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SOME PERTURBATION THEORY
FOR LINEAR PROGRAMMING

by

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1 Introduction

This paper examines a few relations between solution characteristics of an LP and the amount by which the LP must be perturbed to obtain either a primal infeasible LP or a dual infeasible LP. We consider such solution characteristics as the size of the optimal solution and the sensitivity of the optimal value to data perturbations. We show, for example, that an LP has a large optimal solution, or has a sensitive optimal value, only if the instance is nearly primal infeasible or dual infeasible. The results are not particularly surprising but they do formalize an interesting viewpoint which apparently has not been made explicit in the linear programming literature.

The results are rather general. Several of the results are valid for linear programs defined in arbitrary real normed spaces. A Hahn-Banach Theorem is the main tool employed in the analysis; given a closed convex set in a normed vector space and a point in the space but not in the set, there exists a continuous linear functional strictly separating the set from the point.

We introduce notation, then the results.

Let X, Y denote real vector spaces, each with a norm. We use the same notation (i.e. $\| \cdot \|$) for all norms, it being clear from context which norm is referred to.

Let X^* denote the dual space for X ; this is the space of all continuous linear functionals $c^* : X \rightarrow \mathbf{R}$ (continuous with respect to the norm topology). Endow X^* with the operator norm; if $c^* \in X^*$ then

$$\|c^*\| := \sup\{c^*x : \|x\| = 1\}$$

Define Y^* and its norm analogously.

Let X^{**} denote the dual space of X^* . Note that X can be viewed as a subset of X^{**} ; $x \in X$ induces the continuous linear functional on X^* given by $c^* \mapsto c^*x$. If $X = X^{**}$ then X is said to be *reflexive*.

To make this introductory section expositively clean we assume throughout it that X is reflexive; no such restriction is placed on Y ; in later sections the requirement that X be reflexive is sometimes removed. Many important normed spaces are reflexive, e.g., finite dimensional spaces regardless of the norm, Hilbert spaces.

Let $\mathcal{L}(X, Y)$ denote the space of bounded (i.e. continuous) linear operators from X to Y . Endow this space with the usual operator norm; if $A \in \mathcal{L}(X, Y)$ then

$$\|A\| := \sup\{\|Ax\|; \|x\| = 1\}.$$

Let C_X, C_Y be convex cones in X, Y , each with vertex at the origin, i.e., each is closed under multiplication by non-negative scalars and under addition. The cone C_X induces an "ordering" on X by

$$x' \geq x'' \leftrightarrow x' - x'' \in C_X.$$

(It is easily verified that we obtain a partial ordering iff the cone C_X is pointed; we do not assume pointedness.) Similarly, C_Y induces an “ordering” on Y .

In this introductory section we assume C_X and C_Y are closed.

Given $A \in \mathcal{L}(X, Y)$, $b \in Y$, $c^* \in X^*$ we define the LP instance $d := (A, b, c^*)$ by

$$\begin{array}{ll} \sup_{x \in X} & c^* x \\ \text{s.t.} & Ax \leq b \\ & x \geq 0. \end{array}$$

Many researchers have studied linear programming in this generality (cf. [1], [2], [3], [5]). Although linear programming from this vantage point is generally referred to by names such as “infinite linear programming” we prefer the phrase “analytic linear programming” because of close connections to functional analysis.

Although we use the symbol “ \leq ” the reader should note that all common forms of LP are covered by the general setting. For example, what one typically writes as “ $Ax = b$ ” is obtained by letting $C_Y = \{0\}$. Similarly, what one typically expresses as “no non-negativity constraints” is obtained by letting $C_X = X$.

The LP instance d has a natural dual which we now define.

First, the cone C_X has a dual cone in X^* defined by

$$C_X^* := \{\tilde{c}^* \in X^*; x \in C_X \Rightarrow \tilde{c}^* x \geq 0\}.$$

Define C_Y^* analogously.

The closed cones C_X^* and C_Y^* induce orderings on X^* and Y^* just as C_X and C_Y induced orderings on X and Y . Relying on these orderings, the dual d^* of $d := (A, b, c^*)$ is defined as the following LP:

$$\begin{array}{ll} \inf_{y^* \in Y^*} & y^* b \\ \text{s.t.} & y^* A \geq c^* \\ & y^* \geq 0. \end{array}$$

(Here, $y^* A$ is the linear functional $x \mapsto y^* Ax$. The linear transformation $y^* \mapsto y^* A$ is an element of $\mathcal{L}(Y^*, X^*)$; it is the dual operator of A .)

Let $val(d)$, $val(d^*)$ denote the optimal value for d , d^* ; if d is infeasible define $val(d) = -\infty$; if d^* is infeasible define $val(d^*) = \infty$.

It is a simple exercise to verify the weak duality relation $val(d) \leq val(d^*)$. However, unlike finite dimensional polyhedral linear programming, strong duality may be absent in analytic linear programming; it can happen that $val(d) < val(d^*)$ even when d and/or d^* is feasible. Conditions guaranteeing that no such “duality gap” occurs are of interest.

Consider the space \mathbf{D} consisting of all instances $d = (A, b, c^*)$; we view the cones C_X, C_Y (and hence the orderings) to be fixed independently of d .

For $d = (A, b, c^*) \in \mathbf{D}$ define the norm of d as the value

$$\|d\| := \max\{\|A\|, \|b\|, \|c^*\|\}.$$

Let $Pri\emptyset$ denote the set of all $d \in \mathbf{D}$ which are (primal) infeasible. Let $Dual\emptyset$ denote the set of all $d \in \mathbf{D}$ for which the dual d^* is infeasible.

Given $d \in \mathbf{D}$ define

$$dist(d, Pri\emptyset) := \inf\{\|d - \tilde{d}\|; \tilde{d} \in Pri\emptyset\},$$

the distance from d to the set of primal infeasible LP's. Similarly, define

$$dist(d, Dual\emptyset) := \inf\{\|d - \tilde{d}\|; \tilde{d} \in Dual\emptyset\}.$$

Let $Feas(d)$ denote the set of feasible points for d , and let $Opt(d)$ denote the optimal solution set. It can happen that $Opt(d) = \emptyset$ even when $val(d)$ is finite and the cones C_X and C_Y are closed; in general, “sup” cannot be replaced by “max” in specifying the objective of d .

