined Q_n for $n > N$, let $Q_{n+1} = M_{n+1} \cap R_{n+1}^{-1}(Q_n)$; the same arguments show Q_{n+1} is a non-empty compact subset of M_{n+1}. If $N = 0$ we are finished; otherwise, define $Q_{N-1} = R_N(Q_N), \ldots, Q_0 = R_1(Q_1)$. The sequence $\{Q_n : n = 0, 1, \ldots\}$ contradicts the maximality of $\{M_n : n = 0, 1, \ldots\}$, completing the proof. ∎

We now prove a uniform boundedness result using this theorem.

Theorem 4. *Let $\{E_n: n = 0, 1, \ldots\}$ be a sequence of either complete metric or locally compact Hausdorff spaces, and assume that, for $n \geqq 1$, $R_n: E_n \to E_{n-1}$ is a continuous map with dense range. Let Y be a topological space, $\{Y_n: n = 1, 2, \ldots\}$ an increasing closed cover of Y. Let $\{T_a: a \in A\}$ be a pointwise-bounded family of maps of E_0 into Y. Assume that, for each $a \in A$, there is an integer n such that $T_a R_1 \ldots R_n$ is continuous. Then*

(a) *there is a dense subset X_0 of E_0 and an open subset U_0 of E_0 such that $\{T_a: a \in A\}$ is uniformly bounded on $X_0 \cap U_0$.*

(b) *If each R_n is onto, then X_0 can be chosen to be E_0.*

Proof. We prove both (a) and (b) simultaneously, choosing as the dense subset X_0 the set guaranteed by the Mittag-Leffler Inverse Limit Theorem. Note that $X_0 = E_0$ if each R_n is onto. We start with the case where the sets are complete metric spaces.

I Assume the theorem is false. Choose $x_1 \in X_0$ and $a_1 \in A$ such that $T_{a_1} x_1 \notin Y_1$; choose an open $W_1 \subset Y$ such that $W_1 \cap Y_1 = \emptyset$ and $T_{a_1} x_1 \in W_1$. Choose n_1 so $T_{a_1} R_1 \ldots R_{n_1}$ is continuous. Choose a neighborhood U_{01} of x_1 with diam $\bar{U}_{01} < 1$; since $x_1 \in X_0$ choose $u_1 \in E_{n_1}$ such that $R_1 \ldots R_{n_1} u_1 = x_1$. Then $T_{a_1} R_1 \ldots R_{n_1} u_1 = T_{a_1} x_1 \in W_1$. Since both $R_1 \ldots R_{n_1}$ and $T_{a_1} R_1 \ldots R_{n_1}$ are continuous, choose a neighborhood U_{11} of of u_1 such that diam $\bar{U}_{11} < 1$, $R_1 \ldots R_{n_1}(U_{11}) \subset U_{01}$, and $T_{a_1} R_1 \ldots R_{n_1}(U_{11}) \subset W_1$. Let $n_0 = 0$.

After p steps, we have chosen integers $n_0 < \ldots < n_p$, indices $a_1, \ldots, a_p \in A$, open subsets $W_1, \ldots, W_p$ of Y such that $W_k \cap Y_k = \emptyset$ for $1 \leqq k \leqq p$, and a nested collection of open subsets $\{U_{jk}: 0 \leqq j \leqq k \leqq p\}$ such that $U_{jk} \subset E_{n_j}$, diam $\bar{U}_{jk} < 1/k$, and

(1) $$U_{jk} \subset \bar{U}_{jk} \subset U_{j,k-1} \quad \text{for} \quad j < k \leqq p,$$

(2) $$R_{n_i+1} \ldots R_{n_k}(U_{kj}) \subset U_{ij} \quad \text{for} \quad i < k \leqq j \leqq p,$$

(3) $$T_{a_k} R_1 \ldots R_{n_k}: U_{kk} \to W_k \quad \text{for} \quad 1 \leqq k \leqq p,$$

(4) $$T_{a_k} R_1 \ldots R_{n_k} \quad \text{is continuous for} \quad 1 \leqq k \leqq p.$$

