Some Orthogonal Designs and complex Hadamard matrices by using two Hadamard matrices

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May 20, 1991

Abstract

We prove that if there exist Hadamard matrices of order h and n divisible by 4 then there exist two disjoint $W(\frac{1}{4}hn, \frac{1}{8}hn)$, whose sum is a $(1,-1)$ matrix and a complex Hadamard matrix of order $\frac{1}{4}hn$, furthermore, if there exists an $OD(m; s_1, s_2, \cdots, s_l)$ for even m then there exists an $OD(\frac{1}{4}hnm; \frac{1}{4}hns_1, \frac{1}{4}hns_2, \cdots, \frac{1}{4}hns_l)$.

1 Introduction and Basic Definitions

A *complex Hadamard matrix* (see [4]), say C, of order c is a matrix with elements $1, -1, i, -i$ satisfying $CC^* = cI$, where C^* is the Hermitian conjugate of C. From [4], any complex Hadamard matrix has order 1 or order divisible by 2. Let $C = X + iY$, where X, Y consist of $1, -1, 0$ and $X \wedge Y = 0$ where $\wedge$ is the Hadamard product. Clearly, if C is an complex Hadamard matrix then $XX^T + YY^T = cI$, $XY^T = YX^T$.

A *weighing matrix* [2] of order n with weight k, denoted by $W = W(n,k)$, is a $(1,-1,0)$ matrix satisfying $WW^T = kI_n$. $W(n,n)$ is an Hadamard matrix.

Let A_j be a $(1,-1,0)$ matrix of order m and $A_jA_j^T = s_jI_m$. An *orthogonal design* $D = x_1A_1 + x_2A_2 + \cdots + x_lA_l$ of order m and type $(s_1, s_2, \cdots, s_l)$, written $OD(m; s_1, s_2, \cdots, s_l)$, on the commuting variables $x_1, x_2, \cdots, x_l$ is a square matrix with entries $0, \pm x_1, \pm x_2, \cdots, \pm x_l$ where x_i or $-x_i$ occurs s_i times in each row and column and distinct rows are formally orthogonal. That is

$$DD^T = (\sum_{j=1}^{l} s_i x_j^2) I_m$$

Let M be a matrix of order tm. Then M can be expressed as

$$M = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1t} \\ M_{21} & M_{22} & \cdots & M_{2t} \\ & & \vdots & \\ M_{t1} & M_{t2} & \cdots & M_{tt} \end{bmatrix}$$

where M_{ij} is of order m $(i, j = 1, 2, \cdots, t)$. Analogously with Seberry and Yamada [3], we call this a t^2 *block M-structure* when M is an orthogonal matrix.

To emphasize the block structure , we use the notation $M_{(t)}$, where $M_{(t)} = M$ but in the form of t^2 blocks, each of which has order m.

Let N be a matrix of order tn. Then, write

$$N_{(t)} = \begin{bmatrix} N_{11} & N_{12} & \cdots & N_{1t} \\ N_{21} & N_{22} & \cdots & N_{2t} \\ & & \cdots & \\ N_{t1} & N_{t2} & \cdots & N_{tt} \end{bmatrix}$$

where N_{ij} is of order n $(i, j = 1, 2, \cdots, t)$.

We now define the operation $\bigcirc$ as the following:

$$M_{(t)} \bigcirc N_{(t)} = \begin{bmatrix} L_{11} & L_{12} & \cdots & L_{1t} \\ L_{21} & L_{22} & \cdots & L_{2t} \\ & & \cdots & \\ L_{t1} & L_{t2} & \cdots & L_{tt} \end{bmatrix}$$

where M_{ij}, N_{ij} and L_{ij} are of order of m, n and mn, respectively and

$$L_{ij} = M_{i1} \times N_{1j} + M_{i2} \times N_{2j} + \cdots + M_{it} \times N_{tj},$$

$i, j = 1, 2, \cdots, t$. We call this the *strong Kronecker* multiplication of two matrices.

