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Some operatorial properties of the generalized hypergeometric coherent states

Dušan Popov[1] and Miodrag Popov[2]

[1] Politehnica University Timisoara, Department of Physical Foundations of Engineering, Bd. V. Parvan No. 2, 300223 Timisoara, Romania

[2] Politehnica University Timisoara, Department of Steel Structures and Building Mechanics, Traian Lalescu Street, No. 2/A, 300223 Timisoara, Romania

E-mail: miodrag.popov@up-tim.org and popov.mio@gmail.com

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Abstract

The technique regarding the integration within a normally ordered product of operators, which refers to the creation and annihilation operators of the harmonic oscillator coherent states, has proved to be very fruitful for different operator identities and applications in quantum optics. In this paper we propose a generalization of this technique by introducing a new operatorial approach—the diagonal ordering operation technique (DOOT)—regarding the calculations connected with the normally ordered product of generalized creation and annihilation operators that generate the generalized hypergeometric coherent states. We have pointed out a number of properties of these coherent states, including the case of mixed (thermal) states. At the same time, by particularizing the obtained results to the one-dimensional harmonic and pseudoharmonic oscillators, we get the well-known results achieved through other methods in the corresponding coherent states representation.

Keywords: normally ordered product, hypergeometric coherent states, density operator

1. Introduction

The concept of coherent state was introduced more than nine decades ago (in 1926) by Schrödinger [1] for the quantum harmonic oscillator (HO) as the specific quantum state that has dynamical behavior that is most similar to that of the classical harmonic oscillator. Throughout this period, the applicability area of the one-dimensional harmonic oscillator (HO-1D) coherent states (CSs), also called canonical CSs, has been considerably expanded, including very different fields, such as physics and mathematical physics, to signal theory and quantum information. Apart from the harmonic oscillator, different kinds of coherent states have also been built for the anharmonic oscillators, either as the lowering operator eigenstate (the so-called CSs of the Barut–Girardello (BG) kind) or by applying the displacement operator on a ground state (Klauder–Perelomov CSs) or CSs of the Gazeau–Klauder kind, including the nonlinear CSs, squeezed states and deformed CSs. There are many significant books and review papers on the CSs and their applications [2–6].

On the one hand, an interesting category of CSs is the so-called generalized hypergeometric coherent states (GH-CSs); these states have an appellation from their normalization function, which is given by a generalized hypergeometric function. These kinds of states were firstly introduced by Appl and Schiller [7] and applied to the thermal states of the pseudoharmonic oscillator in one of our previous papers [8].

On the other hand, Hong-yi Fan and coauthors have elaborated on the integration within an ordered product (IWOP) technique for Bose operators, referring to the CSs of the HO-1D. Using this technique a number of known as well as new results were obtained; however, much simpler mathematical calculations were used (see [9–16] and the references therein). This useful technique can be particularized by the so-called double dot operation $: \; :$ (in the acceptance of Blasiak *et al* [17]), which consists of applying the normal ordered IWOP rules without taking into account the commutation relation between the annihilation and creation operators. Simply, inside the double dot operation $: \; :$ these two operators can be commuted as two commutable operators, i.e. such as numbers. The normally ordered technique is very useful in the calculations, which imply the CSs of the BG kind, as we will see later.

The aim of the present paper is to recover the characteristics of the GH-CSs by introducing a new approach of ordering the operators which generates these states. We call this operation the DOOT and denote it with a new symbol # #. The name is justified by the fact that here we are dealing only with operators (more precisely, with normal ordered products of operators) that are diagonal in the basis of the Hamiltonian's eigenvectors (Fock vectors). This technique differs somewhat from the double dot operations technique, which can be considered as the particular case of the IWOP technique, proposed and developed by Hong-yi Fan and coauthors. In this way we want to show that the DOOT technique can be formulated and used not only for the CSs of the HO-1D but can also be generalized and extended to the CSs of other types of oscillators, which are the particular cases of more general HG-CSs.

The paper is organized as follows: In section 2 we quote briefly some of the basic elements of the DOOT in order to use it for the generalization of this technique to other operators and finally to compare the results obtained with those known from literature. In section 3 we present the definition and some important properties of the generalized hypergeometric BG coherent states, respectively, the generalized creation and annihilation operators which generate these coherent states. In section 4 we examine the statistical properties of mixed (thermal) states corresponding to a canonical distributed system in the light of using the DOOT deduced in the previous section. In section 5 we present two examples that illustrate the usefulness of the results obtained in the previous two sections by referring to the HO-1D and also to the pseudoharmonic oscillator (PHO). Section 6 is devoted to some concluding remarks, while in the appendix we have inserted some mathematical relationships on the generalized hypergeometric functions and also on the Meijer's G functions that we have used in previous sections.

2. Basics of the diagonal ordering operation technique (DOOT)

Before introducing the basics of the DOOT, we examine some properties of the GH-CSs, which will help to introduce this new way of performing calculations. For this purpose, following [7], we consider two hermitic conjugate operators $A_-^{(p,q)} \equiv A_-$ and $A_+^{(p,q)} \equiv A_+$, where $A_- = (A_+)^+$, which acts in the infinite dimensional Hilbert space of the Fock vectors $|n;\ \lambda>, n = 0,\ 1,\ 2,\ \ldots,$ where p and q are positive integers, and λ is a real parameter; its physical significance will be evinced later. These densely defined operators act as the generalized annihilation or, respectively, as creation operators which generate the GH-CSs and are defined in the following manner [7, 8]

$$A_- = \sum_{n=0}^{\infty} f_{p,q}(n)|n;\ \lambda\rangle\langle n+1;\ \lambda|, \tag{2.1}$$

$$A_+ = \sum_{n=0}^{\infty} f_{p,q}(n)|n+1;\ \lambda\rangle\langle n;\ \lambda| \tag{2.2}$$

For brevity, we will omit writing the entire positive numbers $(p,\ q)$ and also the adjective word 'generalized' when we refer to the operators $A_-^{(p,q)} \equiv A_-$ and $A_+^{(p,q)} \equiv A_+$, but we will use this word if we refer to the hypergeometric functions. Integers p and q appear in the definition of positive functions $f_{p,q}(m)$ [7, 8]; the choice and meaning of these functions will be explained below

$$f_{p,q}(m) = \sqrt{(m+1)\frac{\prod_{j=1}^{q}\left(b_j + m\right)}{\prod_{i=1}^{p}(a_i + m)}},$$
$$\prod_{m=0}^{n-1} f_{p,q}(m) = \sqrt{n!\frac{\prod_{j=1}^{q}\left(b_j\right)_n}{\prod_{i=1}^{p}(a_i)_n}} \equiv \sqrt{\rho_{p,q}(n)} \tag{2.3}$$

where $(a_i)_n = \Gamma(a_i + n)/\Gamma(a_i)$ are Pochhammer's symbols, expressed through Euler gamma functions $\Gamma(a_i)$ and where the sequence of coefficients $a_1, a_2, \ldots, a_p \equiv \{a_i\}_1^p$ and $b_1, b_2, \ldots, b_q \equiv \left\{b_j\right\}_1^q$ are real numbers. To simplify writing the formulas, in the following we will use this abbreviated notation for these coefficients.

We point out the following recurrence relationships, which we will use in what follows

$$\rho_{p,q}(0) = 1;$$
$$\rho_{p,q}(n+1) = \rho_{p,q}(n)\left[f_{p,q}(n)\right]^2 \tag{2.4}$$

Using the above definitions of operators $A_-^{(p,q)} \equiv A_-$ and $A_+^{(p,q)} \equiv A_+$, as well as the orthogonality and completeness relations of the Fock vectors,

$$<n;\ \lambda|n';\ \lambda\rangle = \delta_{nn'},$$
$$\sum_{n=0}^{\infty}|n;\ \lambda\rangle\langle n;\ \lambda| = 1, \tag{2.5}$$

it follows that the below relations are valid

$$A_-|n;\ \lambda\rangle = f_{p,q}(n-1)|n-1;\ \lambda\rangle, \tag{2.6}$$

$$A_+|n;\ \lambda\rangle = f_{p,q}(n)|n+1;\ \lambda\rangle, \tag{2.7}$$

i.e. the operator A_- acts as a lowering operator, while its conjugate operator A_+ acts as a raising operator. Their product operator in the normal ordered manner A_+A_- is a diagonal operator in the Fock-vectors basis, and this property we will exploit in the following

$$A_+A_-|n;\ \lambda\rangle = \left[f_{p,q}(n-1)\right]^2|n;\ \lambda\rangle,$$
$$\langle n;\ \lambda|A_+A_-|n;\ \lambda\rangle = \left[f_{p,q}(n-1)\right]^2 \tag{2.8}$$

These two operators do not commute; therefore, their commutator is

$$\left[A_-, A_+\right] = \sum_{n=0}^{\infty}\left(\left[f_{p,q}(n)\right]^2 - \left[f_{p,q}(n-1)\right]^2\right) \times |n;\,\lambda\rangle\langle n;\,\lambda|. \tag{2.9}$$

In what follows, we deal only with general functions depending on the normal ordered operator product A_+A_-, say $F(A_+A_-)$, and we will use a new operation, which we will call the DOOT. We will denote this new introduced operation by the symbols # #. Generally, this procedure differs and may yield different results than those for the case of normal ordering in the ordinary sense [17] or the IWOP technique [9].

We assume that the DOOT with the symbols # # consists of simply applying the normal ordering rules without taking into account the above commutation relation. In other words, inside the symbol # #, all the annihilation operators A_- can be moved to the right as if they commuted with the creation operator A_+ in order to obtain the normal ordered operator product A_+A_- or a function of this product $F(A_+A_-)$.

In a certain sense, from the operational point of view, the DOOT calculus can be considered as a particular case of the IWOP technique (namely, their normal ordered branch), with, however, some specific particularities.

Namely, we introduce and adopt the following rules for the DOOT calculus:

(I)* Inside the symbol # #, the order of the operators A_- and A_+ can be permuted like commutable operators, but they are permuted in a way that finally will result in an operator function that depends only on the powers of the normally ordered operator product A_+A_-, i.e.

$$\begin{gathered} \#\,A_-A_+\# = \#A_+A_-\# = A_+A_-, \\ \#\left(A_-\right)^n\left(A_+\right)^n\# = \#\left(A_+\right)^n\left(A_-\right)^n\# = \left(A_+A_-\right)^n \end{gathered} \tag{2.10}$$

(II)* A symbol # #inside of another symbol # # can be deleted.

(III)* If the integration is convergent, a normally ordered product of operators can be integrated or differentiated with respect to c-numbers, according to the usual rules. In addition, the c-numbers can be taken out from the symbol # #.

(IV)* The projector $|0><0|$ of the normalized vacuum state $|0>$ in the frame of the DOOT has the following normal ordered form

$$|0;\,\lambda\rangle\langle 0;\,\lambda| = \#\frac{1}{{}_pF_q\left(\{a_i\}_1^p;\,\left\{b_j\right\}_1^q;\,A_+A_-\right)}\# \tag{2.11}$$

The last expression will be demonstrated below and at the end of section 2 using the coherent states representation.