Theorem 1.1 *Assume X is reflexive, C_X and C_Y are closed. Assume $d = (A, b, c^*) \in \mathbf{D}$. If d satisfies $dist(d, Pri\emptyset) > 0$ then statements (1) and (2) are true:*

(1) *There exists $x \in Feas(d)$ satisfying*

$$\|x\| \leq \frac{\|b\|}{dist(d, Pri\emptyset)}$$

(2) *If $x' \in Feas(d + \Delta d)$ where $\Delta d := (\vec{0}, \Delta b, \vec{0})$ (i.e. perturbation of b alone), then there exists $x \in Feas(d)$ satisfying*

$$\|x - x'\| \leq \|\Delta b\| \frac{\max\{1, \|x'\|\}}{dist(d, Pri\emptyset)}$$

If d satisfies both $dist(d, Pri\emptyset) > 0$ and $dist(d, Dual\emptyset) > 0$ then statements (3), (4) and (5) are true:

(3)

$$\frac{-\|b\| \|c^*\|}{\text{dist}(d, \text{Pri}\emptyset)} \leq \text{val}(d) = \text{val}(d^*) \leq \frac{\|b\| \|c^*\|}{\text{dist}(d, \text{Dual}\emptyset)}$$

(4) $\text{Opt}(d) \neq \emptyset$; moreover, if $x \in \text{Opt}(d)$ then

$$\|x\| \leq \left[\frac{\|b\|}{\text{dist}(d, \text{Dual}\emptyset)} \right] \left[\frac{\|d\|}{\text{dist}(d, \text{Pri}\emptyset)} \right]$$

(5) If $\Delta d := (\Delta A, \Delta b, \Delta c^*)$ satisfies both $\|\Delta d\| < \text{dist}(d, \text{Pri}\emptyset)$ and $\|\Delta d\| < \text{dist}(d, \text{Dual}\emptyset)$, then

$$\begin{aligned} |\text{val}(d + \Delta d) - \text{val}(d)| &\leq \|\Delta A\| \left[\frac{\|b\| + \|\Delta b\|}{\text{dist}(d, \text{Dual}\emptyset) - \|\Delta d\|} \right] \left[\frac{\|c^*\| + \|\Delta c^*\|}{\text{dist}(d, \text{Pri}\emptyset) - \|\Delta d\|} \right] \\ &\quad \left[\frac{\|d\|}{\text{dist}(d, \text{Pri}\emptyset \cup \text{Dual}\emptyset)} \right] \\ &+ \|\Delta b\| \left[\frac{\|c^*\| + \|\Delta c^*\|}{\text{dist}(d, \text{Pri}\emptyset) - \|\Delta d\|} \right] \left[\frac{\|d\|}{\text{dist}(d, \text{Dual}\emptyset)} \right] \\ &+ \|\Delta c^*\| \left[\frac{\|b\| + \|\Delta b\|}{\text{dist}(d, \text{Dual}\emptyset) - \|\Delta d\|} \right] \left[\frac{\|d\|}{\text{dist}(d, \text{Pri}\emptyset)} \right] \quad \square \end{aligned}$$

The propositions in Section 3 provide bounds which are more detailed; several of the bounds allow both X and Y to be arbitrary normed vector spaces.

Note that the first order terms of the quantity on the right of the inequality in assertion (5) of the theorem are bounded above by

$$\frac{\|\Delta A\|}{d_1 d_2 d_3} + \frac{\|\Delta b\|}{d_1 d_2} + \frac{\|\Delta c^*\|}{d_1 d_2} \quad (1.1)$$

where

$$d_1 := \frac{\text{dist}(d, \text{Pri}\emptyset)}{\|d\|}, \quad d_2 := \frac{\text{dist}(d, \text{Dual}\emptyset)}{\|d\|}, \quad d_3 := \min\{d_1, d_2\}.$$

This bound depends cubically on the inverses of the relative distances d_1 and d_2 . In Section 5 we show by way of examples that this bound cannot be improved in general; similarly for the other bounds in the theorem. However, the results of Section 3 can be used to obtain better bounds for many special cases, e.g., if $|\text{val}(d)| \leq \|d\|$ then one obtains a bound analogous to (1.1) depending quadratically on the inverses of d_1 and d_2 rather than cubically.

The theorem focuses on solution characteristics of d rather than d^* ; analogous results pertaining to d^* are discussed in Section 3.

Assertions (1) and (2) in the theorem are similar to results one finds in the literature on linear equations, i.e., when the cones C_X and C_Y are subspaces. (See [11]). In this restricted context the term “ $\max\{1, \|x'\|\}$ ” occurring in assertion (2) can be replaced simply with “1”; in fact, this replacement is valid for arbitrary closed cones C_X and C_Y if $d = (A, b, c^*)$ has the property that

$$\text{dist}((A, tb, c^*), \text{Pri}\emptyset) \quad (1.2)$$

is independent of t satisfying $0 < t \leq 1$ (as it is if C_X and C_Y are closed subspaces).¹

Assertion (2) of the theorem can be easily extended to allow perturbations in A as well as b . If $x' \in \text{Feas}(d + \Delta d)$ where $\Delta d = (\Delta A, \Delta b, \bar{0})$ then $x' \in \text{Feas}(d + \Delta' d)$ where $\Delta' d := (\bar{0}, \Delta b - (\Delta A)x', \bar{0})$ and hence assertion (2) implies there exists $x \in \text{Feas}(d)$ satisfying

$$\|x - x'\| \leq (\|\Delta b\| + \|\Delta A\| \|x'\|) \frac{\max\{1, \|x'\|\}}{\text{dist}(d, \text{Pri}\emptyset)}.$$

The bounds asserted by the theorem will be useful in developing a complexity theory for linear programming where problem instance “size” is defined using quantities similar to condition numbers; see Renegar [3] and Vera [5] for work in this direction.

Others have studied perturbations of linear programs but not in terms of the quantities $\text{dist}(d, \text{Pri}\emptyset)$ and $\text{dist}(d, \text{Dual}\emptyset)$; cf. Hoffman [4], Mangasarian [6], [7] and Robinson [9].

In the sections that follow we do not assume X or Y is reflexive unless stated. However, we do assume X and Y are indeed normed as is natural for perturbation theory.

Whenever we write “cone” we mean “convex cone with vertex at the origin”.

We do not assume the cones C_X and C_Y are closed unless stated.

2 Duality Gaps

In this section we prove that if X is reflexive, C_X and C_Y are closed, $\text{dist}(d, \text{Pri}\emptyset) > 0$ and $\text{dist}(d, \text{Dual}\emptyset) > 0$ then $\text{val}(d) = \text{val}(d^*)$, i.e. no duality gap. We begin with well-known propositions from which the proof follows easily. For completeness we include short proofs of the well-known propositions.

¹To see this fact note we may assume that $\|x'\| > 1$. Let $t := 1/\|x'\|$ and observe that $x/\|x'\|$ is feasible for (A, tb, c^*) and $x'/\|x'\|$ is feasible for $(A, t(b + \Delta b), c^*)$. Applying assertion (2) to these “scaled” LP’s and using the assumption that (1.2) is independent of $0 < t \leq 1$ gives the desired conclusion.

The exposition throughout the paper allows the reader to skip all proofs yet still follow the main thread of the development.

