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Some Oscillation Theorems for a Class of Quasilinear Elliptic Equations (*).

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Abstract. – *Oscillation criteria are obtained for quasilinear elliptic equations of the form* (E) *below. We are mainly interested in the case where the coefficient function oscillates near infinity. Generalized Riccati inequalities are employed to establish our results.*

1. – Introduction.

In this paper we treat quasilinear elliptic equations of the form

$$\text{div}\,(|Du|^{m-2}Du) + a(x)\,|u|^{m-2}u = 0 \tag{E}$$

in an exterior domain $\Omega \subset \mathbb{R}^N$, where $x = (x_i)$, $Du = (\partial u/\partial x_i)$. We note that the exponent of the leading term of (E) coincides with that of the nonlinear term. Such quasilinear equations are sometimes called half-linear equations. We always assume that $N \geqslant 2$, $m > 1$, and $a \in C(\Omega)$. By a solution of (E) we mean a function u which is of class C^1 together with $|Du|^{m-2}Du$, and satisfies (E) near ∞.

DEFINITION. – (i) A nontrivial solution u of (E) (defined near ∞) is called *oscillatory* if the set $\{x \in \Omega \cap \text{dom}\, u\colon u(x) = 0\}$ is unbounded.

(ii) Equation (E) is called *oscillatory* if every nontrivial solution (defined near ∞) of (E) is oscillatory.

When $m = 2$ or $N = 1$, there are many works in which oscillatory properties of (E) are treated; see e.g. [1, 5, 6, 7]. However, as far as the author knows, there are few results concerning equation (E) with $m \neq 2$ and $N \geqslant 2$, even though a is nonnegative near ∞. Motivated by this fact, we intend here to establish sufficient conditions for (E) to be oscillatory. We are especially interested in the case where a may take on negative values for arbitrarily large $|x|$.

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The paper is organized as follows. In Section 2, we show that the oscillatory property of (E) is closely related to the nonexistence of solutions near ∞ of certain one-dimensional generalized Riccati inequalities. We discuss these inequalities in Section 3. Our main results, as well as an illustrative example, are given in Section 4.

2. – Reduction to one-dimesional problems.

In the paper we employ the notation

$$\bar{a}(r) = \frac{1}{\omega_N r^{N-1}} \int\limits_{|x|=r} a(x)\, dS$$

where $\omega_N = \int\limits_{|x|=1} dS$, the area of the unit sphere in $\mathbb{R}^N$. This function is called the spherical mean of a.

Let us begin with a preparatory consideration. Let u be a positive solution of (E) defined for $|x| \geq R$. Then, the change of variable given by

$$v = \log u$$

converts (E) into

$$\text{(E}')\qquad \operatorname{div}(|Dv|^{m-2}Dv) + (m-1)|Dv|^m + a(x) = 0\,.$$

Furthermore, introducing the function w by $w = -|Dv|^{m-2}Dv$, we get from (E′)

$$\text{(1)}\qquad \operatorname{div} w = a(x) + (m-1)|w|^{m/(m-1)}, \qquad |x| \geq R\,.$$

It should be noted that such a simple identity will not be obtainable if the exponent of the leading term and that of the nonlinear term do not coincide. The following lemma is suggested by [8].

LEMMA 1. – *Let u be a positive solution of* (E) *defined for $|x| \geq R$, and w be as above. Then, the function defined by*

$$z(r) = \int\limits_{|x|=r} \left(w(x), \frac{x}{r}\right) dS$$

for $r \geq R$, satisfies the generalized Riccati inequality

$$\text{(2)}\qquad z'(r) \geq \frac{m-1}{(\omega_N r^{N-1})^{1/(m-1)}} |z(r)|^{m/(m-1)} + \omega_N r^{N-1} \bar{a}(r)$$

for $r \geq R$. Here $(,)$ denotes the inner product between vectors.