Generalizing, for a function which depends on the normal operator product A_+A_-, say $F(A_+A_-)$, we have, successively

$$\begin{aligned} \#\,F\left(A_+A_-\right)\#|n;\,\lambda\rangle &= \sum_l c_l\#\left(A_+A_-\right)^l\#|n;\,\lambda\rangle = \\ &= \sum_l c_l\left(\left[f_{p,q}(n-1)\right]^2\right)^l|n;\,\lambda\rangle \\ &= F\left(\left[f_{p,q}(n-1)\right]^2\right)|n;\,\lambda\rangle \end{aligned} \tag{2.12}$$

and consequently

$$\begin{aligned} \langle n;\,\lambda\left|\#F\left(A_+A_-\right)\#\right|n;\,\lambda\rangle &\equiv \left\langle n;\,\lambda\left|F\left(A_+A_-\right)\right|n;\,\lambda\right\rangle \\ &= F\left(\left[f_{p,q}(n-1)\right]^2\right) \end{aligned} \tag{2.13}$$

It may be observed that any function depending on the normal ordered product of operators A_+A_- has a diagonal matrix in the representation of the Fock-vectors basis $|n;\,\lambda>$; in what follows we will exploit fully this property in the sense that we will deal only with the functions which depend on the normal ordered product operators A_+A_-. The reason for this is that the normalized functions of the GH-CSs are expressed in terms of these kinds of functions, as we will see later.

As it is usual, we choose the ground or vacuum state $|0;\,\lambda>$ as the state for which the lowering operator acts in the following manner $A_-|0;\,\lambda> = 0|0;\,\lambda>$, while the repeatable action of the raising operator on the vacuum state is

$$\begin{aligned} \left(A_+\right)^n|0;\,\lambda\rangle &= \prod_{m=0}^{n-1} f_{p,q}(m)|n;\,\lambda\rangle \\ &= \sqrt{\rho_{p,q}(n)}\,|n;\,\lambda\rangle \end{aligned} \tag{2.14}$$

By means of the conjugation property $A_- = (A_+)^+$ we can derive the following relations

$$\begin{aligned} |n;\,\lambda\rangle &= \frac{1}{\sqrt{\rho_{p,q}(n)}}\left(A_+\right)^n|0;\,\lambda\rangle, \\ \langle n;\,\lambda| &= \frac{1}{\sqrt{\rho_{p,q}(n)}}\langle 0;\,\lambda|\left(A_-\right)^n \end{aligned} \tag{2.15}$$

The orthogonality relation of the Fock vectors helps us to get a useful relation for evaluating the overlap of two GH-BG-CSs

$$\left\langle 0;\,\lambda\left|\left(A_-\right)^n\left(A_+\right)^n\right|0;\,\lambda\right\rangle = \rho_{p,q}(n) \tag{2.16}$$

Using the above properties, we can write the completeness relation of the Fock vectors using the DOOT

$$\sum_{n=0}^{\infty}|n;\,\lambda\rangle\langle n;\,\lambda|$$

$$= \sum_{n=0}^{\infty} \frac{1}{\rho_{p,q}(n)} \# \left(A_+\right)^n |0;\,\lambda\rangle\langle 0;\,\lambda| \left(A_-\right)^n \#$$
$$= \#|0;\,\lambda\rangle\langle 0;\,\lambda|\#\#$$
$$\times \sum_{n=0}^{\infty} \frac{1}{\rho_{p,q}(n)} \left(A_+\right)^n \left(A_-\right)^n \# = \#|0;\,\lambda\rangle\langle 0;\,\lambda|\#\,\#$$
$$\times \sum_{n=0}^{\infty} \frac{1}{\rho_{p,q}(n)} \left(A_+ A_-\right)^n \# =$$
$$= \#|0;\,\lambda\rangle\langle 0;\,\lambda|\#\,\#_p F_q\left(\{a_i\}_1^p;\,\left\{b_j\right\}_1^q;\,A_+A_-\right)\# = 1 \quad (2.17)$$

from which follows the expression of the projector of the vacuum state (2.11).

The general proof regarding the inverse operators of functions, which depend on the product of normally ordered generalized operators A_+A_-, can be found in appendix B.

3. Generalized hypergeometric Barut–Girardello (BG) coherent states

Now, let us define the coherent states (CSs) of the annihilation operator A_- in the BG manner

$$A_-|z;\,\lambda\rangle = z|z;\,\lambda\rangle \quad (3.1)$$

where $z = |z|\exp(i\phi)$ is the complex variable which labels the CSs, defined on the entire complex plane $0 \leqslant |z| \leqslant \infty$, $0 \leqslant \phi \leqslant 2\pi$.

These coherent states will be called the generalized hypergeometric BG coherent states (GH-BG-CSs) for the reason that follows: Their expansion in the Fock-vectors basis $|n;\,\lambda>$ is

$$|z;\,\lambda\rangle = \sum_{n=0}^{\infty} <n;\,\lambda|z;\,\lambda\rangle|n;\,\lambda\rangle$$
$$\equiv \sum_{n=0}^{\infty} c_n(z;\,\lambda)|n;\,\lambda\rangle \quad (3.2)$$

From the definition of GH-BG-CSs and the action of the operator A_- on the Fock vectors, we obtain that the expansion functions are

$$c_n(z;\,\lambda) = c_0(z;\,\lambda)\frac{z^n}{\prod_{m=0}^{n-1} f_{p,q}(m)} \equiv c_0(z;\,\lambda)\frac{z^n}{\sqrt{\rho_{p,q}(n)}} \quad (3.3)$$

where $|c_0(z;\,\lambda)|^2 \equiv |<0;\,\lambda|z;\,\lambda>|^2$ is the normalization function, which can be determined if we impose that the GH-BG-CSs must be normalized to unity $<z;\,\lambda|z;\,\lambda> = 1$.

For brevity's sake, we prefer to use the above-introduced notation

$$\rho_{p,q}(n) \equiv \prod_{m=0}^{n-1}\left[f_{p,q}(m)\right]^2 = n!\frac{\prod_{j=1}^{q}\left(b_j\right)_n}{\prod_{i=1}^{p}(a_i)_n} \quad (3.4)$$

The positive constants $\rho_{p,q}(n)$ are assumed to arise as the moments of a probability distribution, i.e. [18, 19]

$$\rho_{p,q}(n) \equiv \int_0^R u^n \rho_{p,q}(u)\mathrm{d}u \geqslant 0 \quad (3.5)$$

and we can talk about GH-BG-CSs only if all moments exist for all n: $\rho_{p,q}(0) = 1$ and $\rho_{p,q}(n) < +\infty$.

Consequently, we obtain

$$c_0(z;\,\lambda) = \left(\sqrt{\sum_{n=0}^{\infty}\frac{1}{\sqrt{\rho_{p,q}(n)}}\left(|z|^2\right)^n}\right)^{-1}$$
$$= \frac{1}{\sqrt{{}_pF_q\left(\{a_i\}_1^p;\,\left\{b_j\right\}_1^q;\,|z|^2\right)}} \quad (3.6)$$

Finally, the expansion of the GH-BG-CSs in the Fock-vectors basis becomes

$$|z;\,\lambda> = \frac{1}{\sqrt{{}_pF_q\left(\{a_i\}_1^p;\,\left\{b_j\right\}_1^q;\,|z|^2\right)}}$$
$$\times \sum_{n=0}^{\infty}\frac{z^n}{\sqrt{\rho_{p,q}(n)}}|n;\,\lambda\rangle, \quad (3.7)$$

and now it becomes clear why the functions $f_{p,\,q}(n)$ have the above structure: the square of the normalization function is exactly the generalized hypergeometric function ${}_pF_q\left(a_1,\,\ldots,\,a_p;\,b_1,\,\ldots,\,b_q;\,|z|^2\right)$ $\equiv {}_pF_q\left(\{a_i\}_1^p;\,\left\{b_j\right\}_1^q;\,|z|^2\right)$ [20, 21]. The counterpart of the GH-BG-CSs is

$$<z;\,\lambda| = \frac{1}{\sqrt{{}_pF_q\left(\{a_i\}_1^p;\,\left\{b_j\right\}_1^q;\,|z|^2\right)}}$$
$$\times \sum_{n=0}^{\infty}\frac{(z^*)^n}{\sqrt{\rho_{p,q}(n)}}\langle n;\,\lambda| \quad (3.8)$$

For this reason the above coherent states are called the GH-BG-CSs.

It goes without saying that the parameters $\{a_i\}_1^p$ and $\left\{b_j\right\}_1^q$ of the generalized hypergeometric function ${}_pF_q\left(\{a_i\}_1^p;\,\left\{b_j\right\}_1^q;\,|z|^2\right)$ must be such that the denominator factors in the terms of the series are never zero. On the other hand, only these hypergeometric functions, which are convergent, can play the role of the normalization function. The convergence of the hypergeometric series can be examined following their general convergence conditions (see, e.g. [20, 21]). Generally, the radius of convergence R of an infinite series is given by [18, 22]

$$R = \frac{1}{\lim_{n\to\infty}\sqrt[n]{\rho_{p,q}(n)}}, \quad (3.9)$$

and the coherent states exist only if the convergence radius is nonzero.

Using equation (2.14) we can rewrite the GH-BG-CSs in a manner that highlights the raising operator

$$|z;\lambda\rangle = \frac{1}{\sqrt{{}_pF_q\left(\{a_i\}_1^p;\ \{b_j\}_1^q;\ |z|^2\right)}} \times \sum_{n=0}^{\infty}\frac{1}{\rho_{p,q}(n)}\left(zA_+\right)^n|0;\lambda\rangle \tag{3.10}$$

or in a more compacted form, using the definition of the generalized hypergeometric functions

$$|z;\lambda\rangle = \frac{1}{\sqrt{{}_pF_q\left(\{a_i\}_1^p;\ \{b_j\}_1^q;\ |z|^2\right)}} \times {}_pF_q\left(\{a_i\}_1^p;\ \{b_j\}_1^q;\ zA_+\right)|0;\lambda\rangle \tag{3.11}$$

This relation shows that the generalized hypergeometric functions generate these coherent states, which is another reason to use the name generalized hypergeometric BG coherent states (GH-BG-CSs) for these kind of coherent states. It is useful to point out that for the particular cases, $p=1$ and $q \geqslant 1$ Dehghani and Mojavery introduced a continuously parametrized family of nonlinear CSs [23] via a generalized analogue of the displacement operator acting on the vacuum state. The same approach was also used to construct the generalized su(1,1) CSs for the PHO [24] of the generalized su(2) CSs for the Landau levels [25] (see also the more recent paper in [26]). It can be concluded that these states can be regarded as the special classes of GH-CSs, introduced by Appl and Schiller [7].