Fix $A \in \mathcal{L}(X, Y)$ and define

$$C(A) := \{b \in Y; \exists x \geq 0 \text{ such that } Ax \leq b\}.$$

It is easily verified that $C(A)$ is a cone. Let $\overline{C(A)}$ denote the closure of $C(A)$; the closure is also a cone. If X and Y are finite dimensional and if the cones C_X and C_Y which define the orderings are polyhedral, then $C(A)$ is polyhedral and hence $C(A) = \overline{C(A)}$. In other cases the closure may contain additional points.

A system of inequalities

$$\begin{array}{rcl} Ax & \leq & b \\ x & \geq & 0 \end{array}$$

is said to be *asymptotically consistent* if $b \in \overline{C(A)}$, i.e. if it can be made consistent by an arbitrarily slight perturbation of b .

The following proposition relies only on the local convexity of the normed space Y . In the case of finite-dimensional polyhedral linear programming the proposition is Farkas' lemma.

Proposition 2.1 (*Duffin [2]*) Assume $A \in \mathcal{L}(X, Y), b \in Y$. Consider the following two systems:

$$\begin{array}{rcl} Ax & \leq & b \\ x & \geq & 0 \end{array} \quad \begin{array}{rcl} y^* A & \geq & 0 \\ y^* & \geq & 0 \\ y^* b & < & 0 \end{array}$$

The first system is asymptotically consistent if and only if the second is inconsistent.

Proof. Let $C(A)^*$ denote the set of all functionals $y^* \in Y^*$ satisfying $y^* \tilde{b} \geq 0$ for all $\tilde{b} \in C(A)$. Since in a normed space any closed convex set can be strictly separated from a point not in the set by a continuous linear functional (separation version of the Hahn-Banach Theorem; cf. [10], Theorem 3.4b), and since $\overline{C(A)}$ is a convex set (because $C(A)$ is), it easily follows that

$$\overline{C(A)} = \{\tilde{b} \in Y; y^* \in C(A)^* \Rightarrow y^* \tilde{b} \geq 0\}. \quad (2.1)$$

Noting that

$$C(A) = \{\tilde{b}; \exists x \geq 0, \hat{b} \geq 0 \text{ such that } \tilde{b} = Ax + \hat{b}\},$$

and recalling that $\{x; x \geq 0\}$ and $\{\hat{b}; \hat{b} \geq 0\}$ are cones, we have

$$C(A)^* = \{y^* \in Y^*; (x \geq 0 \Rightarrow y^* Ax \geq 0) \text{ and } (\hat{b} \geq 0 \Rightarrow y^* \hat{b} \geq 0)\},$$

that is,

$$C(A)^* = \{y^* \in Y^*; (y^* A \geq 0) \text{ and } (y^* \geq 0)\}. \quad (2.2)$$

The proposition follows immediately from (2.1) and (2.2). \square

Shortly, we state a dual analog of the proposition. Before doing so we digress to present a simple technical lemma important for establishing the analog.

Assuming $A \in \mathcal{L}(X, Y)$, the system

$$\begin{array}{rcl} Ax & \leq & b \quad \leftrightarrow \quad b - Ax \in C_Y \\ x & \geq & 0 \quad \leftrightarrow \quad x \in C_X \end{array} \quad (2.3)$$

is naturally associated with a system which we will call its “double-dual-extension”:

$$\begin{array}{rcl} b - A^{**}x^{**} & \in & C_Y^{**} \\ x^{**} & \in & C_X^{**} \end{array} \quad (2.4)$$

where $x^{**} \in X^{**}$, C_X^{**} is the dual cone for C_X^* , C_Y^{**} is the dual cone for C_Y^* , and A^{**} is the dual operator of the dual operator of A . Viewing X as a subset of X^{**} , Y as a subset of Y^{**} , it is trivial to see that $C_X \subseteq C_X^{**}$, $C_Y \subseteq C_Y^{**}$; also, if $x \in X$ then $A^{**}x = Ax$. Hence, any solution of the original system is also a solution of the double-dual-extension. However, the double-dual-extension can have solutions which are not solutions of the original system; it can have solutions in $X^{**} \setminus X$. This possibility is a well-known obstruction to the development of a symmetric duality theory for analytic linear programming.

To gain symmetry we occasionally impose additional assumptions; ironically, these are not symmetric, being restrictive for X but not for Y . The following lemma becomes important.

Lemma 2.2 *Assume X is reflexive, C_X and C_Y are closed. Assume $A \in \mathcal{L}(X, Y)$, $b \in Y$. Then the solutions for the system (2.3) are identical with those for the double-dual-extension (2.4).*

Proof. It is a simple exercise using the separation version of the Hahn-Banach Theorem to prove that $C_Y = C_Y^{**} \cap Y$ since C_Y is closed. Similarly, $C_X = C_X^{**} \cap X = C_X^{**}$ using $X = X^{**}$. Since $Ax = A^{**}x$ for all $x \in X$, the lemma follows. \square

Corollary 2.3 Assume X is reflexive, C_X and C_Y are closed. Assume $A \in \mathcal{L}(X, Y)$, $c^* \in X^*$. Consider the following two systems:

$$\begin{array}{ll} Ax & \leq 0 \\ x & \geq 0 \\ c^*x & > 0 \end{array} \quad \begin{array}{ll} y^*A & \geq c^* \\ y^* & \geq 0 \end{array}$$

The first system is inconsistent if and only if the second is asymptotically consistent (meaning that it can be made consistent by an arbitrarily slight perturbation of c^*).

Proof. Replacing the first system of Proposition 2.1 with the second system of the corollary, note that the appropriate second system for the proposition will then have the same solutions as the first system of the corollary by Lemma 2.2. \square

Letting $d = (A, b, c^*)$ denote an LP instance, the *asymptotic optimal value* $\uparrow val(d)$ is defined to be the supremum of the optimal objective values of LP's obtained from d by perturbing b by an arbitrarily small amount, that is,

$$\uparrow val(d) := \lim_{\delta \downarrow 0} \sup_{\substack{\tilde{b} \\ \|\tilde{b} - b\| < \delta}} val(\tilde{d}).$$

where $\tilde{d} := (A, \tilde{b}, c^*)$. Of course $\uparrow val(d) \geq val(d)$.

Recall that d^* is the dual LP for d .

The following proposition is well-known. The condition $\uparrow val(d) \neq -\infty$ simply means that if d is not itself feasible, then it can be made feasible by an arbitrarily slight perturbation of b .

Proposition 2.4 If $\uparrow val(d) \neq -\infty$ then $\uparrow val(d) = val(d^*)$.