Using property (2), choose $x_{p+1} \in U_{0p} \cap X_0$ and $a_{p+1} \in A$ such that $T_{a_{p+1}} x_{p+1} \notin Y_{p+1}$, and such that x_{p+1} is the image under $R_1 \ldots R_{n_1}$ of a point in $U_{1p}, \ldots,$ which is the image under $R_{n_{p-1}} \ldots R_{n_p}$ of a point in U_{pp}. Let W_{p+1} be open in Y such that $T_{a_{p+1}} x_{p+1} \in W_{p+1}$ and $W_{p+1} \cap Y_{p+1} = \emptyset$. Now choose $n_{p+1} > n_p$ such that $T_{a_{p+1}} R_1 \ldots R_{n_{p+1}}$ is continuous. Choose a neighborhood $U_{0,p+1}$ of x_{p+1} such that diam $\bar{U}_{0,p+1} < 1/(p+1)$ and $\bar{U}_{0,p+1} \subset U_{0p}$. Since x_{p+1} is the image of a point in U_{1p}, choose an open subset $U_{1,p+1}$ of diameter $< 1/(p+1)$, and such that $\bar{U}_{1,p+1} \subset U_{1p}$ and $R_1 \ldots R_{n_1}(U_{1,p+1}) \subset U_{0,p+1}$. This can be continued back to obtain $U_{k,p+1}$, for $1 \leqq k \leqq p$, with the appropriate properties. There exist points $z_1 \in U_{1,p+1}, \ldots, z_p \in U_{p,p+1}$ for which $R_1 \ldots R_{n_1} z_1 = x_{p+1}$, and also such that $R_{n_k+1} \ldots R_{n_{k+1}} z_{k+1} = z_k$ for $1 \leqq k < p$. Since $x_{p+1} \in X_0$, we can choose $z_{p+1} \in E_{n_{p+1}}$

such that $R_{n_p+1} \ldots R_{n_{p+1}} z_{p+1} = z_p$. We note that $T_{a_{p+1}} R_1 \ldots R_{n_{p+1}} z_{p+1} = T_{a_{p+1}} x_{p+1} \in$ $\in W_{p+1}$. By the continuity of $T_{a_{p+1}} R_1 \ldots R_{n_{p+1}}$ and $R_{n_p+1} \ldots R_{n_{p+1}}$, choose a neighborhood $U_{p+1,p+1}$ of z_{p+1} of diameter $<1/(p+1)$ such that $T_{a_{p+1}} R_1 \ldots$ $\ldots R_{n_{p+1}}(U_{p+1,p+1}) \subset W_{p+1}$, and which is mapped by $R_{n_p+1} \ldots R_{n_{p+1}}$ into $U_{p,p+1}$. The denouement now follows as in Theorem 1.

The proof for locally compact Hausdorff spaces combines the elements of the above proof for complete metric spaces with the method of proving the Mittag-Leffler Inverse Limit Theorem for locally compact Hausdorff spaces. We simply sketch the idea. Instead of being able to control the diameters of the closures of certain open sets, as we could in a metric space, we instead require the open sets to have compact closures. We then obtain (using most of the notation from the proof above for complete metric spaces) compact sets $\{K_j: j = 1, 2, \ldots\}$, with $K_j \subset E_{n_j}$, such that $T_{a_j}(K_j) \subset W_j$ and $R_{n_i+1} \ldots R_{n_j}(K_j) \subset K_i$. As in the proof of the Mittag-Leffler Theorem for locally compact Hausdorff spaces, order all such sequences $\{K_j: j = 1, 2, \ldots\}$ of compact sets. Zorn's Lemma can again be used to extract a maximal element, which will be a sequence of singletons. The singleton in this sequence belonging to E_0 will fail to be pointwise-bounded under $\{T: j = 1, 2, \ldots\}$. ∎

The Mittag-Lefler Inverse Limit Theorem can be used to prove the Baire Category Theorem (by showing that the intersection of a countable family of dense open subsets of a complete metric space or locally compact Hausdorff space is dense), so it cannot be cpnsidered surprising that the above theorem does not hold for sequentially compact spaces.

To see a specific example, let $Y = E_0 = E_1 = \ldots$ be the integers with the cofinite topology, in which closed sets are either finite or the entire space. Define $R_n: E_n \to$ $\to E_{n-1}$ by $R_n(k) = n$ if $k \leqq n$, and $R_n(k) = k$ otherwise. Note that the inverse image of any finite set is finite, so R_n is continuous, and since the range of R_n is infinite, it is dense. For each integer n, define $T_n: E_0 \to Y$ by $T_n(n) = n$, $T_n(k) = 1$ if $k \neq n$. Note that $R_1 \ldots R_{n+1}(k) = n + 1$ if $1 \leqq k \leqq n + 1$, and $R_1 \ldots R_{n+1}(k) =$ $= k$ if $k > n + 1$, so $T_n R_1 \ldots R_{n+1}$ is constant, hence continuous. However, any dense subset of E_0 must be infinite, and any open subset of E_0 is cofinite. Therefore, the intersection of a dense subset and an open subset of E_0 must be infinite. If we let $Y_n = \{1, 2, \ldots, n\}$ be the increasing closed cover of Y, we see that $\{T_n: n = 1, 2, \ldots\}$ is pointwise-bounded. But $\{T_n: n = 1, 2, \ldots\}$ is unbounded on any infinite subset of E_0, and so the previous theorem cannot hold for sequentially compact spaces.