2 Preliminaries

Theorem 1 *Let A be an $OD(tm; p_1, \cdots, p_l)$ with entries $x_1, \cdots, x_l$ and B be an $OD(tn; q_1, \cdots, q_s)$ with entries $y_1, \cdots, y_s$ then*

$$(A_{(t)} \bigcirc B_{(t)})(A_{(t)} \bigcirc B_{(t)})^T = (\sum_{j=1}^{l} p_j x_j^2)(\sum_{j=1}^{s} q_j y_j^2) I_{tmn}.$$

$(A_{(t)} \bigcirc B_{(t)}$is not an orthogonal design but an orthogonal matrix.)

Proof.

$$A_{(t)} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1t} \\ A_{21} & A_{22} & \cdots & A_{2t} \\ & & \cdots & \\ A_{t1} & A_{t2} & \cdots & A_{tt} \end{bmatrix}$$

and

$$B_{(t)} = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1t} \\ B_{21} & B_{22} & \cdots & B_{2t} \\ & & \cdots & \\ B_{t1} & B_{t2} & \cdots & B_{tt} \end{bmatrix}$$

where A_{ij} and B_{ij} are of orders m and n respectively $(i, j = 1, 2, \cdots, t)$.

Write

$$C = (A_{(t)} \bigcirc B_{(t)})(A_{(t)} \bigcirc B_{(t)})^T = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1t} \\ C_{21} & C_{22} & \cdots & C_{2t} \\ & & \cdots & \\ C_{t1} & C_{t2} & \cdots & C_{tt} \end{bmatrix}$$

where C_{ij} is of order mn.

We first prove $C_{13} = 0$. It is easy to calculate $C_{13} =$

$$= \sum_{j=1}^{t} (A_{11} \times B_{1j} + A_{12} \times B_{2j} + \cdots + A_{1t} \times B_{tj})(A_{31}^T \times B_{1j}^T + A_{32}^T \times B_{2j}^T + \cdots + A_{3t}^T \times B_{tj}^T)$$

$$= \sum_{j=1}^{t} [(A_{11}A_{31}^T) \times (B_{1j}B_{1j}^T) + (A_{12}A_{32}^T) \times (B_{2j}B_{2j}^T) + \cdots + (A_{1t}A_{3t}^T) \times (B_{tj}B_{tj}^T)]$$

$$= (A_{11}A_{31}^T + A_{12}A_{32}^T + \cdots + A_{1t}A_{3t}^T) \times (\sum_{j=1}^{s} q_j y_j^2) I_n.$$

But

$$A_{11}A_{31}^T + A_{12}A_{32}^T + \cdots + A_{1t}A_{3t}^T = 0,$$

so

$$C_{13} = 0.$$

Similarly,

$$C_{ij} = 0 \, (i \neq j).$$

We now calculate C_{ii}.

$$C_{ii} = \sum_{j=1}^{t} (A_{i1} \times B_{1j} + A_{i2} \times B_{2j} + \cdots + A_{it} \times B_{tj})(A_{i1}^T \times B_{1j}^T + A_{i2}^T \times B_{2j}^T + \cdots + A_{it}^T \times B_{tj}^T)$$

$$= \sum_{j=1}^{t} [(A_{i1}A_{i1}^T) \times (B_{1j}B_{1j}^T) + (A_{i2}A_{i2}^T) \times (B_{2j}B_{2j}^T) + \cdots + (A_{it}A_{it}^T) \times (B_{tj}B_{tj}^T)]$$

$$= (A_{i1}A_{i1}^T + A_{i2}A_{i2}^T + \cdots + A_{it}A_{it}^T) \times (\sum_{j=1}^{s} q_j y_j^2) I_n.$$

$$= (\sum_{j=1}^{l} p_j x_j^2) I_m \times (\sum_{j=1}^{s} q_j y_j^2) I_n$$

$$= (\sum_{j=1}^{l} p_j x_j^2)(\sum_{j=1}^{s} q_j y_j^2) I_{mn}.$$

Thus

$$(A_{(t)} \bigcirc B_{(t)})(A_{(t)} \bigcirc B_{(t)})^T = (\sum_{j=1}^{l} p_j x_j^2)(\sum_{j=1}^{s} q_j y_j^2) I_{tmn}.$$

Corollary 2 *Let A and B be the matrices of orders tm and tn respectively, consist of $1, -1, 0$ satisfying $AA^T = pI_{mt}$ and $BB^T = qI_{nt}$. Then*

$$(A_{(t)} \bigcirc B_{(t)})(A_{(t)} \bigcirc B_{(t)})^T = pqI_{tmn}.$$

Proof. In this case, $A = OD(tm;p)$, $B = OD(tn;q)$ and $x_1 = y_1 = 1$.