Proof. Let $k \in \mathbf{R}$ and consider the following two systems:

$$\begin{array}{ll} I) \quad Ax & \leq b \\ \quad -c^*x & \leq -k \\ \quad x & \geq 0 \end{array} \quad \begin{array}{ll} II) \quad y^*A - tc^* & \geq 0 \\ \quad y^* & \geq 0, \quad t \geq 0 \\ \quad y^*b - tk & < 0 \end{array}$$

where $t \in \mathbf{R}$. Note the following three facts:

- i) I is asymptotically consistent iff II is inconsistent (by Proposition 2.1, replacing Y there with $Y \times \mathbf{R}$, the operator $x \mapsto Ax$ with $x \mapsto (Ax, -c^*x)$, etc.);
- ii) If y^* is feasible for d^* then $(y^*, 1)$ is feasible for all constraints in II except perhaps $y^*b - tk < 0$;

iii) If $(y^*, 0)$ is feasible for II then $\uparrow val(d) = -\infty$ (because if $(y^*, 0)$ is feasible for II , then y^* is feasible for the second system of Proposition 2.1 and hence that proposition implies the system $Ax \leq b, x \geq 0$ is not asymptotically consistent.)

If $k \leq \uparrow val(d)$ then I is asymptotically consistent and hence, by (i), II is inconsistent. Thus, by (ii), if y^* is feasible for d^* then $y^*b \geq k$. It follows that $\uparrow val(d) \leq val(d^*)$.

If $-\infty < \uparrow val(d) < k$ then I is not asymptotically consistent and hence, by (i), II is consistent. Assume (y^*, t) is feasible for II . By (iii), $t \neq 0$. Hence y^*/t is feasible for d^* and satisfies $(y^*/t)b < k$. It follows that $\uparrow val(d) \geq val(d^*)$. \square

Continuing to let $d = (A, b, c^*)$ denote an LP instance, and d^* its dual, define

$$\downarrow val(d^*) = \lim_{\delta \downarrow 0} \inf_{\substack{\tilde{c}^* \\ \|\tilde{c}^* - c^*\| < \delta}} val(\tilde{d}^*)$$

where \tilde{d}^* is the dual of $\tilde{d} := (A, b, \tilde{c}^*)$.

Proposition 2.5 *Assume X is reflexive, C_X and C_Y are closed. If $\downarrow val(d^*) \neq \infty$ then $\downarrow val(d^*) = val(d)$.*

Proof. The proof is analogous to that of Proposition 2.4, using Corollary 2.3 rather than Proposition 2.1, noting that $X \times \mathbb{R}$ is reflexive if X is. \square

Proposition 2.6 *Assume X is reflexive, C_X and C_Y are closed. If $dist(d, Pri\emptyset) > 0$ and $dist(d, Dual\emptyset) > 0$ then $val(d) = val(d^*)$.*

Proof. We know $val(d) \leq val(d^*)$ by weak duality. It thus suffices to prove that if d is any LP instance satisfying

$$-\infty < val(d) < val(d^*) < \infty \tag{2.5}$$

then either $dist(d, Pri\emptyset) = 0$ or $dist(d, Dual\emptyset) = 0$. So assume $d = (A, b, c^*)$ satisfies (2.5).

Proposition 2.4 implies there exists a sequence of pairs $\{(x_i, b_i)\}$ such that

$$\begin{aligned} Ax_i &\leq b_i \\ x_i &\geq 0, \end{aligned}$$

$\|b_i - b\| \rightarrow 0$ and $c^*x_i \rightarrow val(d^*)$. Similarly, Proposition 2.5 implies there exists a sequence of pairs $\{(y_i^*, c_i^*)\}$ such that

$$\begin{aligned} y_i^* A &\geq c_i^* \\ y_i^* &\geq 0, \end{aligned}$$

$\|c_i^* - c^*\| \rightarrow 0$ and $y_i^* b \rightarrow \text{val}(d)$.

We claim that either $\|x_i\| \rightarrow \infty$ or $\|y_i^*\| \rightarrow \infty$. For assume otherwise. Then, restricting to a subsequence if necessary,

$$\lim c_i^* x_i = \lim c^* x_i = \text{val}(d^*), \quad (2.6)$$

$$\lim y_i^* b_i = \lim y_i^* b = \text{val}(d). \quad (2.7)$$

Consider the LP instances $d_i := (A, b_i, c_i^*)$. Noting that x_i is feasible for d_i and y_i^* is feasible for d_i^* , we have by weak duality

$$c_i^* x_i \leq \text{val}(d_i) \leq \text{val}(d_i^*) \leq y_i^* b_i \quad (2.8)$$

From (2.6), (2.7) and (2.8) we find that $\text{val}(d^*) \leq \text{val}(d)$ contradicting (2.5).

Assuming $\|x_i\| \rightarrow \infty$, $x_i \neq \bar{0}$, we now prove that $\text{dist}(d, \text{Dual}\emptyset) = 0$. In doing so we re-use notation.

By the extension version of the Hahn-Banach Theorem there exists $c_i^* \in X^*$ such that $\|c_i^*\| = 1$ and $c_i^* x_i = \|x_i\|$.

Let $\epsilon > 0$ and consider the LP instances

$$d_i := \left(A - \left(\frac{1}{\|x_i\|} \right) b_i c_i^*, \bar{0}, c^* + \left(\frac{\epsilon - c^* x_i}{\|x_i\|} \right) c_i^* \right).$$

Note that x_i is feasible for d_i ; in fact, tx_i is feasible for d_i for all $t \geq 0$. Since the objective value for d_i at x_i is positive, it follows that $\text{val}(d_i) = \infty$. Hence $d_i \in \text{Dual}\emptyset$ by weak duality.

Let d_i' be obtained from d_i by replacing the RHS vector $\bar{0}$ with b . Then $d_i' \in \text{Dual}\emptyset$ since $d_i \in \text{Dual}\emptyset$. Noting that $\|d_i' - d\| \rightarrow 0$ since $\|x_i\| \rightarrow \infty$ and $c^* x_i \rightarrow \text{val}(d^*) < \infty$, we finally have $\text{dist}(d, \text{Dual}\emptyset) = 0$.

In similar fashion one proves that if $\|y_i^*\| \rightarrow \infty$ then $\text{dist}(d, \text{Pri}\emptyset) = 0$. \square

3 Bounds

We prove various bounds depending on the quantities $\text{dist}(d, \text{Pri}\emptyset)$ and $\text{dist}(d, \text{Dual}\emptyset)$. For most propositions presented in this section, a dual analog is also presented; of the two, we first present the one with fewest assumptions.

We always assume X and Y are normed spaces, $C_X \subseteq X$ and $C_Y \subseteq Y$ are cones, $A \in \mathcal{L}(X, Y)$, $b \in Y$, $c^* \in X^*$. No other assumptions are made unless stated.

Given an LP $d = (A, b, c^*)$, we continue to let $Feas(d)$ refer to the set of feasible points for d and $Opt(d)$ refer to the set of optimal points; similarly we have $Feas(d^*)$ and $Opt(d^*)$ with respect to the dual d^*

In general one can have $Opt(d) = \emptyset$ and $-\infty < val(d) < \infty$, i.e., the supremum, although finite, is not attained. Similarly, one can have $Opt(d^*) = \emptyset$ and $-\infty < val(d^*) < \infty$.