PROOF. – Integrating both sides of (1) over the sphere $|x| = r$, we obtain

$$\int_{|x|=r} \operatorname{div} w \, dS = \omega_N r^{N-1} \bar{a}(r) + (m-1) \int_{|x|=r} |w|^{m/(m-1)} dS . \tag{3}$$

provided $r \geq R$. The divergence theorem shows that

$$z'(r) = \frac{d}{dr} \int_{|x|=r} \left(w(r), \frac{x}{r} \right) dS = \tag{4}$$

$$= \frac{d}{dr} \left(\int_{R \leq |x| \leq r} \operatorname{div} w(x) \, dx + \int_{|x|=R} \left(w(x), \frac{r}{R} \right) dS \right) = \int_{|x|=r} \operatorname{div} w(x) \, dS ,$$

if $r \geq R$. On the other hand, Hölder's inequality implies that

$$|z(r)| \leq \int_{|x|=r} |w| \cdot 1 \, dS \leq \left(\int_{|x|=r} |w|^{m/(m-1)} dS \right)^{(m-1)/m} \left(\int_{|x|=r} dS \right)^{1/m} ,$$

which is equivalent to

$$\left(\int_{|x|=r} |w|^{m/(m-1)} dS \right)^{(m-1)/m} \geq (\omega_N r^{N-1})^{-1/m} |z(r)| , \qquad r \geq R . \tag{5}$$

Inequality (2) follows from (3), (4) and (5). The proof is complete.

Lemma 1 immediately gives the following key result on which our oscillation theory is based. When $N = 1$ and $m = 2$, it is well known that a similar relationship holds between (E) and the Riccati equation associated to (E); see e.g. [3, Theorem 7.2].

PROPOSITION 2. – *Equation* (E) *is oscillatory if the generalized Riccati inequality* (2) *has no solutions near* $+\infty$.

3. – Generalized Riccati inequalities.

By Proposition 2, oscillation criteria for (E) follow from conditions which imply the nonexistence of solutions near $+\infty$ of inequality (2). For the sake of completeness, we will consider the inequality

$$h' \geq \frac{|h|^\alpha}{p(r)} + q(r), \tag{6}$$

more general than (2). We assume that $\alpha > 1$, p is a positive continuous function, and q is a continuous function defined near $+\infty$. We emphasize that q need not be nonnegative. Infinite integrals appearing in the sequel are improper: $\int^\infty = \lim_{r \to \infty} \int^r$.

PROPOSITION 3. – *Inequality* (6) *has no solutions defined near* $+\infty$ *if there is a positive* C^1*-function* φ *satisfying*

$$\int^{\infty}\left(\frac{p(r)\,|\varphi'(r)|^{\alpha}}{\varphi(r)}\right)^{1/(\alpha-1)} dr < \infty\,, \tag{7}$$

$$\int^{\infty}\frac{dr}{p(r)[\varphi(r)]^{\alpha-1}} = \infty\,, \tag{8}$$

and

$$\int^{\infty}\varphi(r)\,q(r)\,dr = \infty\,. \tag{9}$$

PROOF. – Suppose on the contrary that (6) admits a solution $h\in C^1[R,\infty)$. We may assume that φ is defined for $r\geq R$. Multiplying (6) by φ, and integrating both sides of the resulting inequality on $[R, r]$, we have

$$h(r)\,\varphi(r)\geq c_1+\int_R^r h\varphi'\,ds+\int_R^r\frac{\varphi|h|^{\alpha}}{p}\,ds+\int_R^r\varphi q\,ds\,, \tag{10}$$

where c_1 is a constant. By Hölder's inequality we have

$$\int_R^r|h\varphi'|\,ds=\int_R^r\left(\frac{\varphi}{p}\right)^{1/\alpha}|h|\cdot\left(\frac{p}{\varphi}\right)^{1/\alpha}|\varphi'|\,ds\leq c_2\left(\int_R^r\frac{\varphi|h|^{\alpha}}{p}\,ds\right)^{1/\alpha}\equiv c_2[H(r)]^{1/\alpha}\,,$$

where

$$c_2=\left(\int_R^{\infty}\left(\frac{p}{\varphi}\right)^{1/(\alpha-1)}|\varphi'|^{\alpha/(\alpha-1)}\,ds\right)^{(\alpha-1)/\alpha}$$

is finite by (7), and the function H is defined by the last equality. Hence (10) implies that

$$h(r)\varphi(r)\geq c_1-c_2[H(r)]^{1/\alpha}+\frac{1}{2}H(r)+\frac{1}{2}\int_R^r\frac{\varphi|h|^{\alpha}}{p}\,ds+\int_R^r\varphi q\,ds\,, \tag{11}$$

for every $r\geq R$. Since $\alpha>1$, the function $-c_2\xi^{1/\alpha}+\xi/2$ is bounded from below on $[0,\infty)$. Then, assumption (9) shows that the right hand side of (11) tends to $+\infty$ as $r\to$ $\to+\infty$. It follows that $h(r)>0$, if $r\geq r_0$, for some sufficiently large r_0. Again from (11) we have