Their counterpart then is

$$\langle z;\lambda| = \frac{1}{\sqrt{{}_pF_q\left(\{a_i\}_1^p;\ \{b_j\}_1^q;\ |z|^2\right)}} \times \langle 0;\lambda|\,{}_pF_q\left(\{a_i\}_1^p;\ \{b_j\}_1^q;\ z^*A_-\right) \tag{3.12}$$

Using the DOOT, the projector onto a GH-BG-CS then becomes

$$|z;\lambda\rangle\langle z;\lambda| = \frac{1}{{}_pF_q\left(\{a_i\}_1^p;\ \{b_j\}_1^q;\ |z|^2\right)}\# \frac{{}_pF_q\left(\{a_i\}_1^p;\ \{b_j\}_1^q;\ zA_+\right){}_pF_q\left(\{a_i\}_1^p;\ \{b_j\}_1^q;\ z^*A_-\right)}{{}_pF_q\left(\{a_i\}_1^p;\ \{b_j\}_1^q;\ A_+A_-\right)}\# \tag{3.13}$$

When we put $z=0$ in this relation, we recover the projector on the vacuum state (2.11).

We will show that the above-defined states accomplish all of the Klauder's prescriptions regarding the coherent states [2].

The overlap (scalar product) of two GH-BG-CSs can be expressed using equations (3.7) and (3.8) and also the normalization condition of the Fock vectors (2.5)

$$\langle z;\lambda|z';\lambda\rangle = \left({}_pF_q\left(\{a_i\}_1^p;\ \{b_j\}_1^q;\ z^*z'\right)\right)/ \left(\sqrt{{}_pF_q\left(\{a_i\}_1^p;\ \{b_j\}_1^q;\ |z|^2\right)\,{}_pF_q\left(\{a_i\}_1^p;\ \{b_j\}_1^q;\ |z'|^2\right)}\right) \tag{3.14}$$

The continuity in the complex variable z follows from the overlap

$$\lim_{z'\to z}\left\||z';\lambda\rangle - |z;\lambda\rangle\right\|^2 = 2\left[1 - \lim_{z'\to z}\left(\mathrm{Re}\langle z;\lambda|z';\lambda\rangle\right)\right] = 0 \tag{3.15}$$

Next, we will find the weight function ${}_ph_q(|z|)$ of the integration measure's expression $\mathrm{d}\mu_{p,q}(z) = \frac{\mathrm{d}\phi}{2\pi}\mathrm{d}(|z|^2)\,{}_ph_q(|z|)$ in order to demonstrate that the GH-BG-CSs fulfill the resolution of the unity operator

$$\int \mathrm{d}\mu_{p,q}(z)|z;\lambda\rangle\langle z;\lambda| = 1 \tag{3.16}$$

Replacing the expression of the integration measure, as well as the GH-BG-CSs projector (3.13), we obtain

$$\int_0^\infty \mathrm{d}\left(|z|^2\right)\frac{h_{p,q}(|z|)}{{}_pF_q\left(\{a_i\}_1^p;\ \{b_j\}_1^q;\ |z|^2\right)} \int_0^{2\pi}\frac{\mathrm{d}\phi}{2\pi}\#\left({}_pF_q\left(\{a_i\}_1^p;\ \{b_j\}_1^q;\ zA_+\right)\,{}_pF_q\left(\{a_i\}_1^p;\ \{b_j\}_1^q;\ z^*A_-\right)\right)/ \left({}_pF_q\left(\{a_i\}_1^p;\ \{b_j\}_1^q;\ A_+A_-\right)\right)\# = 1 \tag{3.17}$$

First of all, we must perform the following function change

$$\tilde{h}_{p,q}(|z|) \equiv \frac{h_{p,q}(|z|)}{{}_pF_q\left(\{a_i\}_1^p;\ \{b_j\}_1^q;\ |z|^2\right)} \tag{3.18}$$

The angular integration inside the symbol # # is easy to perform if we use the definition of the hypergeometric functions (equation (A.2)) and, of course, if we apply the IWOP rules

$$\int_0^{2\pi}\frac{\mathrm{d}\phi}{2\pi}\#\,{}_pF_q\left(\{a_i\}_1^p;\ \{b_j\}_1^q;\ zA_+\right)\,{}_pF_q\left(\{a_i\}_1^p;\ \{b_j\}_1^q;\ z^*A_-\right)\# = \sum_{n',n=0}^{\infty}\frac{1}{\rho_{p,q}(n')}\frac{1}{\rho_{p,q}(n)}$$

$$\times \#\left(A_+\right)^{n'}\left(A_-\right)^n\# \int_0^{2\pi} \frac{d\phi}{2\pi}(z)^{n'}(z^*)^n$$
$$= \sum_{n=0}^{\infty} \frac{1}{\left[\rho_{p,q}(n)\right]^2} \#\left(A_+A_-\right)^n\#\left(|z|^2\right)^n \tag{3.19}$$

where we took into account that the angular integral from the right-hand side is $|z|^{2n}\delta_{nn'}$ so that we have to solve only the radial integral

$$\sum_{n=0}^{\infty} \frac{1}{\left[\rho_{p,q}(n)\right]^2} \#\left(A_+A_-\right)^n\#$$
$$\times \int_0^{\infty} d\left(|z|^2\right)\tilde{h}_{p,q}(|z|)\left(|z|^2\right)^n$$
$$= \#_pF_q\left(\{a_i\}_1^p;\ \left\{b_j\right\}_1^q;\ A_+A_-\right)\# \tag{3.20}$$

To obtain the hypergeometric function on the right-hand side of the relation above, it is obvious that the following equality must be fulfilled

$$\int_0^{\infty} d\left(|z|^2\right)\tilde{h}_{p,q}(|z|)\left(|z|^2\right)^n = \rho_{p,q}(n) \tag{3.21}$$

The above integral is actually the Stieltjes moment problem [18]. After the exponent change, i.e. $n = s - 1$, the solution of this integral equation can be expressed through Meijer's G-functions [20, 21]

$$\tilde{h}_{p,q}(|z|) = \frac{\prod_{i=1}^{p}\Gamma(a_i)}{\prod_{j=1}^{q}\Gamma\left(b_j\right)}$$
$$\times G_{p,q+1}^{q+1,0}\left(|z|^2 \,\middle|\, \begin{matrix} /; & \left\{a_i - 1\right\}_1^p \\ 0,\ \left\{b_j - 1\right\}_1^q; & / \end{matrix}\right) \tag{3.22}$$

Using the relationship between the hypergeometric functions and the Meijer's G-functions [20, 21] (see equation (A.3)), the integration measure becomes

$$d\mu_{p,q}(z) = \frac{d\phi}{2\pi} d\left(|z|^2\right)\frac{\prod_{i=1}^{p}\Gamma(a_i)}{\prod_{j=1}^{q}\Gamma\left(b_j\right)}$$
$$\times {}_pF_q\left(\{a_i\}_1^p;\ \left\{b_j\right\}_1^q;\ |z|^2\right)$$
$$\times G_{p,\,q+1}^{q+1,0}\left(|z|^2 \,\middle|\, \begin{matrix} /; & \left\{a_i - 1\right\}_1^p \\ 0,\ \left\{b_j - 1\right\}_1^q; & / \end{matrix}\right)$$
$$= \frac{d\phi}{2\pi} d\left(|z|^2\right) G_{p,q+1}^{1,p} \times \left(-|z|^2 \,\middle|\, \begin{matrix} \left\{1 - a_i\right\}_1^p; & / \\ 0; & \left\{1 - b_j\right\}_1^q \end{matrix}\right)$$
$$\times G_{p,q+1}^{q+1,0}\left(|z|^2 \,\middle|\, \begin{matrix} /; & \left\{a_i - 1\right\}_1^p \\ 0,\ \left\{b_j - 1\right\}_1^q; & / \end{matrix}\right) \tag{3.23}$$

The weight function of the integration measure must be a nonoscillatory positive defined function, and it must be unique. The sufficient condition for unicity is given by the Carleman condition [18, 22]: if the solution of the Stieltjes moment problem exists, then

$$S \overset{def}{=} \sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{\left|\rho_{p,q}(n)\right|}}$$
$$= \begin{cases} +\infty, & \text{the solution is unique} \\ < +\infty, & \text{non-unique solutions exist} \end{cases} \tag{3.24}$$

For finding the value of S, as well as the convergence radius R, it is convenient to use, e.g. the logarithmic or d'Alembert convergence test.

The expectation value of an operator $\boldsymbol{A}$ in the GH-BG-CSs representation is

$$\langle z;\ \lambda|\boldsymbol{A}|z;\ \lambda\rangle \equiv \langle \boldsymbol{A}\rangle_{z;\lambda}$$
$$= \frac{1}{{}_pF_q\left(\{a_i\}_1^p;\ \left\{b_j\right\}_1^q;\ |z|^2\right)}$$
$$\times \sum_{n',n=0}^{\infty} \frac{(z^*)^{n'}z^n}{\sqrt{\rho_{p,q}(n')\rho_{p,q}(n)}}\langle n';\ \lambda\,|\boldsymbol{A}|\ n;\ \lambda\rangle \tag{3.25}$$

In particular, if $\boldsymbol{A} = \boldsymbol{N}^s$, where $s = 1,\ 2,\ \ldots$, and $\boldsymbol{N}$ is the particle number operator, i. e. $\boldsymbol{N}|n;\ \lambda\rangle = n|n;\ \lambda\rangle$, the expectation value can be calculated as follows

$$\langle \boldsymbol{N}^s\rangle_{z;\lambda} = \frac{1}{{}_pF_q\left(\{a_i\}_1^p;\ \left\{b_j\right\}_1^q;\ |z|^2\right)}$$
$$\times \left[|z|^2 \frac{d}{d\left(|z|^2\right)}\right]^s {}_pF_q\left(\{a_i\}_1^p;\ \left\{b_j\right\}_1^q;\ |z|^2\right) \tag{3.26}$$

In order to compare the statistics of different GH-BG-CSs we need to compute the Mandel parameter, defined as follows [27]

$$Q_{|z|;\lambda} = \frac{\langle \boldsymbol{N}^2\rangle_{z;\lambda} - \left(\langle \boldsymbol{N}\rangle_{z;\lambda}\right)^2}{\langle \boldsymbol{N}\rangle_{z;\lambda}} - 1 \tag{3.27}$$

which shows how the relative difference between the variance $\Delta \boldsymbol{N} = \langle \boldsymbol{N}^2\rangle_{z;\lambda} - \left(\langle \boldsymbol{N}\rangle_{z;\lambda}\right)^2$ and the expectation value $\langle \boldsymbol{N}\rangle_{z;\lambda}$, both calculated in the representation of GH-BG-CSs.