Lemma 3.1 *Assume $d = (A, b, c^*)$ satisfies $dist(d, Pri\emptyset) > 0$. If $y^* \in Feas(d^*)$ then*

$$\|y^*\| \leq \frac{\max\{\|c^*\|, y^*b\}}{dist(d, Pri\emptyset)}.$$

Proof. Let $\rho \in \mathbf{R}$ be such that $0 \leq \rho < 1$ and there exists $\tilde{b} \in Y$ satisfying $y^*\tilde{b} = (1 - \rho)\|y^*\|$, $\|\tilde{b}\| = 1$; note that ρ can be chosen arbitrarily near zero if not zero itself.

Let $\rho' \in \mathbf{R}$ satisfy $\rho' > 0$. Consider the LP

$$d + \Delta d := (A + \Delta A, b + \Delta b, c^*)$$

where

$$\begin{aligned} \Delta A &:= - \left(\frac{1}{(1 - \rho)\|y^*\|} \right) \tilde{b}c^* \\ \Delta b &:= - \left(\frac{\max\{0, y^*b + \rho'\}}{(1 - \rho)\|y^*\|} \right) \tilde{b} \end{aligned}$$

Note that since $y^* \in Feas(d^*)$,

$$\begin{aligned} y^*(A + \Delta A) &\geq 0 \\ y^* &\geq 0 \\ y^*(b + \Delta b) &< 0. \end{aligned}$$

Proposition 2.1 thus implies $d + \Delta d \in Pri\emptyset$. Since

$$\|\Delta d\| \leq \frac{\max\{\|c^*\|, y^*b + \rho'\}}{(1 - \rho)\|y^*\|}$$

and since ρ can be chosen arbitrarily near zero, the proposition follows by letting $\rho' \downarrow 0$. \square

Lemma 3.2 Assume X is reflexive, C_X and C_Y are closed. Assume $d = (A, b, c^*)$ satisfies $\text{dist}(d, \text{Dual}\emptyset) > 0$. If $x \in \text{Feas}(d)$ then

$$\|x\| \leq \frac{\max\{\|b\|, -c^*x\}}{\text{dist}(d, \text{Dual}\emptyset)}.$$

Proof. Analogous to the proof of Lemma 3.1, using Proposition 2.3; begin by noting the extension form of the Hahn-Banach theorem implies there exists $\tilde{c}^* \in X^*$ satisfying $\tilde{c}^*x = 1$ and $\|\tilde{c}^*\| = 1$. \square

Proposition 3.3 Assume $d = (A, b, c^*)$ satisfies $\text{dist}(d, \text{Pri}\emptyset) > 0$. If $\{y_i^*\} \subseteq \text{Feas}(d^*)$ satisfies $y_i^*b \rightarrow \text{val}(d^*)$ then

$$\limsup \|y_i^*\| \leq \frac{\max\{\|c^*\|, \text{val}(d^*)\}}{\text{dist}(d, \text{Pri}\emptyset)} \quad (3.1)$$

Proof. Immediate from Lemma 3.1. \square

Proposition 3.4 Assume Y is reflexive. Assume $d = (A, b, c^*)$ satisfies $\text{dist}(d, \text{Pri}\emptyset) > 0$ and $\text{Feas}(d^*) \neq \emptyset$. Then $\text{Opt}(d^*) \neq \emptyset$; moreover, if $y^* \in \text{Opt}(d^*)$ then

$$\|y^*\| \leq \frac{\max\{\|c^*\|, \text{val}(d^*)\}}{\text{dist}(d, \text{Pri}\emptyset)}$$

Proof. Proposition 3.3 shows that an additional constraint of the form $\|y^*\| \leq R$ can be added to d^* without changing the optimal value. The resulting feasible region is bounded, closed and convex. Since Y^* is reflexive (equivalent to Y being reflexive), the Banach-Alaoglu Theorem (c.f. [10], Theorem 3.15) implies the optimal value is attained; thus $\text{Opt}(d^*) \neq \emptyset$.

The final bound is immediate from Proposition 3.3. \square

Proposition 3.5 Assume X is reflexive, C_X and C_Y are closed. Assume $d = (A, b, c^*)$ satisfies $\text{dist}(d, \text{Dual}\emptyset) > 0$ and $\text{Feas}(d) \neq \emptyset$. Then $\text{Opt}(d) \neq \emptyset$; moreover, if $x \in \text{Opt}(d)$ then

$$\|x\| \leq \frac{\max\{\|b\|, -\text{val}(d)\}}{\text{dist}(d, \text{Dual}\emptyset)} \quad \square$$

Proof. Analogous to those of Propositions 3.3 and 3.4, using Lemma 3.2. \square

Proposition 3.6 Assume $d = (A, b, c^*)$ satisfies $\text{dist}(d, \text{Pri}\emptyset) > 0$. Then

$$\frac{-\|b\| \|c^*\|}{\text{dist}(d, \text{Pri}\emptyset)} \leq \text{val}(d^*).$$

Proof. We may assume $val(d^*) < 0$. Letting $\{y_i^*\} \subseteq Feas(d^*)$ satisfy $y_i^* b \rightarrow val(d^*)$, Proposition 3.3 implies

$$\frac{-\|b\| \|c^*\|}{dist(d, Pri\emptyset)} \leq -\|b\| \limsup \|y_i^*\| \leq val(d^*). \quad \square$$

Proposition 3.7 Assume X is reflexive, C_X and C_Y are closed. Assume $d = (A, b, c^*)$ satisfies $dist(d, Dual\emptyset) > 0$. Then

$$val(d) \leq \frac{\|b\| \|c^*\|}{dist(d, Dual\emptyset)}.$$

Proof. Analogous to the proof of Proposition 3.6 using Proposition 3.5. \square

Proposition 3.8 Assume X is reflexive, C_X and C_Y are closed. Assume $d = (A, b, c^*)$ satisfies $dist(d, Pri\emptyset) > 0$ and $dist(d, Dual\emptyset) > 0$. Then

$$\frac{-\|b\| \|c^*\|}{dist(d, Pri\emptyset)} \leq val(d) = val(d^*) \leq \frac{\|b\| \|c^*\|}{dist(d, Dual\emptyset)}$$

Proof. Combine Propositions 2.6, 3.6 and 3.7. \square

Lemma 3.9 Assume $d = (A, b, c^*)$ satisfies $dist(d, Pri\emptyset) > 0$ and $Feas(d^*) \neq \emptyset$. Assume $\Delta d := (0, \Delta b, 0)$. Then

$$val([d + \Delta d]^*) - val(d^*) \leq \|\Delta b\| \frac{\max\{\|c^*\|, val(d^*)\}}{dist(d, Pri\emptyset)}$$

Proof. Assume $\{y_i^*\} \subseteq Feas(d^*)$ satisfies $y_i^* b \rightarrow val(d^*)$. Note that $\{y_i^*\} \subseteq Feas([d + \Delta d]^*)$. Hence,

$$\begin{aligned} val([d + \Delta d]^*) &\leq \liminf y_i^*(b + \Delta b) = val(d^*) + \liminf y_i^*(\Delta b) \\ &\leq val(d^*) + \|\Delta b\| \limsup \|y_i^*\|. \end{aligned}$$

Substituting the bound of Proposition 3.3 completes the proof. \square

A dual analog of Lemma 3.9 is obtained by an analogous proof using Proposition 3.5. We leave this to the reader.