In the remainder of this paper let $H = (H_{ij})$ and $N = (N_{ij})$ of order h and n respectively be 16 block M-structures [3]. So

$$H = \begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{bmatrix}$$

where

$$\sum_{j=1}^{4} H_{ij} H_{ij}^T = hI_h = \sum_{j=1}^{4} H_{ji} H_{ji}^T,$$

for $i = 1, 2, 3, 4$ and

$$\sum_{j=1}^{4} H_{ij} H_{kj}^T = 0 = \sum_{j=1}^{4} H_{ji} H_{jk}^T,$$

for $i \neq k$, $i, k = 1, 2, 3, 4$.

Similarly, let

$$N = \begin{bmatrix} N_{11} & N_{12} & N_{13} & N_{14} \\ N_{21} & N_{22} & N_{23} & N_{24} \\ N_{31} & N_{32} & N_{33} & N_{34} \\ N_{41} & N_{42} & N_{43} & N_{44} \end{bmatrix}$$

where

$$\sum_{j=1}^{4} N_{ij} N_{ij}^T = nI_n = \sum_{j=1}^{4} N_{ji} N_{ji}^T,$$

for $i = 1, 2, 3, 4$ and

$$\sum_{j=1}^{4} N_{ij} N_{kj}^T = 0 = \sum_{j=1}^{4} N_{ji} N_{jk}^T,$$

for $i \neq k$, $i, k = 1,2,3,4$.

For ease of writing we define $X_i = \frac{1}{2}(H_{i1} + H_{i2})$, $Y_i = \frac{1}{2}(H_{i1} - H_{i2})$, $Z_i = \frac{1}{2}(H_{i3} + H_{i4})$, $W_i = \frac{1}{2}(H_{i3} - H_{i4})$, where $i = 1,2,3,4$. Then both $X_i \pm Y_i$ and $Z_i \pm W_i$ are $(1,-1)$-matrices with $X_i \wedge Y_i = 0$ and $Z_i \wedge W_i = 0$, $\wedge$ the Hadamard product.

Let

$$S = \frac{1}{2}\begin{bmatrix} H_{11}+H_{12} & -H_{11}+H_{12} & H_{13}+H_{14} & -H_{13}+H_{14} \\ H_{21}+H_{22} & -H_{21}+H_{22} & H_{23}+H_{24} & -H_{23}+H_{24} \\ H_{31}+H_{32} & -H_{31}+H_{32} & H_{33}+H_{34} & -H_{33}+H_{34} \\ H_{41}+H_{42} & -H_{41}+H_{42} & H_{43}+H_{43} & -H_{43}+H_{44} \end{bmatrix}.$$

Then S can be rewritten as

$$S = \frac{1}{2}\begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{bmatrix} \bigcirc \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & +1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & +1 \end{bmatrix}$$

or

$$S = \begin{bmatrix} X_1 & -Y_1 & Z_1 & -W_1 \\ X_2 & -Y_2 & Z_2 & -W_2 \\ X_3 & -Y_3 & Z_3 & -W_3 \\ X_4 & -Y_4 & Z_4 & -W_4 \end{bmatrix}.$$

Obviously, S is a $(0,1,-1)$ matrix.

Write

$$R = \begin{bmatrix} Y_1 & X_1 & W_1 & Z_1 \\ Y_2 & X_2 & W_2 & Z_2 \\ Y_3 & X_3 & W_3 & Z_3 \\ Y_4 & X_4 & W_4 & Z_4 \end{bmatrix},$$

also a $(0,1,-1)$ matrix.