After some simple algebraic calculations it is not difficult to calculate $\langle \boldsymbol{N}\rangle_{z;\lambda}$ and $\langle \boldsymbol{N}^2\rangle_{z;\lambda}$ and to prove that the Mandel parameter can be expressed as follows (for brevity, we used $|z|^2 \equiv x$)

$$Q_{z;\lambda} = x\left[\frac{\left(\frac{d}{dx}\right)^2 {}_pF_q\left(\{a_i\}_1^p;\ \left\{b_j\right\}_1^q;\ x\right)}{\frac{d}{dx}{}_pF_q\left(\{a_i\}_1^p;\ \left\{b_j\right\}_1^q;\ x\right)} - \frac{\frac{d}{dx}{}_pF_q\left(\{a_i\}_1^p;\ \left\{b_j\right\}_1^q;\ x\right)}{{}_pF_q\left(\{a_i\}_1^p;\ \left\{b_j\right\}_1^q;\ x\right)}\right] \tag{3.28}$$

In order to verify the behavior of the GH-BG-CSs, i.e. to see if they are sub-Poissonian (for which, $Q_{z;\lambda} < 0$), Poissonian (with $Q_{z;\lambda} = 0$) or super-Poissonian (with $Q_{z;\lambda} > 0$), the behavior of the Mandel parameter $Q_{z;\lambda}$ can be examined with respect to the variable $x = |z|^2$. Thus, the statistical properties of the GH-BG-CSs are dependent on the analytical properties of the mathematical operations involving hypergeometric functions ${}_pF_q\left(\{a\}_1^p;\ \{b\}_1^q;\ x\right)$ and their derivatives. Generally, because the hypergeometric functions are present, the Mandel parameter must be evaluated numerically for different variable values in order to establish that $Q_{|z|;\lambda}$ will be <0, =0 or > 0. The Mandel parameter can be evaluated analytically in only a few particular cases.

At the end of this chapter, we will point out a useful relation. According to equations (3.28) and (2.8), the expectation value of the normally ordered operator $\boldsymbol{A} = A_+A_-$ is then

$$\begin{aligned}
&\langle z;\lambda|A_+A_-|z;\lambda\rangle \equiv \langle A_+A_-\rangle_{z;\lambda} = \\
&= \frac{1}{{}_pF_q\left(\{a_i\}_1^p;\ \left\{b_j\right\}_1^q;\ |z|^2\right)} \\
&\quad\times\sum_{n',n=0}^{\infty}\frac{(z^*)^{n'}z^n}{\sqrt{\rho_{p,q}(n')\,\rho_{p,q}(n)}}\langle n';\lambda|A_+A_-|n;\lambda\rangle \\
&= \frac{1}{{}_pF_q\left(\{a_i\}_1^p;\ \left\{b_j\right\}_1^q;\ |z|^2\right)} \\
&\quad\times\sum_{n=0}^{\infty}\frac{\left(|z|^2\right)^n}{\rho_{p,q}(n)}\left[f_{p,q}(n-1)\right]^2 = |z|^2
\end{aligned} \tag{3.29}$$

To obtain this result we used the recurrence relation (2.4) for coefficients $\rho_{p,q}(n)$ by eliminating the term with $n-1$ and changing the summation index $n' = n - 1$.

This expectation value can be obtained in an easier way by writing the conjugate relation of the definition of GH-BG-CSs (3.15)

$$\langle z;\lambda|A_+ = z^*\langle z;\lambda| \tag{3.30}$$

and by calculating the inner product of vectors $\langle z;\lambda|A_+$ and $A_-|z;\lambda\rangle$.

As we have pointed out at the beginning of this paper, the DOOT is very fruitful in the calculations, which imply CSs of the BG kind. Due to the definition of the BG-CSs (3.1) and also of their counterpart (3.30), the calculations are reduced to the substitutions $A_- \to z$ and also $A_+ \to z^*$ so that we have

$$\langle z;\lambda|\#A_+A_-\#|z;\lambda\rangle = |z|^2 \tag{3.31}$$

Consequently, for a function of the normal ordered operators A_+A_-, it follows that

$$\langle z;\lambda|\#F\left(A_+A_-\right)\#|z;\lambda\rangle = F\left(|z|^2\right) \tag{3.32}$$

This is one of the main results of the DOOT procedure for the generalized operators A_+ and A_-: in the GH-BG-CSs representation the matrix element $\langle z;\lambda|\#F\left(A_+A_-\right)\#|z;\lambda\rangle$ of any operator valued function which depends only on the normally ordered product of operators A_+A_- can be replaced with a function of the same algebraic structure obtained by replacing A_+A_- with $|z|^2$, i.e. $F\left(|z|^2\right)$.

So, we can verify the correctness of vacuum state projector expression (2.11) by using the GH-BG-CSs representation. By multiplying equation (2.11) on the left/right with $\langle z;\lambda|$, respectively $|z;\lambda\rangle$, we obtain just the expression (3.6) for the normalization function $|c_0(z;\lambda)|^2$

$$\begin{aligned}
|\langle 0;\lambda|z;\lambda\rangle|^2 &= \langle z;\lambda|\#\frac{1}{{}_pF_q\left(\{a_i\}_1^p;\ \left\{b_j\right\}_1^q;\ A_+A_-\right)} \\
\#|z;\lambda\rangle &= \frac{1}{{}_pF_q\left(\{a_i\}_1^p;\ \left\{b_j\right\}_1^q;\ |z|^2\right)}
\end{aligned} \tag{3.33}$$

4. Statistical properties

Now, let us consider a quantum system of oscillators with a linear energy spectrum $E_{n,\lambda} = \hbar\omega n + E_{0,\lambda}$, which is in the thermodynamic equilibrium with the environment at temperature $T = (k_B\beta)^{-1}$, i.e. it obeys a canonical distribution. The corresponding normalized density operator is then [27]

$$\begin{aligned}
\rho &= \frac{1}{Z(\beta)}\sum_{n=0}^{\infty}e^{-\beta\left(\hbar\omega n + E_{0,\lambda}\right)}|n;\lambda\rangle\langle n;\lambda| \\
&= \frac{1}{\bar{n}+1}\sum_{n=0}^{\infty}\left(\frac{\bar{n}}{\bar{n}+1}\right)^n|n;\lambda\rangle\langle n;\lambda|
\end{aligned} \tag{4.1}$$

where $\bar{n} = \left(e^{\beta\hbar\omega} - 1\right)^{-1}$ is the thermal expectation value of the number operator (the Bose–Einstein distribution function for oscillators with angular frequency ω) and also with the partition function $Z(\beta) = \mathrm{Tr}\,\rho = e^{-\beta E_{0,\lambda}}(\bar{n}+1)$.

Using equations (2.15) and (2.11) and the DOOT rules, the following expression yields

$$\begin{aligned}
\rho &= \frac{e^{-\beta E_{0,\lambda}}}{Z(\beta)}\#\frac{1}{{}_pF_q\left(\{a_i\}_1^p;\ \left\{b_j\right\}_1^q;\ A_+A_-\right)} \\
&\quad\times\sum_{n=0}^{\infty}\frac{1}{\sqrt{\rho_{p,q}(n)}}\left(\frac{\bar{n}}{\bar{n}+1}A_+A_-\right)^n\#
\end{aligned} \tag{4.2}$$

which leads to the expression

$$\rho = \frac{e^{-\beta E_{0,\lambda}}}{Z(\beta)}\#\frac{{}_pF_q\left(\{a_i\}_1^p;\ \left\{b_j\right\}_1^q;\ \frac{\bar{n}}{\bar{n}+1}A_+A_-\right)}{{}_pF_q\left(\{a_i\}_1^p;\ \left\{b_j\right\}_1^q;\ A_+A_-\right)}\# \tag{4.3}$$

By normalizing to unity the density operator ρ, we obtain the partition function $Z(\beta)$

$$1 = \mathrm{Tr}\,\rho = \int d\mu_{p,q}(z')\langle z';\lambda|\rho|z';\lambda\rangle$$
$$= \frac{e^{-\beta E_{0,\lambda}}}{Z(\beta)} \times \int d\mu_{p,q}(z')\langle z';\lambda|\#\left({}_pF_q\left(\{a_i\}_1^p;\ \left\{b_j\right\}_1^q;\ \frac{\bar{n}}{\bar{n}+1}A_+A_-\right)\right)/\left({}_pF_q\left(\{a_i\}_1^p;\ \left\{b_j\right\}_1^q;\ A_+A_-\right)\right)\#|z';\lambda\rangle$$
$$= \frac{e^{-\beta E_{0,\lambda}}}{Z(\beta)}\int d\mu_{p,q}(z')\left({}_pF_q\left(\{a_i\}_1^p;\ \left\{b_j\right\}_1^q;\ \frac{\bar{n}}{\bar{n}+1}|z'|^2\right)\right)/\left({}_pF_q\left(\{a_i\}_1^p;\ \left\{b_j\right\}_1^q;\ |z'|^2\right)\right) \tag{4.4}$$

The last equality is obtained using the property (3.32). By inserting the expression of the integration measure and also by inserting the connection between the generalized hypergeometric function and the Meijer's G-function (A.2), after the angular integration, we obtain the integral

$$1 = \frac{e^{-\beta E_{0,\lambda}}}{Z(\beta)} \times \int_0^\infty d\left(|z'|^2\right) G_{p,q+1}^{1,p}\left(-\frac{\bar{n}}{\bar{n}+1}|z'|^2\ \middle|\ \begin{matrix}\left\{1-a_i\right\}_1^p; & /\\ 0; & \left\{1-b_j\right\}_1^q\end{matrix}\right) G_{p,q+1}^{q+1,0}\left(|z'|^2\ \middle|\ \begin{matrix}/; & \left\{a_i-1\right\}_1^p\\ 0,\ \left\{b_j-1\right\}_1^q; & /\end{matrix}\right) \tag{4.5}$$

This integral can be solved using the integral from two Meijer's G-functions (A.6) and their properties [20, 28], bringing us to

$$1 = \frac{e^{-\beta E_{0,\lambda}}}{Z(\beta)} G_{1,1}^{1,1}\left(-\frac{\bar{n}}{\bar{n}+1}\ \middle|\ \begin{matrix}0; & /\\ 0; & /\end{matrix}\right) = \frac{e^{-\beta E_{0,\lambda}}}{Z(\beta)}(\bar{n}+1) \tag{4.6}$$

from which we obtain the partition function $Z(\beta)$

$$Z(\beta) = e^{-\beta E_{0,\lambda}}(\bar{n}+1) \tag{4.7}$$

Finally, the normalized density operator in a normal ordered form reads

$$\rho = \frac{1}{\bar{n}+1}\#\frac{{}_pF_q\left(\{a_i\}_1^p;\ \left\{b_j\right\}_1^q;\ \frac{\bar{n}}{\bar{n}+1}A_+A_-\right)}{{}_pF_q\left(\{a_i\}_1^p;\ \left\{b_j\right\}_1^q;\ A_+A_-\right)}\#. \tag{4.8}$$

The Q-distribution function, defined as the diagonal elements of the normalized density operator in the CSs representation [27, 30], particularly for the GH-BG-CSs, is

$$Q_{p,q}\left(|z|^2\right) \equiv \langle z;\lambda|\rho|z;\lambda\rangle = \frac{1}{\bar{n}+1}\frac{{}_pF_q\left(\{a_i\}_1^p;\ \left\{b_j\right\}_1^q;\ \frac{\bar{n}}{\bar{n}+1}|z|^2\right)}{{}_pF_q\left(\{a_i\}_1^p;\ \left\{b_j\right\}_1^q;\ |z|^2\right)} \tag{4.9}$$

where we have also used the property (3.32).