Proposition 3.10 Assume X is reflexive, C_X and C_Y are closed. Assume $d = (A, b, c^*)$ satisfies $dist(d, Pri\emptyset) > 0$ and $dist(d, Dual\emptyset) > 0$. Assume $\Delta d := (\Delta A, \Delta b, \Delta c^*)$ satisfies $dist(d + \Delta d, Pri\emptyset) > 0$ and $dist(d + \Delta d, Dual\emptyset) > 0$. Then

$$\begin{aligned} val(d + \Delta d) - val(d) &\leq \|\Delta A\| \left[\frac{\max\{\|c^*\|, val(d)\}}{dist(d, Pri\emptyset)} \right] \left[\frac{\max\{\|b + \Delta b\|, -val(d + \Delta d)\}}{dist(d + \Delta d, Dual\emptyset)} \right] \\ &+ \|\Delta b\| \left[\frac{\max\{\|c^*\|, val(d)\}}{dist(d, Pri\emptyset)} \right] \\ &+ \|\Delta c^*\| \left[\frac{\max\{\|b + \Delta b\|, -val(d + \Delta d)\}}{dist(d + \Delta d, Dual\emptyset)} \right] \end{aligned}$$

An analogous lower bound on $val(d + \Delta d) - val(d)$ is obtained by interchanging the roles of d and $d + \Delta d$.

Remarks. The value $-val(d + \Delta d)$ occurring on the right side of the inequality can be replaced with $-val(d)$; this follows immediately from the inequality by considering the two cases $val(d + \Delta d) \leq val(d)$ and $val(d) \leq val(d + \Delta d)$. Similarly, the value $val(d + \Delta d)$ appearing in the analogous lower bound can be replaced with $val(d)$.

Proof. Proposition 3.5 implies $d + \Delta d$ has an optimal solution x . Let $\Delta'd := (0, \Delta'b, 0)$ where

$$\Delta'b := \Delta b - (\Delta A)x.$$

Note that x is feasible for $d + \Delta'd$ and hence $c^*x \leq val(d + \Delta'd)$. Thus,

$$val(d + \Delta d) - val(d + \Delta'd) \leq (\Delta c^*)x \leq \|\Delta c^*\| \|x\|$$

and hence

$$val(d + \Delta d) - val(d) \leq \|\Delta c^*\| \|x\| + [val(d + \Delta'd) - val(d)].$$

Noting that $\|\Delta'b\| \leq \|\Delta b\| + \|\Delta A\| \|x\|$, the proof is now easily completed by substituting the bounds of Proposition 3.5 and Lemma 3.9, using the facts that $val(d) = val(d^*)$ by Proposition 2.6 and $val(d + \Delta'd) \leq val([d + \Delta'd]^*)$ by weak duality. \square

The following two propositions regard perturbations of the feasible region; the objective functional c^* plays no role. For consistency we retain the same notation although c^* is irrelevant to the results.

Proposition 3.11 Assume X is reflexive, C_X and C_Y are closed. Assume $d = (A, b, c^*)$ satisfies $dist(d, Pri\emptyset) > 0$. Then there exists $x \in Feas(d)$ satisfying

$$\|x\| \leq \frac{\|b\|}{dist(d, Pri\emptyset)}.$$

Proposition 3.12 Assume X is reflexive, C_X and C_Y are closed. Assume $d = (A, b, c^*)$ satisfies $dist(d, Pri\emptyset) > 0$. Assume $x' \in Feas(d + \Delta d)$ where $\Delta d := (0, \Delta b, 0)$. Then there exists $x \in Feas(d)$ satisfying

$$\|x - x'\| \leq \|\Delta b\| \frac{\max\{1, \|x'\|\}}{dist(d, Pri\emptyset)}.$$

The two propositions are proven via the following lemma.

Lemma 3.13 Assume X is reflexive, C_X and C_Y are closed, $x' \in X$. Assume $d = (A, b, c^*)$ satisfies $\uparrow val(d) \neq -\infty$ and

$$Feas(d) \cap \{x; \|x - x'\| \leq r\} = \emptyset \quad (3.2)$$

Then there exists $\tilde{c}^* \in X^*$ such that the dual \tilde{d}^* of $\tilde{d} := (A, b, \tilde{c}^*)$ satisfies

$$val(\tilde{d}^*) < \tilde{c}^* x' - \|\tilde{c}^*\| r \quad (3.3)$$

Proof. There exists $r' > r$ such that (3.2) remains valid if r' is substituted for r . For otherwise there would exist a sequence $\{x_i\} \subseteq Feas(d)$ such that $\|x_i - x'\| \downarrow r$; the sequence would have a weakly convergent subsequence by the Eberlein-Shmul'yan Theorem (cf. [14]); because the weak-closure of a convex set in a normed space is identical to the strong closure (by the Hahn-Banach Theorem) it is easily argued that the weak limit x of the subsequence would satisfy $\|x - x'\| \leq r$, $x \in C_X$ and $b - Ax \in C_Y$ contradicting (3.2).

There exists $\delta' > 0$ such that

$$\|b' - b\| \leq \delta' \Rightarrow Feas(d') \cap \{x; \|x - x'\| \leq r'\} = \emptyset$$

where $d' := (A, b', c^*)$. For otherwise there would exist a sequence of pairs $\{(x_i, b_i)\}$ such that $\|b_i - b\| \downarrow 0$, $\|x_i - x'\| \leq r'$ and $x_i \in Feas(d_i)$ where $d_i = (A, b_i, c^*)$; the sequence $\{x_i\}$ would have a weakly convergent subsequence by the Eberlein-Shmul'yan Theorem; the weak limit x of the subsequence would be easily argued to satisfy both $x \in Feas(d)$ and $\|x - x'\| \leq r'$ contradicting the definition of r' .