We note $S \pm R$ is a $(1,-1)$ matrix, $R \wedge S = 0$ and by Corollary 1

$$SS^T = RR^T = \frac{1}{2}hI_h.$$

Lemma 3 *If there exists an Hadamard matrix of order h divisible by 4, there exists an $OD(h; \frac{1}{2}h, \frac{1}{2}h)$.*

Proof. From S and R as above. Now $H = S + R$. Note $HH^T = SS^T + RR^T + SR^T + RS^T = hI_h$ and $SS^T = RR^T = \frac{1}{2}hI_h$. Hence $SR^T + RS^T = 0$. Let x and y be commuting variables then $E = xS + yR$ is the required orthogonal design.

3 Weighing Matrices

Lemma 4 *If there exist Hadamard matrices of order h and n divisible by 4, there exists a* $W(\frac{1}{4}hn, \frac{1}{8}hn)$.

Proof. Let H and N as above be the Hadamard matrices of order h and n respectively. Let

$$P = \frac{1}{2}\begin{bmatrix} X_1 & Y_1 & Z_1 & W_1 \\ X_2 & Y_2 & Z_2 & W_2 \\ X_3 & Y_3 & Z_3 & W_3 \\ X_4 & Y_4 & Z_4 & W_4 \end{bmatrix} \bigcirc \begin{bmatrix} N_{11} & N_{12} & N_{13} & N_{14} \\ N_{21} & N_{22} & N_{23} & N_{24} \\ N_{31} & N_{32} & N_{33} & N_{34} \\ N_{41} & N_{42} & N_{43} & N_{44} \end{bmatrix}.$$

Rewrite

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \\ P_{41} & P_{42} & P_{43} & P_{44} \end{bmatrix}.$$

Consider

$$P_{11} = \frac{1}{2}(X_1 \times N_{11} + Y_1 \times N_{21} + Z_1 \times N_{31} + W_1 \times N_{41}),$$

where both $X_1 \times N_{11} + Y_1 \times N_{21}$ and $Z_1 \times N_{31} + W_1 \times N_{41}$ are $(1,-1)$ matrices. So P_{11} has entries $1, -1, 0$ and similarly for other P_{ij}. By Lemma 1,

$$PP^T = \frac{1}{8}hnI_{\frac{1}{4}hn}.$$

Then P is a $W(\frac{1}{4}hn, \frac{1}{8}hn)$.

Corollary 5 *There exists a* $W(h, \frac{1}{2}h)$ $(h > 1)$ *if there exists an Hadamard matrix of order* h.

Proof. If $h > 2$ let $n = 4$ in Theorem 1. For the case $h = 2$, note $W(2,1)$ is the identity matrix.

We also note that if

$$Q = \frac{1}{2}\begin{bmatrix} X_1 & Y_1 & Z_1 & W_1 \\ X_2 & Y_2 & Z_2 & W_2 \\ X_3 & Y_3 & Z_3 & W_3 \\ X_4 & Y_4 & Z_4 & W_4 \end{bmatrix} \bigcirc \begin{bmatrix} N_{11} & N_{12} & N_{13} & N_{14} \\ N_{21} & N_{22} & N_{23} & N_{24} \\ -N_{31} & -N_{32} & -N_{23} & -N_{34} \\ -N_{41} & -N_{42} & -N_{43} & -N_{44} \end{bmatrix}.$$

Then Q is also a $W(\frac{1}{4}hn, \frac{1}{8}hn)$.

Theorem 6 *Suppose h and n divisible by 4, are the orders of Hadamard matrices then there exist two disjoint* $W(\frac{1}{4}hn, \frac{1}{8}hn)$, *whose sum and difference are* $(1,-1)$ *matrices.*

Rewrite

$$Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} \\ Q_{21} & Q_{22} & Q_{23} & Q_{24} \\ Q_{31} & Q_{32} & Q_{33} & Q_{34} \\ Q_{41} & Q_{42} & Q_{43} & Q_{44} \end{bmatrix}.$$