It is not difficult to prove that the Q-distribution function is normalized to unity

$$\int d\mu_{p,q}(z) Q\left(|z|^2\right) = 1 \tag{4.10}$$

The diagonal expansion of the normalized canonical density operator in terms of the GH-BG-CSs projectors is

$$\rho = \int d\mu_{p,q}(z) P_{p,q}\left(|z|^2\right)|z;\lambda\rangle\langle z;\lambda| \tag{4.11}$$

In order to determine the quasi-distribution function $P_{p,q}(|z|^2)$ we must compare two expressions for the density operator ρ, i.e. equations (4.10) and (4.13), and we must have

$$\frac{1}{\bar{n}+1}\#\frac{{}_pF_q\left(\{a_i\}_1^p;\ \left\{b_j\right\}_1^q;\ \frac{\bar{n}}{\bar{n}+1}A_+A_-\right)}{{}_pF_q\left(\{a_i\}_1^p;\ \left\{b_j\right\}_1^q;\ A_+A_-\right)}\# = \int d\mu_{p,q}(z) P_{p,q}\left(|z|^2\right)\#|z;\lambda\rangle\langle z;\lambda|\# \tag{4.12}$$

Substituting equations (3.23) and (3.13), the right-hand side becomes

$$\mathrm{R.H.S.} = \#\frac{1}{{}_pF_q\left(\{a_i\}_1^p;\ \left\{b_j\right\}_1^q;\ A_+A_-\right)} \times \int_0^\infty d\left(|z|^2\right) P_{p,q}\left(|z|^2\right) G_{p,q+1}^{q+1,0}\left(|z|^2\ \middle|\ \begin{matrix}/; & \left\{a_i-1\right\}_1^p\\ 0,\ \left\{b_j-1\right\}_1^q; & /\end{matrix}\right) \times \frac{\prod_{i=1}^p \Gamma(a_i)}{\prod_{j=1}^q \Gamma\left(b_j\right)} \times \int_0^{2\pi}\frac{d\phi}{2\pi}\,{}_pF_q\left(\{a_i\}_1^p;\ \left\{b_j\right\}_1^q;\ zA_+\right)\,{}_pF_q\left(\{a_i\}_1^p;\ \left\{b_j\right\}_1^q;\ z^*A_-\right)\# \tag{4.13}$$

After performing the angular integration (the result is $|z|^{2n}\delta_{nn'}$) we get the expression

$$\text{R.H.S.} = \#\frac{1}{{}_pF_q\left(\{a_i\}_1^p;\ \{b_j\}_1^q;\ A_+A_-\right)} \times \sum_{n=0}^{\infty}\frac{1}{\sqrt{\rho_{p,q}(n)}}\left(\frac{\bar{n}}{\bar{n}+1}A_+A_-\right)^n \# \left[\frac{1}{\left(\frac{\bar{n}}{\bar{n}+1}\right)^n}\frac{1}{n!}\frac{\prod_{i=1}^{p}\Gamma(a_i+n)}{\prod_{j=1}^{q}\Gamma(b_j+n)}\right] \times \int_0^\infty d\left(|z|^2\right)P_{p,q}\left(|z|^2\right) \times G_{p,\,q+1}^{q+1,0}\left(|z|^2\left|\begin{matrix}/; & \{a_i-1\}_1^p\\ 0,\ \{b_j-1\}_1^q; & /\end{matrix}\right.\right)\left(|z|^2\right)^n \quad (4.14)$$

We proceed in a manner similar to those used for deducing the integration measure $d\mu_{p,q}(z)$ (3.23), i.e. we perform the suitable function change

$$\tilde{P}_{p,q}\left(|z|^2\right) = P_{p,q}\left(|z|^2\right) \times G_{p,q+1}^{q+1,0}\left(|z|^2\left|\begin{matrix}/; & \{a_i-1\}_1^p\\ 0,\ \{b_j-1\}_1^q; & /\end{matrix}\right.\right) \quad (4.15)$$

From the definition of the hypergeometric function (A.2) and comparing it with the left-hand side of equation (4.12), we see that the result of the integration must be

$$\int_0^\infty d\left(|z|^2\right)\tilde{P}_{p,q}\left(|z|^2\right)\left(|z|^2\right)^n = \frac{1}{\bar{n}+1}\left(\frac{\bar{n}}{\bar{n}+1}\right)^n n!\frac{\prod_{j=1}^{q}\Gamma(b_j+n)}{\prod_{i=1}^{p}\Gamma(a_i+n)} \quad (4.16)$$

In order to get to the Stieltjes moment problem we perform the exponent change, i.e. $n = s - 1$, and we get the integral

$$\int_0^\infty d\left(|z|^2\right)\tilde{P}_{p,q}\left(|z|^2\right)\left(|z|^2\right)^{s-1} = \frac{1}{\bar{n}}\frac{1}{\left(\frac{\bar{n}+1}{\bar{n}}\right)^s}\Gamma(s)\frac{\prod_{j=1}^{q}\Gamma(b_j-1+s)}{\prod_{i=1}^{p}\Gamma(a_i-1+s)} \quad (4.17)$$

This integral is fulfilled if the checked function $\tilde{P}_{p,q}(|z|^2)$ is [20]

$$\tilde{P}_{p,q}\left(|z|^2\right) = \times\frac{1}{\bar{n}}G_{p,q+1}^{q+1,0}\left(\frac{\bar{n}+1}{\bar{n}}|z|^2\left|\begin{matrix}/; & \{a_i-1\}_1^p\\ 0,\ \{b_j-1\}_1^q; & /\end{matrix}\right.\right) \quad (4.18)$$

so that finally the quasi-distribution function $P_{p,q}(|z|^2)$ is

$$P_{p,q}\left(|z|^2\right) = \frac{1}{\bar{n}} \times \frac{G_{p,q+1}^{q+1,0}\left(\frac{\bar{n}+1}{\bar{n}}|z|^2\left|\begin{matrix}/; & \{a_i-1\}_1^p\\ 0,\ \{b_j-1\}_1^q; & /\end{matrix}\right.\right)}{G_{p,q+1}^{q+1,0}\left(|z|^2\left|\begin{matrix}/; & \{a_i-1\}_1^p\\ 0,\ \{b_j-1\}_1^q; & /\end{matrix}\right.\right)} \quad (4.19)$$

The diagonal expansion of the normalized canonical density operator in terms of the GH-BG-CSs projectors is useful in order to calculate the thermal expectation values (thermal averages) of the operators $\#\boldsymbol{A}\#$, which characterize the quantum system. The thermal expectation values are defined by

$$\langle\#\boldsymbol{A}\#\rangle = \mathrm{Tr}\,(\#\rho\boldsymbol{A}\#) = \int d\mu_{p,q}(z)P_{p,q}\left(|z|^2\right)\langle z;\lambda|\#\boldsymbol{A}\#|z;\lambda\rangle \quad (4.20)$$

If $\boldsymbol{A} = I$ (unity operator), we obtain that the function $P_{p,q}(|z|^2)$ is normalized to unity

$$\int d\mu_{p,q}(z)P_{p,q}\left(|z|^2\right) = 1 \quad (4.21)$$

On the other hand, if $\boldsymbol{A} = \#(A_+A_-)^s\#$, we have, successively

$$\left\langle\left(A_+A_-\right)^s\right\rangle = \int d\mu_{p,q}(z)P_{p,q}\left(|z|^2\right)\left\langle z;\lambda\left|\left(A_+A_-\right)^s\right|z;\lambda\right\rangle = \int d\mu_{p,q}(z)P_{p,q}\left(|z|^2\right)\left(|z|^2\right)^s = \frac{1}{\bar{n}}\frac{\prod_{i=1}^{p}\Gamma(a_i)}{\prod_{j=1}^{q}\Gamma(b_j)}\int_0^\infty d\left(|z|^2\right)\left(|z|^2\right)^s \times {}_pF_q\left(\{a_i\}_1^p;\ \{b_j\}_1^q;\ |z|^2\right) \times G_{p,\,q+1}^{q+1,0}\left(\frac{\bar{n}+1}{\bar{n}}|z|^2 \times\left|\begin{matrix}/; & \{a_i-1\}_1^p\\ 0,\ \{b_j-1\}_1^q; & /\end{matrix}\right.\right) \quad (4.22)$$

Using the definition of the hypergeometric functions (A.2) and also the integral formula (A.6), we obtain

$$\left\langle\left(A_+A_-\right)^s\right\rangle = \frac{1}{\bar{n}}\frac{\prod_{i=1}^{p}\Gamma(a_i)}{\prod_{j=1}^{q}\Gamma(b_j)}\sum_{n=0}^{\infty}\frac{1}{\sqrt{\rho_{p,q}(n)}} \times \int_0^\infty d\left(|z|^2\right)\left(|z|^2\right)^{(n+s+1)-1}$$

$$\times G^{q+1,0}_{p,q+1}\left(\frac{\bar{n}+1}{\bar{n}}|z|^2 \left| \begin{matrix} /; & \{a_i-1\}_1^p \\ 0,\ \{b_j-1\}_1^q; & / \end{matrix} \right.\right)$$

$$= \frac{1}{\bar{n}+1}\left(\frac{\bar{n}}{\bar{n}+1}\right)^s \rho_{p,q}(s)\sum_{n=0}^{\infty}(s+1)_n$$

$$\times \frac{\prod_{i=1}^p (a_i)_n \prod_{j=1}^q \left(b_j+s\right)_n}{\prod_{i=1}^p (a_i+s)_n \prod_{j=1}^q \left(b_j\right)_n}\frac{\left(\frac{\bar{n}}{\bar{n}+1}\right)^n}{n!} \tag{4.23}$$

The final result is then

$$\left\langle \left(A_+A_-\right)^s\right\rangle = \frac{1}{\bar{n}+1}\left(\frac{\bar{n}}{\bar{n}+1}\right)^s \rho_{p,q}(s)_{p+q+1} F_{p+q}\left(s+1,\ \{a_i\}_1^p,\ \left\{b_j+s\right\}_1^q;\ \{a_i+s\}_1^p,\ \left\{b_j\right\}_1^q;\ \frac{\bar{n}}{\bar{n}+1}\right) \tag{4.24}$$

Let us verify this result in two cases: firstly, if we take $s = 0$, we implicitly verify the normalization condition of the quasi-distribution function $P_{p,q}(|z|^2)$

$$1 = \frac{1}{\bar{n}+1}\rho_{1,0}(0)_1 F_0\left(1;\ ;\ \frac{\bar{n}}{\bar{n}+1}\right) = \frac{1}{\bar{n}+1}\frac{1}{1-\frac{1}{\bar{n}+1}} \tag{4.25}$$

Secondly, for $s = 1$ we obtain

$$\left\langle A_+A_-\right\rangle = \frac{1}{\bar{n}+1}\frac{\bar{n}}{\bar{n}+1}\frac{\prod_{j=1}^q \left(b_j\right)_1}{\prod_{i=1}^p (a_i)_1}{}_{p+q+1}F_{p+q} \times\left(2,\ \{a_i\}_1^p,\ \left\{b_j+1\right\}_1^q;\ \{a_i+1\}_1^p,\ \times\left\{b_j\right\}_1^q;\ \frac{\bar{n}}{\bar{n}+1}\right) \tag{4.26}$$

where $(b_j)_1 = \Gamma(b_j+1)/\Gamma(b_j) = b_j$, and so on.