Let K denote the closed, convex set consisting of all points of distance at most r from the convex set

$$\bigcup_{\substack{b' \\ \|b' - b\| \leq \delta'}} Feas(d').$$

By definition of δ' , $x' \notin K$; thus, by the Hahn-Banach Theorem there exists $\tilde{c}^* \in X^*$ such that

$$\sup\{\tilde{c}^* x; x \in K\} < \tilde{c}^* x'$$

and hence

$$\|b' - b\| \leq \delta' \Rightarrow \sup\{\tilde{c}^* x; x \in Feas(d')\} < \tilde{c}^* x' - \|\tilde{c}^*\| r.$$

Letting $\tilde{d} := (A, b, \tilde{c}^*)$, it follows that the asymptotic optimal value $\uparrow val(\tilde{d})$ satisfies

$$\uparrow val(\tilde{d}) < \tilde{c}^* x' - \|\tilde{c}^*\| r.$$

Hence, by Proposition 2.4 and the fact that $\uparrow val(\tilde{d}) \neq \infty$ (because $\uparrow val(d) \neq -\infty$), (3.3) is valid. \square

Proof of Proposition 3.11. Assume the conclusion is not true, i.e., assume

$$Feas(d) \cap \{x; \|x - x'\| \leq r\} = \emptyset$$

where

$$x' := \bar{0} \quad \text{and} \quad r := \frac{\|b\|}{dist(d, Pri\emptyset)}.$$

Then d, x' and r satisfy the assumptions of Lemma 3.13. Let \tilde{d} be as in the conclusion of that lemma; thus

$$val(\tilde{d}^*) < \tilde{c}^* x' - \|\tilde{c}^*\| r = -\frac{\|b\| \|\tilde{c}^*\|}{dist(d, Pri\emptyset)}.$$

However, this contradicts Proposition 3.6 applied to \tilde{d} since $dist(\tilde{d}, Pri\emptyset) = dist(d, Pri\emptyset)$. \square

Proof of Proposition 3.12. Assume the conclusion is not true, i.e., assume

$$Feas(d) \cap \{x; \|x - x'\| \leq r\} = \emptyset$$

where

$$r := \|\Delta b\| \frac{\max\{1, \|x'\|\}}{dist(d, Pri\emptyset)}$$

Then d, x' and r satisfy the assumptions of Lemma 3.13. Let \tilde{d} be as in the conclusion of that lemma; thus

$$val(\tilde{d}^*) < \tilde{c}^* x' - \|\tilde{c}^*\| r = \tilde{c}^* x' - \|\Delta b\| \|\tilde{c}^*\| \frac{\max\{1, \|x'\|\}}{dist(d, Pri\emptyset)} \quad (3.4)$$

Note that $x' \in Feas(\tilde{d} + \Delta d)$ implying, by weak duality,

$$\tilde{c}^* x' \leq val([\tilde{d} + \Delta d]^*). \quad (3.5)$$

Combining (3.4) and (3.5),

$$val([\tilde{d} + \Delta d]^*) - val(\tilde{d}^*) > \|\Delta b\| \|\tilde{c}^*\| \frac{\max\{1, \|x'\|\}}{dist(d, Pri\emptyset)}. \quad (3.6)$$

Since $dist(\tilde{d}, Pri\emptyset) = dist(d, Pri\emptyset) > 0$, (3.6) and Lemma 3.9 yield

$$\|\tilde{c}^*\| \max\{1, \|x'\|\} < \max\{\|\tilde{c}^*\|, val(\tilde{d}^*)\}$$

This gives a contradiction since $val(\tilde{d}^*) \leq \|\tilde{c}^*\| \|x'\|$ by (3.4). \square

4 Proof of Theorem 1.1

We now collect our results to prove the theorem. As mentioned in the introduction, the results of Section 3 provide for many special cases better bounds than those asserted by the theorem.

Proposition 3.11 establishes (1) of the theorem, Proposition 3.12 establishes (2), and Proposition 3.8 establishes (3).

Proposition 3.5 together with the lower bound provided by (3) of the theorem show that in the setting of (4) we have $Opt(d) \neq \emptyset$ and

$$x \in Opt(d) \Rightarrow \|x\| \leq \left[\frac{\|b\|}{dist(d, Dual\emptyset)} \right] \max \left\{ 1, \frac{\|c^*\|}{dist(d, Pri\emptyset)} \right\}.$$

Since $\|c^*\| \leq \|d\|$, to establish (4) it thus suffices to show that under the assumptions of (4) at least one of the following two relations is true:

$$\frac{\|d\|}{dist(d, Pri\emptyset)} \geq 1 \quad (4.1)$$

$$Opt(d) = \{\bar{0}\}. \quad (4.2)$$

Assume (4.1) is not true. Then $dist(\bar{0}, Pri\emptyset)$ (i.e., the distance from $Pri\emptyset$ to the identically zero LP) satisfies $dist(\bar{0}, Pri\emptyset) > 0$ and hence, since $Pri\emptyset$ is closed under multiplication by positive scalars, $Pri\emptyset = \emptyset$. Note that $Pri\emptyset = \emptyset$ implies $\{\tilde{b} \in Y; \tilde{b} \geq 0\} = Y$; otherwise the RHS of the identically zero LP could be perturbed to obtain an infeasible LP, a contradiction. Hence, $Pri\emptyset = \emptyset$ implies that the feasible region for all LP's is precisely the closed cone $\{x; x \geq 0\}$. Consequently, if (4.1) and (4.2) are not true then a slight perturbation of c^* in $d = (A, b, c^*)$ creates an LP with unbounded optimal solution, contradicting $dist(d, Dual\emptyset) > 0$ as is assumed in (4) of the Theorem.

In all, either (4.1) or (4.2) is true, concluding the proof of (4).

Towards proving (5) first note we may assume that

$$\frac{\|d\|}{dist(d, Pri\emptyset \cup Dual\emptyset)} \geq 1. \quad (4.3)$$

Otherwise, by arguments analogous to the preceding, we deduce that $\{\tilde{c}^* \in X^*; \tilde{c}^* \geq 0\} = X^*$ (hence $\{x \in X; x \geq 0\} = \{\bar{0}\}$) and $\{\tilde{b} \in Y; \tilde{b} \geq 0\} = Y$. Consequently $\bar{0}$ is an optimal solution for all LP's; then (5) follows trivially. So we may assume (4.3).

Assertion (5) is established with (3), Proposition 3.10 and tedious consideration of several cases. For example, consider the case

$$|val(d + \Delta d) - val(d)| = val(d + \Delta d) - val(d) \quad (4.4)$$

$$val(d + \Delta d) < 0. \quad (4.5)$$

Then the factor for $\|\Delta A\|$ in the bound of Proposition 3.10 is

$$\left[\frac{\|c^*\|}{dist(d, Pri\emptyset)} \right] \left[\frac{\max\{\|b + \Delta b\|, -val(d + \Delta d)\}}{dist(d + \Delta d, Dual\emptyset)} \right] \quad (4.6)$$

The crucial point is that at most one of the two quantities $val(d)$ and $val(d + \Delta d)$ appear in this expression; the same is true for all cases.

As noted in the remark following the statement of Proposition 3.10, the value $-val(d + \Delta d)$ in (4.6) can be replaced with $-val(d)$; in turn, $-val(d)$ can be replaced with the upper bound $\|b\| \|c^*\| / dist(d, Pri\emptyset)$ (provided by statement (3) of the theorem), which clearly does not exceed

$$\frac{(\|b\| + \|\Delta b\|)\|d\|}{dist(d, Pri\emptyset \cup Dual\emptyset)}. \quad (4.7)$$

Substituting (4.7) for $-val(d + \Delta d)$ in (4.6) and using (4.3), one arrives at a quantity which is obviously bounded from above by the factor for $\|\Delta A\|$ in (5). Similarly, one argues that the other factors are correct in the case that (4.4) and (4.5) are valid.