We note

$$P_{ij} = \frac{1}{2}(X_i \times N_{1j} + Y_i \times N_{2j} + Z_i \times N_{3j} + W_i \times N_{4j}),$$

and

$$Q_{ij} = \frac{1}{2}(X_i \times N_{1j} + Y_i \times N_{2j} - Z_i \times N_{3j} - W_i \times N_{4j}).$$

Since $P_{ij} + Q_{ij} = X_i \times N_{1j} + Y_i \times N_{2j}$ and $P_{ij} - Q_{ij} = Z_i \times N_{3j} + W_i \times N_{4j}$ we conclude that $P_{ij} \pm Q_{ij}$ are $(1,-1)$ matrices and $P_{ij} \wedge Q_{ij} = 0$. Thus $P \pm Q$ is a $(1,-1)$ matrix and $P \wedge Q = 0$. P and Q are both $W(\frac{1}{4}hn, \frac{1}{8}hn)$ by Corollary 1.

4 Complex Hadamard Matrices

Lemma 7 $PQ^T = QP^T$.

Proof. Write

$$PQ^T = \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} \\ E_{21} & E_{22} & E_{23} & E_{24} \\ E_{31} & E_{32} & E_{33} & E_{34} \\ E_{41} & E_{42} & E_{43} & E_{44} \end{bmatrix}$$

and

$$QP^T = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix}.$$

We first prove $E_{13} = F_{13}$.
We note

$$E_{13} =$$

$$= \frac{1}{4}\sum_{j=1}^{4}(X_1 \times N_{1j} + Y_1 \times N_{2j} + Z_1 \times N_{3j} + W_1 \times N_{4j})(X_3^T \times N_{1j}^T + Y_3^T \times N_{2j}^T - Z_3^T \times N_{3j}^T - W_3^T \times N_{4j}^T)$$

and

$$F_{13} =$$

$$= \frac{1}{4}\sum_{j=1}^{4}(X_1 \times N_{1j} + Y_1 \times N_{2j} - Z_1 \times N_{3j} - W_1 \times N_{4j})(X_3^T \times N_{1j}^T + Y_3^T \times N_{2j}^T + Z_3^T \times N_{3j}^T + W_3^T \times N_{4j}^T).$$

Obviously, $E_{13} = F_{13}$ if and only if

$$\sum_{j=1}^{4}(X_1 \times N_{1j} + Y_1 \times N_{2j})(Z_3^T \times N_{3j}^T + W_3^T \times N_{4j}^T) \quad (1)$$

$$= \sum_{j=1}^{4}(Z_1 \times N_{3j} + W_1 \times N_{4j})(X_3^T \times N_{1j}^T + Y_3^T \times N_{2j}^T). \quad (2)$$

To show this, note

$$\sum_{j=1}^{4}(X_1 \times N_{1j})(Z_3^T \times N_{3j}^T) = \sum_{j=1}^{4}(X_1 Z_3^T) \times (N_{1j}N_{3j}^T) = X_1 Z_3^T \times \sum_{j=1}^{4} N_{1j}N_{3j}^T = 0,$$

and similarly for other parts in (1) and (2). Thus $E_{13} = F_{13}$. Similarly, $E_{ij} = F_{ij}$, for other $i \neq j$.

We now prove $E_{ii} = F_{ii}$. We see

$$E_{ii} =$$

$$= \frac{1}{4}\sum_{j=1}^{4}(X_i \times N_{1j} + Y_i \times N_{2j} + Z_i \times N_{3j} + W_i \times N_{4j})(X_i^T \times N_{1j}^T + Y_i^T \times N_{2j}^T - Z_i^T \times N_{3j}^T - W_i^T \times N_{4j}^T)$$

and

$$F_{ii} =$$

$$= \frac{1}{4}\sum_{j=1}^{4}(X_i \times N_{1j} + Y_i \times N_{2j} - Z_i \times N_{3j} - W_i \times N_{4j})(X_i^T \times N_{1j}^T + Y_i^T \times N_{2j}^T + Z_i^T \times N_{3j}^T + W_i^T \times N_{4j}^T).$$

Obviously, $E_{ii} = F_{ii}$ if and only if

$$\sum_{j=1}^{4}(X_i \times N_{1j} + Y_i \times N_{2j})(Z_i^T \times N_{3j}^T + W_i^T \times N_{4j}^T) \quad (3)$$

$$= \sum_{j=1}^{4}(Z_i \times N_{3j} + W_i \times N_{4j})(X_i^T \times N_{1j}^T + Y_i^T \times N_{2j}^T). \quad (4)$$

The proof is the same as in (1) and (2). Hence $E_{ii} = F_{ii}$. Finally, we conclude $PQ^T = QP^T$.