In the definition of the hypergeometric function the ratios will appear as

$$\frac{\prod_{i=1}^p (a_i)_n}{\prod_{i=1}^p \left(a_i+1\right)_n} = \prod_{i=1}^p \frac{\Gamma(a_i+n)}{\Gamma(a_i)}\frac{\Gamma\left(a_i+1\right)}{\Gamma\left(a_i+1+n\right)} = \frac{\prod_{i=1}^p a_i}{\prod_{i=1}^p (a_i+n)}, \tag{4.27}$$

Which is similar for the coefficients b_j. In addition, $(2)_n = \Gamma(2+n) = (n+1)!$; so, we obtain

$$\left\langle A_+A_-\right\rangle = \frac{1}{\bar{n}+1}\frac{\bar{n}}{\bar{n}+1}\sum_{n=0}^{\infty}\left(\frac{\bar{n}}{\bar{n}+1}\right)^n (n+1) \times\frac{\prod_{j=1}^q \left(b_j+n\right)}{\prod_{i=1}^p (a_i+n)} \tag{4.28}$$

This result can be obtained in another way, i.e. if we calculate the thermal expectation in the Fock vectors representation successively using equations and (2.3), (2.5), (2.8), (4.1)

$$\langle A_+A_-\rangle = \mathrm{Tr}\left(\rho\, A_+A_-\right) = \sum_{n'=0}^{\infty}\langle n';\lambda|\rho A_+A_-|n';\lambda\rangle = \frac{1}{\bar{n}+1}\sum_{n=0}^{\infty}\left(\frac{\bar{n}}{\bar{n}+1}\right)^n\langle n;\lambda|A_+A_-|n;\lambda\rangle = \frac{1}{\bar{n}+1}\sum_{n=0}^{\infty}\left(\frac{\bar{n}}{\bar{n}+1}\right)^n n\frac{\prod_{j=1}^q \left(b_j+n-1\right)}{\prod_{i=1}^p \left(a_i+n-1\right)} \tag{4.29}$$

If we again eliminate the term with $n = 0$, which has no effect, and by changing the summation index $n' = n - 1$, we get the result above.

5. Some illustrative examples

In order to illustrate the applicability of the DOOT for the generalized hypergeometric operators A_- and A_+, we apply it to two cases: the HO-1D and the PHO. In this manner we recover, in another way, some previously obtained results (see, e.g. [7, 8, 11, 13, 22–27, 29]). For each result we will indicate the corresponding above equation which generates this result.

5.1. The one-dimensional harmonic oscillator (HO-1D)

If we choose $p = 0$, $q = 0$, $a_i = 0$, $b_j = 0$, we obtain, successively, $f_{0,0}(n) = \sqrt{n+1}$, $\rho_{0,0}(n) = n!$, $A_- = a$, $A_+ = a^+$, $\lambda = 0$, $|n;\lambda = 0\rangle \equiv |n\rangle$, $|z;\lambda = 0\rangle \equiv |z\rangle$, $E_{n,0} = \hbar\omega\, n + E_{0,0} = \hbar\omega\, n + \frac{1}{2}\hbar\omega$ and ${}_0F_0(;\ ;\ |z|^2) = \exp(|z|^2)$.

➢ The annihilation and creation operators (2.1) and (2.2)

$$a = \sum_{n=0}^{\infty}\sqrt{n+1}\,|n\rangle\langle n+1|, \quad a^+ = \sum_{n=0}^{\infty}\sqrt{n+1}\,|n+1\rangle\langle n| \tag{5.1}$$

For the HO-1D the DOOT coincides with the IWOP technique [9].

- Expectation value of the normally ordered product in the Fock-vector state (2.8)

$$\langle n|\#a^+a\,\#|n\rangle = \langle n|a^+a|n\rangle = n \tag{5.2}$$

- Expectation value in the Fock-vector state for a function of normal ordered operators product a^+a (2.13)

$$\langle n|\#F\left(a^+a\right)\#|n\rangle = F(n) \tag{5.3}$$

- The projector of the vacuum state (2.11)

$$|0;\lambda\rangle\langle 0\,;\lambda| = \#\frac{1}{{}_0F_0\left(;\ ;\ a^+a\right)}\# = \#e^{-a^+a}\# \tag{5.4}$$

- Definition of the HO-1D CSs [2, 4], (3.1)

$$a\mid z\rangle = z\mid z\rangle \tag{5.5}$$

- The expansion of the BG-CSs in the Fock-vectors basis [2, 4], (3.7)

$$\mid z\rangle = e^{-\frac{1}{2}|z|^2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n\,!}}\mid n\rangle \tag{5.6}$$

- The BG-CSs written in an operatorial manner (for the HO-CSs this is equivalent to the Klauder–Perelomov CSs) [4], (3.12)

$$\mid z;\lambda\rangle = e^{-\frac{1}{2}|z|^2}e^{z\,a^+}\mid 0\rangle \tag{5.7}$$

- The projector onto a BG-CS [12], (3.13)

$$\begin{aligned}|z\rangle\langle z| &= e^{-|z|^2}\#e^{-a^+a}e^{-z\,a^+}\\ &\times e^{-z^*a}\# = \#e^{-\left(z^*-a^+\right)(z-a)}\#\end{aligned} \tag{5.8}$$

where we have used the canonical commutation relation $[a\,,\ a^+] = 1$ and also the disentangled Baker–Campbell–Hausdorff formula (see, e. g. [30, 31])

$$\begin{aligned}e^{A+B} &= e^Ae^Be^{-\frac{1}{2}[A,B]},\ \text{if}[[A,B],A]\\ &= [[A,B],B] = 0,\end{aligned} \tag{5.9}$$

- The overlap of two BG-CSs (3.14) lead to

$$\begin{aligned}\langle z|z'\rangle &= \frac{{}_0F_0(\ ;\ ;\ z^*z')}{\sqrt{{}_0F_0\left(;\ ;\ |z|^2\right){}_0F_0\left(\ ;\ ;\ |z'|^2\right)}}\\ &= e^{-\frac{1}{2}\left(|z|^2+|z'|^2\right)+z^*z'}.\end{aligned} \tag{5.10}$$

- The integration measure (3.23) becomes

$$\begin{aligned}\mathrm{d}\mu_{0,0}(z) &= \frac{\mathrm{d}\phi}{2\pi}\mathrm{d}\left(|z|^2\right){}_0F_0\left(;\,;\,|z|^2\right)G_{0,1}^{1,0}\left(|z|^2\,\middle|\,\begin{matrix}/; & /\\ 0,\,; & /\end{matrix}\right)\\ &= \frac{\mathrm{d}\phi}{2\pi}\mathrm{d}\left(|z|^2\right)e^{|z|^2}e^{-|z|^2} = \frac{\mathrm{d}\phi}{2\pi}\mathrm{d}\left(|z|^2\right)\end{aligned} \tag{5.11}$$

- The expectation value of an operator $\boldsymbol{A}$ in the BG-CSs representation (3.25) is

$$\begin{aligned}\langle z|\boldsymbol{A}|z\rangle &\equiv \langle\boldsymbol{A}\rangle_z\\ &= e^{-|z|^2}\sum_{n',n=0}^{\infty}\frac{(z^*)^{n'}z^n}{\sqrt{n'!}\sqrt{n!}}\langle n'|\boldsymbol{A}|\,n\rangle\end{aligned} \tag{5.12}$$

- The Mandel parameter (3.28) can be expressed as follows, with $x = |z\,|^2$ [30]

$$\begin{aligned}Q_{|z|} &= x\left[\frac{\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2{}_0F_0(;\ \ ;\ x)}{\frac{\mathrm{d}}{\mathrm{d}x}{}_0F_0(;\ \ ;\ x)} - \frac{\frac{\mathrm{d}}{\mathrm{d}x}{}_0F_0(;\ \ ;\ x)}{{}_0F_0(;\ \ ;\ x)}\right]\\ &= x\left[\frac{\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2e^x}{\frac{\mathrm{d}}{\mathrm{d}x}e^x} - \frac{\frac{\mathrm{d}}{\mathrm{d}x}e^x}{e^x}\right] = 0.\end{aligned} \tag{5.13}$$

which means that the CSs of the HO-1D have a Poissonian behavior.

- For a function of normal ordered operators $a^+\,a$, according to (3.32), we have

$$\langle z|\#F\left(a^+\,a\right)\#|z\rangle = F\left(|z|^2\right) \tag{5.14}$$

- The normalized density operator in a normal ordered form (4.8) reads [27]

$$\begin{aligned}\rho &= \frac{1}{\bar{n}+1}\#\frac{{}_0F_0\left(;\ \ ;\ \frac{\bar{n}}{\bar{n}+1}a^+a\right)}{{}_0F_0\left(;\ \ ;\ a^+a\right)}\#\\ &= \frac{1}{\bar{n}+1}\#e^{-\frac{1}{\bar{n}+1}a^+a}\#.\end{aligned} \tag{5.15}$$

- The Q-distribution function in the representation of BG-CSs (4.9) for HO-1D is [27]

$$Q_{0,0}\left(|z|^2\right) \equiv \langle z|\rho|z\rangle = \frac{1}{\bar{n}+1}e^{-\frac{1}{\bar{n}+1}|z|^2} \tag{5.16}$$

- The P quasi-distribution function (4.19) is then [27]

$$\begin{aligned}P_{0,0}\left(|z|^2\right) &= \frac{1}{\bar{n}}\frac{G_{0,1}^{1,0}\left(\frac{\bar{n}+1}{\bar{n}}|z|^2\,\middle|\,\begin{matrix}/; & /\\ 0,\,/; & /\end{matrix}\right)}{G_{0,1}^{1,0}\left(|z|^2\,\middle|\,\begin{matrix}/; & /\\ 0,\,/; & /\end{matrix}\right)}\\ &= \frac{1}{\bar{n}}e^{-\frac{1}{\bar{n}}|z|^2}.\end{aligned} \tag{5.17}$$

- Thermal expectation value (4.26) of normally ordered operator $\boldsymbol{A} = \#a^+\,a\# = a^+\,a$ is

$$\begin{aligned}\langle a^+\,a\rangle &= \frac{1}{\bar{n}+1}\frac{\bar{n}}{\bar{n}+1}{}_1F_0\left(2;\ \ ;\ \frac{\bar{n}}{\bar{n}+1}\right)\\ &= \frac{1}{\bar{n}+1}\frac{\bar{n}}{\bar{n}+1}\left(\bar{n}+1\right)^2 = \bar{n}.\end{aligned} \tag{5.18}$$

This result can also be found by using equation (4.29), i.e. by calculating the thermal expectation value in the Fock-vectors basis.