SECTION 2. CATEGORY THEOREMS AND LOCAL UNIFORM BOUNDEDNESS

A Baire space is one which has the property that at least one member of a countable closed cover of the space must contain a non-empty open subset; i.e., it is a space in which the Baire Category Theorem can be proved. That this property results in a local uniform boundedness principle (Osgood's Theorem) is elementary. What is equally elementary, but perhaps not so well known, is that the correct phraseology of the local uniform boundedness principle leads to an equivalence between it and the Baire Category Theorem.

Theorem 5. *The following conditions are equivalent.*

(1) *X is a Baire space.*

(2) *Let Y be a topological space and $\{Y_n: n = 1, 2, \ldots\}$ be an increasing closed cover of Y. Let A be a set, and for each integer n, A_n is a subset of A. Let $\{T_a: a \in A\}$ be a collection of continuous maps of X into Y. Suppose that for each $x \in X$, there exist integers $n = n(x)$ and $k = k(x)$ such that $a \in A_k \Rightarrow T_a x \in Y_n$. Then there exist integers N and M, and a non-empty open subset U of X, such that $x \in U$ and $a \in A_M \Rightarrow T_a x \in Y_N$.*

(3) *Let Y be a topological space and $\{Y_n: n = 1, 2, \ldots\}$ be an increasing closed cover of Y. Let A be a set, and let $\{T_a: a \in A\}$ be a collection of continuous maps of X into Y. Suppose that for each $x \in X$, there exists an integer $n = n(x)$ such that $a \in A \Rightarrow T_a x \in Y_n$. Then there exists an integer N and a non-empty open subset U of X, such that $x \in U$ and $a \in A \Rightarrow T_a x \in Y_N$.*

Proof. (1) $\Rightarrow$ (2): Define $X_{kn} = \bigcap_{a \in A_k} T_a^{-1}(Y_n)$. By continuity, each X_{kn} is closed, and by assumption, $\{X_{kn}: k, n = 1, 2, \ldots\}$ forms a countable closed cover of X. Since X is a Baire space, for some integers M and N, there exists a non-empty open set U such that $U \subset X_{MN}$, as desired.

(2) $\Rightarrow$ (3) Let $A_n = A$ for $n = 1, 2, \ldots$.

(3) $\Rightarrow$ (1) We begin by observing that if E and F are closed sets and $E \cup F$ contains a non-empty open subset U, then either E or F must contain a non-empty open subset. If $U \subset E$, we are done. If not, the intersection of the complement of E and U is a nonempty open subset of F.

Assume that X satisfies (3), and $\{X_n: n = 1, 2, \ldots\}$ is a countable closed cover of X. Let $Y = X$, and let $Y_n = \bigcup_{k=1}^{n} X_k$; $\{Y_n: n = 1, 2, \ldots\}$ is an increasing countable closed cover of Y. Let $i: X \to Y$ be the identity. The family $\{i\}$ trivially satisfies the hypotheses of (3), and so there exists an integer N and a non-empty open set U such that $U = i(U) \subset Y_N$. The result now follows from a repeated application of the observation of the preceding paragraph. ■

In Theorem 5, condition (3) is simply the conclusion of Osgood's Theorem stated for a more general type of boundedness than usual. Theorem 5 also shows that Theorem 1 is equivalent to the Baire Category Theorem in a complete metric space.

To show that the conclusion of Theorem 2 cannot be "beefed up" to the full strength of the conclusion of Theorem 1, we need merely exhibit a sequentially compact space which is not a Baire space. Let X be the integers with the cofinite topology. Given a sequence from X, if the range of the sequence is finite, pick a subsequence whose range is a singleton (this subsequence converges to the singleton), and if the range of the sequence is infinite, pick a subsequence in which no term recurs (this subsequence converges to each point of X). Thus X is sequentially compact. However, X is not a Baire space, since it is the countable union of singletons, and each singleton is closed but not open.

In view of the fact that boundedness theorems are sometimes equivalent to category theorems, one might try to abstract the basic idea of a category theorem. In a very broad framework, a category theorem for a set X can be said to be a theorem in which there are several different parameters: two collections of subsets of X, and various types of coverings. A category theorem would then state that, if X is covered in an acceptable way by subsets from the first collection, then some member of the second collection is also covered (possibly in a different way) by those subsets. It is possible to show that quite general theorems, of which Theorem 5 would be a special case, can be proved in a very abstract setting, involving only generalizations of the properties that the intersection of closed sets is closed and that, under a continuous map, the inverse image of a closed set is closed. The value of such theorems to an analyst is not clear.

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Author's address: Department of Mathematics, California State University, Long Beach, Long Beach, Ca. 90840. U.S.A.