The other cases are handled with equally tedious and obvious arguments. \square

5 Examples

In this section we display simple examples indicating that the bounds of Theorem 1.1 cannot be improved in general without relying on additional parameters.

The examples form a family of two variable LP's depending on parameters s and t :

$$\begin{array}{rcll} \max & x_1 & & \\ s.t. & (st)x_1 & + & x_2 \leq s \\ & tx_1 & + & x_2 \leq 1 \\ & x_1, & x_2 & \geq 0 \end{array}$$

We use the notation $d(s, t) = (A(s, t), b(s, t), c^*(s, t))$ when referring to the family.

We assume

$$0 < s \leq 1, \quad 0 < t \leq 1$$

Endow the domain $X = \mathbf{R}^2$ with the ℓ_1 -norm and endow the range $Y = \mathbf{R}^2$ with the ℓ_∞ -norm. So $X^* = \mathbf{R}^2$ is given the ℓ_∞ -norm and $Y^* = \mathbf{R}^2$ is given the ℓ_1 -norm.

Lemma 5.1

$$\begin{aligned}\|A(s, t)\| &= \|b(s, t)\| = \|c^*(s, t)\| = 1 \\ \text{dist}(d(s, t), \text{Pri}\emptyset) &= s \\ \text{dist}(d(s, t), \text{Dual}\emptyset) &= t\end{aligned}$$

Proof. A straightforward exercise left to the reader. \square

Lemma 5.1 elucidates the roles of the parameters s and t ; they allow us to choose the values

$$\frac{\|d\|}{\text{dist}(d, \text{Pri}\emptyset)}, \quad \frac{\|d\|}{\text{dist}(d, \text{Dual}\emptyset)}$$

independently between 1 and ∞ . In light of this the following lemma indicates the “optimality” of several of the bounds of Theorem 1.1 or their dual versions.

Lemma 5.2 (1) *If y^* is feasible for the dual $d(s, t^*)$ then*

$$\|y^*\| \geq \frac{1}{t}$$

(2) *Let $d(r, s, t)$ denote the LP obtained from $d(s, t)$ by replacing the first coefficient of $b(s, t)$ with $s - r$. Letting $x' = (\frac{1}{t}, 0)$, which is feasible for $d(s, t)$, and assuming $r \geq 0$, all feasible points x for $d(r, s, t)$, satisfy*

$$\|x - x'\| \geq \frac{r}{st}.$$

(3)

$$\text{val}(d(s, t)) = \frac{1}{t}$$

(4)

$$\exists y^* \in \text{Opt}(d^*(s, t)) \ni \|y^*\| = \frac{1}{st}$$

(namely, $y^* = (\frac{1}{st}, 0)$.)

(5a) *If $\tilde{d}(r, s, t)$ is the LP obtained from $d(s, t)$ by replacing the $(1, 1)$ coefficient of $A(s, t)$ with $st + r$ then*

$$\lim_{r \downarrow 0} \frac{|\text{val}(\tilde{d}(r, s, t)) - \text{val}(d(s, t))|}{\|\tilde{A}(r, s, t) - A(s, t)\|} = \frac{1}{st^2}$$

(5b) If $d(r, s, t)$ is as defined in (2) then

$$\lim_{r \downarrow 0} \frac{|val(d(r, s, t)) - val(d(s, t))|}{\|b(r, s, t) - b(s, t)\|} = \frac{1}{st}.$$

Proof. A straightforward exercise left to the reader. \square

The “optimality” of the bounds of Theorem 1.1 not addressed by Lemma 5.2, and the “optimality” of the remaining dual version bounds, are verified by considering the dual of $d(s, t)$ as the primal, multiplying the objective by -1 to obtain a maximization problem.

The reader might be confused as to why Lemma 5.2(2) indicates the optimality of Theorem 1.1(2); in the notation of the theorem, let $d = d(r, s, t)$, $d + \Delta d = d(s, t)$ and consider $r \downarrow 0$ so $dist(d, Pri\emptyset) \rightarrow s$.

References

- [1] E.J. Anderson and P. Nash, Linear Programming in Infinite Dimensional Spaces: Theory and Applications, Wiley, Chichester, 1987.
- [2] R.J. Duffin, "Infinite programs," in Linear Equalities and Related Systems, H.W. Kuhn and A.W. Tucker, eds., Princeton University Press, Princeton, 1956, 157-170.
- [3] A.V. Fiacco and K.O. Kortanek, eds., Semi-Infinite Programming and Applications, Lecture Notes in Economics and Mathematical Systems 215, Springer-Verlag, New York, 1983.
- [4] A.J. Hoffman, "On approximate solutions of systems of linear inequalities," Journal of Research of the National Bureau of Standards 49, 1952, 263-265.
- [5] C. Kallina and A.C. Williams, "Linear programming in reflexive spaces," SIAM Review (13) 1971, 350-376.
- [6] O.L. Mangasarian, "A condition number for linear inequalities and linear programs," in Methods of Operations Research (43), Proceedings of the 6th Symposium on Operations Research, Universität Augsburg, G. Bamberg and O. Opitz, eds., Verlagsgruppe Athenäum/Hain/Scriptor/Hanstein, Königstein 1981, 3-15.
- [7] O.L. Mangasarian, "Lipschitz continuity of solutions of linear inequalities, programs and complementarity problems," SIAM J. Control and Optimization, Vol. 25, No. 3, May 1987.
- [8] J. Renegar, "Incorporating condition measures into the complexity theory of linear programming," preprint, School of Operations Research and Industrial Engineering, Cornell University (renegar@orie.cornell.edu).
- [9] S.M. Robinson, "Bounds for error in the solution set of a perturbed linear program," Linear Algebra and Its Applications 6, 1973, 69-81.
- [10] W. Rudin, Functional Analysis, McGraw-Hill, New York, 1973.
- [11] G.W. Stewart and J. Sun, Matrix Perturbation Theory, Academic Press, Boston, 1990.
- [12] J. Vera, "Ill-posedness and the computation of solutions to linear programs with approximate data," preprint, Dept. Ingeniería Industrial, Universidad de Chile, Republica 701, Casilla 2777, Santiago, Chile (jvera@uchcecvm.bitnet).

- [13] J. Vera, “Ill-posedness and the complexity of deciding existence of solutions to linear programs,” preprint, Dept. Ingenieria Industrial, Universidad de Chile, Republica 701, Casilla 2777, Santiago, Chile (jvera@uchcecvm.bitnet).
- [14] K. Yosida, Functional Analysis, Springer-Verlag, New York, 1966.