5.2. The pseudoharmonic oscillator (PHO)

If we choose $p = 0$, $q = 1$, $a_i = 0$, $b_1 = 2k$, $b_j = 0$, $j = 2, 3, \ldots, q$, we obtain, successively [32] $f_{0,1}(n) = \sqrt{(n+1)(2k+n)}$, $A_- = K_-$, $A_+ = K_+$, $\lambda = k$, $|n;\lambda\rangle \equiv |n;\ k\rangle$, $|z;\ \lambda\rangle \equiv |z;\ k\rangle$, with energy eigenvalues $E_{n,J} = \hbar\omega\, n + E_{0,J} = \hbar\omega\, n + \hbar\omega\, k - \frac{m\omega^2}{4} r_0^2$ and the characteristic hypergeometric function ${}_0F_1(;\ 2k;\ |z|^2) = \Gamma(2k)\frac{I_{2k-1}(2|z|)}{|z|^{2k-1}}$, where $I_{2k-1}(2|z|)$ is the modified Bessel function of the first kind [20, 33].

- The annihilation and creation operators (2.1) and (2.2) are

$$K_- = \sum_{n=0}^{\infty} \sqrt{(n+1)(2k+n)}\,|n;\ k\rangle\langle n+1;\ k|,$$
$$K_+ = \sum_{n=0}^{\infty} \sqrt{(n+1)(2k+n)}\,|n+1;\ k\rangle\langle n;\ k| \qquad (5.19)$$

- The expectation value of the normally ordered product in the state $|n;\ k>$; see (2.8)

$$\langle n;\ k\left|K_+K_-\right|n;\ k\rangle = n(2k+n-1) \qquad (5.20)$$

- The expectation value of a function of normal operator product $F(K_+K_-)$ (2.13)

$$\langle n;\ k\left|\#F\left(K_+K_-\right)\#\right|n;\ k\rangle = F\left[n(2k+n-1)\right] \qquad (5.21)$$

- The projector of vacuum state (2.11)

$$|0;\ k\rangle\langle 0;\ k| = \#\frac{1}{{}_0F_1\left(;\ 2k;\ K_+K_-\right)}\# = \frac{1}{\Gamma(2k)}\#\frac{\left(\sqrt{K_+K_-}\right)^{2k-1}}{I_{2k-1}\left(2\sqrt{K_+K_-}\right)}\# \qquad (5.22)$$

- The definition of the BG-CSs (3.1) for the PHO [29, 32]

$$K_-|z;\ k\rangle = z|z;\ k\rangle \qquad (5.23)$$

- The expansion of the BG-CSs in the Fock-vectors basis (3.7) [29]

$$|z;\ k\rangle = \sqrt{\frac{|z|^{2k-1}}{\Gamma(2k)I_{2k-1}(2|z|)}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!(2k)_n}}|n;\ k\rangle \qquad (5.24)$$

- The BG-CSs written in the operatorial manner (3.11)

$$|z;\ k\rangle = \frac{1}{\sqrt{{}_0F_1\left(;\ 2k;\ |z|^2\right)}}\,{}_0F_1\left(;\ 2k;\ z\,K_+\right)|0;\ k\rangle = \sqrt{\Gamma(2k)}\sqrt{\frac{e^{-i(2k-1)\phi}}{I_{2k-1}(2|z|)}}\frac{I_{2k-1}\left(2\sqrt{z\,K_+}\right)}{\left(\sqrt{K_+}\right)^{2k-1}}|0;\ k\rangle \qquad (5.25)$$

- The projector onto a BG-CS (3.13)

$$|z;\ k\rangle\langle z;\ k| = \frac{1}{{}_0F_1\left(;\ 2k;\ |z|^2\right)}\#\frac{{}_0F_1\left(;\ 2k;\ z\,K_+\right){}_0F_k\left(;\ 2k;\ z^*K_-\right)}{{}_0F_1\left(;\ 2k;\ K_+K_-\right)}\# = \frac{1}{I_{2k-1}(2|z|)}\#\frac{I_{2k-1}\left(2\sqrt{z\,K_+}\right)I_{2k-1}\left(2\sqrt{z^*K}\right)}{I_{2k-1}\left(2\sqrt{K_+K_-}\right)}\# \qquad (5.26)$$

- The overlap (scalar product) of two BG-CSs (3.14) [32]

$$\langle z;\ k|z';\ k\rangle = \frac{{}_0F_1\left(;\ 2k;\ z^*z'\right)}{\sqrt{{}_0F_1\left(;\ 2k;\ |z|^2\right){}_0F_1\left(;\ 2k;\ |z'|^2\right)}} = \frac{I_{2k-1}\left(2\sqrt{z^*z'}\right)}{I_{2k-1}(2|z|)\,I_{2k-1}(2|z'|)} \qquad (5.27)$$

- The integration measure (3.23) becomes

$$d\mu_{0,1}(z) = \frac{d\phi}{2\pi}d\left(|z|^2\right)\frac{1}{\Gamma(2k)}{}_0F_1\left(;\ 2k;\ |z|^2\right) \times G_{0,2}^{2,0}\left(|z|^2\,\middle|\,\begin{matrix}/; & /\\ 0,\ 2k; & /\end{matrix}\right) = 2\frac{d\phi}{2\pi}d\left(|z|^2\right)I_{2k-1}(2|z|)K_{2k-1}(2|z|) \qquad (5.28)$$

- The expectation value of an operator $\boldsymbol{A}$ in the BG-CSs representation (3.25)

$$\langle z;\ k|\boldsymbol{A}|z;\ k\rangle \equiv \langle\boldsymbol{A}\rangle_{z;k} = \frac{|z|^{2k-1}}{\Gamma(2k)I_{2k-1}(2|z|)} \times \sum_{n',n=0}^{\infty}\frac{\left(z^*\right)^{n'}z^n}{\sqrt{n'!\,(2k)_{n'}\,n!\,(2k)_n}}\langle n';\ \lambda|\boldsymbol{A}|n;\ \lambda\rangle \qquad (5.29)$$

- Using the equality ${}_0F_1(;\ 2k;\ |z|^2) = \Gamma(2k)\frac{I_{2k-1}(2|z|)}{|z|^{2k-1}}$ and also the differential properties of the hypergeometric functions (A.3), the Mandel parameter (3.28) can finally be expressed as follows [7, 32]

$$Q_{z;\lambda} = |z|\left[\frac{I_{2k+1}(2|z|)}{I_{2k}(2|z|)} - \frac{I_{2k}(2|z|)}{I_{2k-1}(2|z|)}\right]. \qquad (5.30)$$

- The expectation value of the normal ordered product K_+K_- (3.32) is (see also [34])

$$\langle z;\ k|K_+K_-|z;\ k\rangle = |z|^2 \qquad (5.31)$$

and, consequently, for a function of normal ordered operators K_+K_- (3.32)

$$\langle z;\ k|\#F\left(K_+K_-\right)\#|z;\ k\rangle = F\left(|z|^2\right) \qquad (5.32)$$

- The normalized density operator in a normal ordered form (4.3) reads

$$\rho = \frac{1}{\bar{n}+1} \# \frac{{}_0F_1\left(;\,2k;\,\frac{\bar{n}}{\bar{n}+1}K_+K_-\right)}{{}_0F_1\left(;\,2k\,;\,K_+K_-\right)} \#$$
$$= \frac{1}{\bar{n}+1} \frac{1}{\left(\sqrt{\frac{\bar{n}}{\bar{n}+1}}\right)^{2k-1}} \# \frac{I_{2k-1}\left(2\sqrt{\frac{\bar{n}}{\bar{n}+1}K_+K_-}\right)}{I_{2k-1}\left(2\sqrt{K_+K_-}\right)} \# \quad (5.33)$$

- The Q-distribution function in the representation of BG-CSs (4.9) is [32]

$$Q_{0,1}\left(|z|^2\right) = \frac{1}{\bar{n}+1} \frac{{}_0F_1\left(;\,2k;\,\frac{\bar{n}}{\bar{n}+1}|z|^2\right)}{{}_0F_1\left(;\,2k\,;\,|z|^2\right)}$$
$$= \frac{1}{\bar{n}+1} \frac{1}{\left(\sqrt{\frac{\bar{n}}{\bar{n}+1}}\right)^{2k-1}} \frac{I_{2k-1}\left(2\,|z|\,\sqrt{\frac{\bar{n}}{\bar{n}+1}}\right)}{I_{2k-1}(2\,|z|)} \quad (5.34)$$

- The quasi-distribution function $P_{p,q}(|z|^2)$ (4.21) is [28, 32]

$$P_{0,1}\left(|z|^2\right) = \frac{1}{\bar{n}} \frac{G^{2,0}_{0,2}\left(\frac{\bar{n}+1}{\bar{n}}|z|^2 \,\middle|\, \begin{matrix} /\,; & / \\ 0, & 2k\,;\ / \end{matrix}\right)}{G^{q+1,0}_{p,q+1}\left(|z|^2 \,\middle|\, \begin{matrix} /\,; & / \\ 0\,, & 2k\,;\ / \end{matrix}\right)}$$
$$= \frac{1}{\bar{n}}\left(\sqrt{\frac{\bar{n}+1}{\bar{n}}}\right)^{2k-1} \frac{K_{2k-1}\left(2|z|\sqrt{\frac{\bar{n}}{\bar{n}+1}}\right)}{K_{2k-1}(2\,|z|)}. \quad (5.35)$$

- The thermal expectation value of $\boldsymbol{A} = K_+\,K_-$ (4.26) is obtained using equation (A.7)

$$\langle K_+K_-\rangle = \frac{1}{\bar{n}+1}\frac{\bar{n}}{\bar{n}+1}2k\,{}_2F_1$$
$$\times\left(2,\ 2k+1;\ 2k\,;\,\frac{\bar{n}}{\bar{n}+1}\right) =$$
$$= \frac{\bar{n}}{(\bar{n}+1)^2}2k\ (\bar{n}+1)^3$$
$$\times\left[1-\left(1-\frac{1}{k}\right)\frac{\bar{n}}{\bar{n}+1}\right] = 2\bar{n}\left(\bar{n}+k\right) \quad (5.36)$$

The same result can be obtained if we use the thermal expectation formula in the Fock-vectors basis (4.29).

6. Concluding remarks

In the last few decades, the CSs approach has been considerably generalized and has become a powerful tool in many applications since mathematical physics, the quantum information theory and quantum communication. In this paper we have extended the boson normal ordering technique (which is elaborated in the frame of the IWOP technique) to the GH-BG-CSs. We achieved this goal by introducing a new computational approach that we have called the DOOT. In essence, this technique is applicable only to functions that depend on normal ordered products of generalized creation A_+ and annihilation A_- operators, i.e. to functions which depend on the operatorial function 'argument' A_+A_-. In this manner we rediscovered, but in a different way, a series of remarkable results obtained by other methods that allow us to perform many calculations of CSs formalism with direct applications in quantum optics.

For this idea we thought that it would be useful to broaden the applicability area of the normally ordered technique on other operators, respectively other CSs, by using the DOOT. We have chosen a large class of holomorphic CSs, namely the GH-CSs, defined in the BG manner, previously deduced by Appl and Schiller [7]. Due to their generality, if their parameters are particularized, these states lead to a series of known CSs, defined on the whole complex z-plane, on the open disk with $|z| < 1$ or on the unit circle ($|z| = 1$).