$$h(r)\,\varphi(r)\geq\frac{1}{2}\int_R^r\frac{\varphi h^{\alpha}}{p}\,ds\,, \tag{12}$$

if $r \geq r_1$, for some sufficiently large $r_1 \geq r_0$. Differentiating H, we obtain by (12)

$$H'(r) = \frac{[h(r)\,\varphi(r)]^\alpha}{p(r)[\varphi(r)]^{\alpha-1}} \geq \frac{2^{-\alpha}[H(r)]^\alpha}{p(r)[\varphi(r)]^{\alpha-1}}, \qquad r \geq r_1 .$$

Dividing the both sides by $[H(r)]^\alpha$ and integrating, we have

$$\frac{1}{\alpha-1}[H(r_1)]^{1-\alpha} \geq 2^{-\alpha}\int_{r_1}^{r} \frac{ds}{p\varphi^{\alpha-1}}, \qquad r \geq r_1 ,$$

which contradicts (8). The proof is complete.

4. – Oscillation criteria for equation (E).

Now we are in a position to state our main results.

THEOREM 4. – *Equation* (E) *is oscillatory if there exists a positive C^1-function ϱ satisfying*

$$\int^\infty \frac{r^{N-1}|\varrho'(r)|^m}{[\varrho(r)]^{m-1}}\,dr < \infty , \qquad \int^\infty \frac{dr}{[r^{N-1}\varrho(r)]^{1/(m-1)}} = \infty ,$$

and

$$\int^\infty r^{N-1}\varrho(r)\,\overline{a}(r)\,dr = \infty .$$

COROLLARY 5. – (i) *Equation* (E) *is oscillatory if*

$$\int^\infty r^{m-1-\varepsilon}\,\overline{a}(r)\,dr = \infty \qquad \textit{for some } \varepsilon > 0 .$$

(ii) *Let $N \leq m$. Then, equation* (E) *is oscillatory if*

$$\int^\infty r^{N-1}\,\overline{a}(r)\,dr = \infty .$$

(iii) *Let $N = m$. Then, equation* (E) *is oscillatory if*

$$\int^\infty r^{m-1}(\log r)^{m-1-\varepsilon}\,\overline{a}(r)\,dr = \infty \qquad \textit{for some } \varepsilon > 0 .$$

Since these results can be easily proved by combining Propositions 2 and 3, the proofs are left to the readers.

REMARK 6. – The assumption $\varepsilon > 0$ in the statement of Corollary 5 can not be weakened to $\varepsilon \geqslant 0$. In fact, if $N+1-2m>0$, the equation

$$\operatorname{div}(|Du|^{m-2}Du) + (N+1-2m)\,|x|^{-m}\,|u|^{m-2}u = 0\,,$$

has the positive solution $u(x) = |x|^{-1}$, and for this equation, obviously

$$\int^{\infty} r^{m-1}\bar{a}(r)\,dr = \infty\,.$$

REMARK 7. – Let us consider the case where a has radial symmetry: $a(x) \equiv a_0(|x|)$ for some $a_0(r)$. In this case, obviously $\bar{a} \equiv a_0$. Suppose moreover that $a_0(r) \geqslant 0$ near $+\infty$. Then, it has been shown [2, 4] that:

(i) Let $N > m$. Then, equation (E) has positive radial solutions defined near ∞ if

$$\int^{\infty} r^{m-1}a_0(r)\,dr < \infty\,.$$

(ii) Let $N = m$. Then, equation (E) has positive radial solutions defined near ∞ if

$$\int^{\infty} r^{m-1}(\log r)^{m-1}a_0(r)\,dr < \infty\,.$$

Comparing Remarks 6 and 7 with Corollary 5, we find that our results are optimal in some sense.

EXAMPLE 8. – Let us consider the equation

$$\operatorname{div}(|Du|^{m-2}Du) + \frac{1+k\sin|x|}{|x|^{\nu}}\,|u|^{m-2}u = 0\,,$$

where $k > 1$ is a fixed constant, and $\nu \in \mathbb{R}$ is a parameter. Notice that the coefficient function oscillates near ∞.

(i) Let $\nu < m$. Then, this equation is oscillatory. This follows from (i) of Corollary 5.

(ii) Let $\nu > m$. Then, as in [2, 4] we can construct a positive (radial) solution of this equation defined near ∞.

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