Theorem 8 *If there exist Hadamard matrices of order h and n divisible by 4 then there exists a complex Hadamard matrix of order $\frac{1}{4}hn$.*

Proof. By the proof of Theorem 2, P and Q are the two disjoint $W(\frac{1}{4}hn, \frac{1}{8}hn)$ i.e. $P \wedge Q = 0$ and $P \pm Q$ is a $(1,-1)$ matrix. Furthermore by Lemma 3, $PQ^T = QP^T$. Thus $P + iQ$ is a complex Hadamard matrix of order $\frac{1}{4}hn$.

5 Orthogonal Designs

Theorem 9 *If there exist Hadamard matrices of order h , n divisible by 4 and an* $OD(m; s_1, s_2, \cdots, s_l)$, *where m is even, then there exists an*

$$OD(\frac{1}{4}hnm; \frac{1}{4}hns_1, \frac{1}{4}hns_2, \cdots, \frac{1}{4}hns_l).$$

Proof. Let

$$D = \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix},$$

be the $OD(m; s_1, s_2, \cdots, s_l)$ on the commuting variables $x_1, \cdots, x_l$, where D_j is of order $\frac{1}{2}m$. Let

$$D' = \begin{bmatrix} P & Q \\ -Q & P \end{bmatrix} \bigcirc \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix}$$

where P and Q, constructed above, are from the Hadamard matrices of order h and n.

Then by Theorem 3 and Corollary 1,

$$D'D'^T = \frac{1}{4}hn(\sum_j^l s_j x_j^2) I_{\frac{1}{4}hnm}.$$

Since $P \wedge Q = 0$, if D consists of $0, \pm x_1, \cdots, \pm x_l$ then D' also consists of $0, \pm x_1, \cdots, \pm x_l$ so D' is an

$$OD(\frac{1}{4}hnm; \frac{1}{4}hns_1, \frac{1}{4}hns_2, \cdots, \frac{1}{4}hns_l).$$

Corollary 10 *If there exist Hadamard matrices of order h and n divisible by 4 then there exists an* $OD(\frac{1}{2}hn; \frac{1}{4}hn, \frac{1}{4}hn)$.

Proof. Let

$$D = \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$$

in the proof of Theorem 4, where x and y are commuting variables, put $m = l = 2$ and $s_1 = s_2 = 1$.

6 Remark

Theorem 1 cannot be replaced by Corollary 1 because the existence of Hadamard matrices of order h and n does not imply the existence of an Hadamard matrix of order $\frac{1}{4}hn$. For example, there exist Hadamard matrices of order $4 \cdot 3$ and $4 \cdot 71$ but no Hadamard matrix of order $4 \cdot 213$ has been found [1], however, by Theorem 1, we have a $W(4 \cdot 213, 2 \cdot 213)$.

By the same result, there exists a $W(4k, 2k)$ and a complex Hadamard matrix of order $4k$, where k is

781	789	917	1315	1349	1441	1633	1703	2059	2227	2489	2515
2627	2733	3013	3273	3453	3479	3715	4061	4331	4435	4757	4781
4899	4979	4997	5001	5109	5371	5433	5467	5515	5533	5609	5755
5767	5793	5893	6009	6059	6177	6209	6333	6377	6497	6539	6575
6801	6881	6887	6943	7233	7277	7387	7513	7555	7663	7739	7811
7989	8023	8057	8189	8549	8591	8611	8633	8809	8879	8927	9055
9097	9167	9557	9563	9573	9659	9727	9753	9757	9869	9913	9991

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