For the GH-BG-CSs, by the DOOT we have deduced the expression of the projector of the vacuum state, as well as the projector on a GH-CS, $|z;\,\lambda\rangle\langle z;\,\lambda|$, which, together with the generalized integration measure, allow the resolution of the unity operator. By means of the deduced formula for the expectation values, we are able to express the Mandel parameter, which characterizes the behavior of the GH-CSs. As the example of mixed states, we have chosen the thermal states with the corresponding canonical density operator. Their expression, deduced with the DOOT, allows us to obtain the expression of the Q-distribution function, as well as the P-quasi distribution function, which appears in the diagonal expansion of the density operator in the GH-BG-CSs basis. Finally, we have calculated the thermal expectation values for some operators which characterize the quantum system under consideration.

In order to illustrate the correctness of the obtained results and formulae, we have applied them to two known CSs for two oscillators, namely the HO-1D and, respectively, the PHO. The obtained results using the DOOT are the same as those known in literature, but they were obtained by alternative calculation methods. This proves the usefulness of the extension of the DOOT to the GH-BG-CSs.

This paper proposes a new, much simpler approach that refers to calculations with operators involved in the CSs formalism instead of ordinary algebraic calculations. The practical usefulness of the DOOT lies in its generality: it is applicable not only to generation a^+ and annihilation a operators corresponding to HO-1D but to more general kinds of generation A_+ and annihilation A_- operators, which generate GH-BG-CSs. Calculations with the DOOT allow us to find expressions of different features of GH-BG-CSs (e.g. integration measure, density operator, Q and P functions, expectation values, and so on). Then, by simply customizing the indices p and q of the hypergeometric function ${}_pF_q\left(\{a_i\}_1^p;\ \left\{b_j\right\}_1^q;\ |z|^2\right)$ one can get these features that correspond to different types of CSs. We consider that our paper can contribute to the enrichment of the theory and to applications of the operator ordered techniques. Implicitly, our approach, which we have called the DOOT, emphasizes the validity and usefulness of the IWOP technique that was perfected and promoted in several previous works by H-Y Fan [9–15].

Finally, we want to make the remark that the GH-BG-CSs can be regarded as non-linear coherent states of the HO-1D if we consider an operator of the following manner $A_- \equiv f_{p,q}(N)\, a$, where N is the particle number operator $N\,|n;\,\lambda\rangle = n\,|n;\,\lambda\rangle$. Then, GH-BG-CSs can be defined as non-linear CSs of the HO-1D $f_{p,q}(N)\, a|z;\,\lambda\rangle = z|z;\,\lambda\rangle$ [23, 35].

In this way, the impressive gallery of non-linear CSs of the HO-1D may be completed (see, e.g. [5] and the references therein).

Appendix A

In what follows, we insert some useful properties of the generalized hypergeometric functions and the Meijer's G-functions [20, 21, 28].

The definition of the generalized hypergeometric function

$$
{}_pF_q\left(a_1,\, ...,\, a_p;\;\; b_1,\; ...,\; b_q;\; x\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n\,(a_2)_n\,...\,(a_p)_n}{(b_1)_n\,(b_2)_n\,...\,(b_q)_n}\frac{x^n}{n\,!} \tag{A.1}
$$

or, using our short notations

$$
{}_pF_q\left(\{a\}_1^p;\;\{b\}_1^q;\;x\right) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{j=1}^{q}(b_j)_n}\frac{x^n}{n!} = \sum_{n=0}^{\infty}\frac{1}{\rho_{p,q}(n)}x^n \tag{A.2}
$$

Their repeatable derivative with respect to the argument is [28]

$$
\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^s {}_pF_q\left(\{a\}_1^p\;;\;\;\{b\}_1^q;\;\;x\right) = \frac{\prod_{i=1}^{p}(a_i)_s}{\prod_{j=1}^{q}(b_j)_s}\,{}_pF_q\left(\{a+s\}_1^p\;;\;\;\{b+s\}_1^q\;;\;\;x\right) \tag{A.3}
$$

The connection between the generalized hypergeometric function and the Meijer's G-function (using the short notation)

$$
{}_pF_q\left(\{a_i\}_1^p\;;\;\;\{b_j\}_1^q\;;\;x\right) = \frac{\prod_{j=1}^{q}\Gamma\left(b_j\right)}{\prod_{i=1}^{p}\Gamma(a_i)} \times G_{p,q+1}^{1,p}\left(-x\;\middle|\;\begin{matrix}\{1-a_i\}_1^p\;; & / \\ 0\;; & \{1-b_j\}_1^q\end{matrix}\right) \tag{A.4}
$$

The classical integral from one Meijer's G-function [20, 21]

$$
\int_0^{\infty}\mathrm{d}x\;x^{s-1}G_{p,q}^{m,n}\left(\beta\,x\;\middle|\;\begin{matrix}\{a_i\}_1^n\;; & \{a_i\}_{n+1}^p \\ \{b_j\}_1^m\;; & \{b_j\}_{m+1}^q\end{matrix}\right) = \frac{1}{\beta^s}\frac{\prod_{j=1}^{m}\Gamma\left(b_j+s\right)\prod_{i=1}^{n}\Gamma\left(1-a_i-s\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j-s\right)\prod_{i=n+1}^{p}\Gamma(a_i+s)} \tag{A.5}
$$

The classical integral from two Meijer's G-functions [20, 21]

$$
\begin{aligned}
&\int_0^{\infty}\mathrm{d}x\;x^{\eta-1}G_{u,v}^{s,t}\left(\alpha\,x\;\middle|\;\begin{matrix}\{c_i\}_1^t\;; & \{c_i\}_{t+1}^u \\ \{d_j\}_1^s\;; & \{d_j\}_{s+1}^v\end{matrix}\right) \times G_{p,q}^{m,n}\left(\beta\,x\;\middle|\;\begin{matrix}\{a_i\}_1^n\;; & \{a_i\}_{n+1}^p \\ \{b_j\}_1^m\;; & \{b_j\}_{m+1}^q\end{matrix}\right) \\
&= \frac{1}{\alpha^\eta}G_{v+p,u+q}^{m+t,n+s}\left(\frac{\beta}{\alpha}\;\middle|\;\begin{matrix}\{a_i\}_1^n,\;\;\{1-\eta-d_j\}_1^s; & \{1-\eta-d_j\}_{s+1}^v,\;\;\{a_i\}_{n+1}^p \\ \{b_j\}_1^m,\;\;\{1-\eta-c_i\}_1^t; & \{1-\eta-c_i\}_{t+1}^u,\;\;\{b_j\}_{m+1}^q\end{matrix}\right)
\end{aligned} \tag{A.6}
$$

The index interchange relation for the Meijer's G-functions [20, 21]

$$
G_{p,q}^{m,n}\left(x\;\middle|\;\begin{matrix}\{a_i\}_1^n; & \{a_i\}_{n+1}^p \\ \{b_j\}_1^m; & \{b_j\}_{m+1}^q\end{matrix}\right) = G_{q,p}^{n,m}\left(\frac{1}{x}\;\middle|\;\begin{matrix}\{1-b_j\}_1^m; & \{1-b_j\}_{m+1}^q \\ \{1-a_i\}_1^n; & \{1-a_i\}_{n+1}^p\end{matrix}\right) \tag{A.7}
$$

A useful property of the Meijer's G-functions [20, 21]

$$
x^\eta G_{p,q}^{m,n}\left(x\;\middle|\;\begin{matrix}\{a_i\}_1^n; & \{a_i\}_{n+1}^p \\ \{b_j\}_1^m; & \{b_j\}_{m+1}^q\end{matrix}\right) = G_{p,q}^{m,n}\left(x\;\middle|\;\begin{matrix}\{a_i+\eta\}_1^n; & \{a_i+\eta\}_{n+1}^p \\ \{b_j+\eta\}_1^m; & \{b_j+\eta\}_{m+1}^q\end{matrix}\right) \tag{A.8}
$$

A particular value of the Gaussian or ordinary hypergeometric function [20, 21]

$$
{}_2F_1(a,\,b;\,b-1;\,x) = \frac{1}{(1-x)^{a+1}}\left[1-\left(1-\frac{a}{b-1}\right)x\right] \tag{A.9}
$$

Appendix B

As is well-known, if F^{-1} is the inverse operator of the operator F, we have $F^{-1}\,F = F\,F^{-1} = 1$. Let us consider an arbitrary creation operator C and an arbitrary annihilation operator A, not necessarily Hermitian conjugate, $C^{+} \neq A$, which act in the infinite dimensional Hilbert space of the Fock vectors $|n;\,\lambda>$ as follows

$$C|n;\ \lambda\rangle = c_{n+1}|n+1;\lambda\rangle,$$
$$A|n;\lambda\rangle = a_n|n-1;\lambda\rangle,$$
$$CA|n;\lambda\rangle = a_n c_n|n;\ \lambda\rangle. \tag{B.1}$$

In the present paper we are interested in a particular case of operator valued function F, which depends on the normally ordered product of operators CA by applying the # # operation

$$\#F\,\# \ = \#F\,(CA)\# = \sum_{l=0}^{\infty}\alpha_l\#\,(CA)^l\,\# \tag{B.2}$$

Let us suppose that their inverse operator is $(\#F\#)^{-1} = \ \#F^{-1}\#$. By multiplying this relation with $\#F\#$ and using rule (II), we have, successively

$$\#F\#\,(\#F\#)^{-1} = \#F\#\,\#F^{-1}\# = \#F\ F^{-1}\# = 1. \tag{B.3}$$

After a series expansion in CA, and according to equations (B.1) and (B.2), we have

$$\#\,F\,(CA)\,\#|n;\lambda\rangle = \sum_{l=0}^{\infty}\alpha_l\,\#(CA)^l\,\#|n;\ \lambda\rangle$$
$$= \sum_{l=0}^{\infty}\alpha_l\,(a_n c_n)^l|n;\ \lambda\rangle = F\,(a_n c_n)|n;\lambda\rangle \tag{B.4}$$

By applying the inverse operator $[\#F\,(CA)\,\#]^{-1}$ and considering (B.3), we obtain

$$[\#F\,(CA)\,\#]^{-1}\#F\,(CA)\#\,|n;\lambda\rangle$$
$$= F\,(a_n c_n)[\#F\,(CA)\,\#]^{-1}|n;\lambda\rangle, \tag{B.5}$$

from which it follows that

$$[\#F\,(CA)\,\#]^{-1}|n;\ \lambda\rangle$$
$$= \frac{1}{F\,(a_n c_n)}|n;\ \lambda\rangle \equiv \frac{1}{\#F\,(CA)\#}|n;\ \lambda\rangle \tag{B.6}$$

Consequently, the inverse operator of $\#F\,(CA)\#$ is

$$[\#F\,(CA)\,\#]^{-1} = \frac{1}{\#F\,(CA)\#} \tag{B.7}$$

In the previous sections we were interested in the case in which the two operators were $C = A_+$ and $A = A_-$ so that the inverse of the operator valued function $\#F\left(A_+A_-\right)\#$ is

$$\left[\#F\left(A_+A_-\right)\#\right]^{-1} = \frac{1}{\#F\left(A_+A_-\right)\#}. \tag{B.8}$$

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