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SOME PREHOMOGENEOUS REPRESENTATIONS DEFINED BY CUBIC FORMS

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Abstract. Using the notion of Jordan pairs, we give an axiomatic construction of some linear representation of an algebraic group over an arbitrary commutative ring. This representation is prehomogeneous in the sense that all the geometric fibers are prehomogeneous vector spaces modulo scalar multiplications. We also determine one generic stabilizer.

Introduction.

1. Let k be a field. Consider a triple (G, θ, M) consisting of a reductive k-group G, and a finite-dimensional linear representation $\theta: G \to \boldsymbol{GL}(M)$ which, after tensoring with $\bar{k}$ and replacing $G \otimes_k \bar{k}$ by its simply connected cover, becomes isomorphic to the semi-simple part of one of the following *prehomogeneous vector spaces* (notation is of [Sato-Kimura, §7, I)]):

$$\begin{aligned}(5) &= (GL(6), \Lambda_3, V(20))\,,\\ (14) &= (GL(1) \times Sp(3), \square \otimes \Lambda_3, V(1) \otimes V(14))\,,\\ (23) &= (GL(1) \times Spin(12), \square \otimes \text{half-spin rep.}, V(1) \otimes V(32))\,, \quad \text{and}\\ (29) &= (GL(1) \times E_7, \square \otimes \Lambda_6, V(1) \otimes V(56))\,.\end{aligned}$$

((5) is the third exterior power of the six-dimensional standard representation of $GL(6)$, and (14) is obtained from (5) via the inclusion $Sp(3) \to GL(6)$.)

2. It is known that those (G, θ, M) in 1 are related to *Jordan algebras* $J = H_3(\mathscr{C})$ of 3×3 Hermitian matrices with coefficients in various composition algebras $\mathscr{C}$ (cf. [Freu, VIII]). Among the many works concerned with such (G, θ, M), [Igusa] and [Baily] are closely related to our research. For (G, θ, M) of type (14), (23), or (29) in 1, Igusa determined the quotient set $G(k)\backslash M$ and the corresponding stabilizers in G, under the assumption that k is an *algebraically closed field of characteristic different from two and three* (cf. [Igusa, §7]). On the other hand, using a $\boldsymbol{Z}$-form of the real octonion division algebra, Baily treated a triple (G, θ, M) over the *ring of rational integers* such that the associated analytic group $G(\boldsymbol{R})$ is a Lie group of type E_7 acting on a bounded symmetric domain in $\boldsymbol{C}^{27}$ (cf. [Baily]).

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To apply the theory of prehomogeneous vector spaces to number theory, we assume k to be a global field. Then the main problem is to determine the quotient set $(k^* \times G(k))\backslash M$ and the corresponding stabilizers in $\boldsymbol{G}_{mk} \times G$ (cf. [Wright-Yukie]). Equivalently, considering the projective representation $G \to \mathbf{Aut}(\boldsymbol{P}(M))$ associated to θ, the quotient set $G(k)\backslash \boldsymbol{P}(M)(k)$ and the corresponding stabilizers in G need to be determined. In general, one prehomogeneous vector space admits several forms over a given field, on the choice of which the above problem depends. Instead of considering *all* the forms of a special prehomogeneous vector space, we are trying to define a triple (G, θ, M) over an arbitrary commutative ring such that some (or all, if possible,) of the fibers have the property stated in 1. The construction of (G, θ, M) and almost all of the calculations work over an arbitrary commutative ring. Also our construction contains Baily's case, which is not of split type and causes special difficulties in characteristic two. In fact, the desire to handle such a case leads us to considering schemes over $\boldsymbol{Z}$, and hence we are obliged to construct everything without assumption on the base ring. In particular, we need to include the case of characteristics two and three, which are avoided in [Igusa]. In this paper, we give an axiomatic construction of (G, θ, M) and determine one stabilizer. More precisely:

3. Let k be an arbitrary *commutative ring*. Consider a quadruple $(J; N, \sharp, T)$ as the data, where J is a *finitely generated projective* k-module, N a *cubic form* on J, i.e., N is a homogeneous element of degree three of the symmetric algebra $\boldsymbol{S}({}^tJ)$ of the k-module ${}^tJ$ dual to J, $\sharp$ a *quadratic map* in J, which is a certain endomorphism of the k-scheme $\operatorname{Spec} \boldsymbol{S}({}^tJ)$, and T a *symmetric bilinear form* on J, satisfying certain conditions (cf. §1). Then:

(a) We define a k-group *sheaf* G with respect to the fppf topology and its *linear representation* $\theta: G \to \boldsymbol{GL}(M)$ in the k-module $M := k \oplus J \oplus k \oplus J$.

(b) We choose one k-valued point u_0 of the projective space $\boldsymbol{P}(M) := \operatorname{Proj} \boldsymbol{S}({}^tM)$ and determine its *stabilizer* $\mathbf{Cent}_G(u_0)$ in G (cf. §3).

(c) We choose a *quartic form* $f \in \boldsymbol{S}^4({}^tM)$ stabilized by G (cf. §4). Then the action of G is induced on the open subscheme $D_+(f)$ of $\boldsymbol{P}(M)$. The point u_0 in (b) belongs to $D_+(f)(k)$.

(d) Under some additional condition on $(J; N, \sharp, T)$, we prove that, for any k-algebra $k \to K$ with K an algebraically closed field, the action of $G(K)$ on $D_+(f)(K)$ is *transitive*.

4. Main tools of our construction are the notion of *Jordan pairs* and the general theory of associated algebraic groups both due to Loos (cf. [LJP], [LAG]). Also there is an axiomatic construction of Jordan algebras $J = H_3(\mathscr{C})$ of 3×3 Hermitian matrices by McCrimmon (cf. [Mc]). We modify McCrimmon's construction to adapt to Loos's theory, and obtain our quadruple $(J; N, \sharp, T)$ (cf. 1.1). To use the general theory of Loos in the construction of (G, θ, M), we have to prove some identities through complicated calculation (cf. 2.6–2.8). Once we get (G, θ, M) and the quartic form f on M, which is taken from [Freu, I], the remaining part of this paper (§§3–4) is reduced

to direct calculations.

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Terminology.

0.1. Throughout this paper, k is an arbitrary commutative associative unitary ring. k-**alg** stands for the category of commutative associative unitary k-algebras. We denote by k-**alg**$^\wedge$ the category of *k-functors*, whose objects are set-valued covariant functors on k-**alg** and morphisms are natural transformations of functors. We follow the general conventions of [D-G]. In particular, the category of *k-schemes* is a full-subcategory of k-**alg**$^\wedge$. By a *k-sheaf*, we understand an fppf k-sheaf. $\boldsymbol{O}_k \in k$-**alg**$^\wedge$ stands for the affine line, i.e., the forgetful functor. $\mu_k \in k$-**alg**$^\wedge$ stands for the functor $R \mapsto R^* := \{$invertible elements of $R\}$, which is an open subfunctor of $\boldsymbol{O}_k$. For an integer $n \geq 0$, we denote by ${}_n\mu_k$ the functor $R \mapsto \{t \in R^* \mid t^n = 1\}$.

0.2. Following [LAG, 1.4], we use the notion of *dense* subfunctors. Namely, for $X \in k$-**alg**$^\wedge$ and a subfunctor $U \subset X$, U is said to be *dense* in X if the following property holds for any scalar extensions: any open subfunctor V of X has no closed subfunctor $Z \subset V$ containing $U \cap V$ other than V. The next lemma is cited from [LAG, 1.5]. It is based on [EGA IV, 11.10.10] and [SGA3, Exp. XVIII, Prop. 1.2].

LEMMA (cf. [LAG, 1.5]). *Let X be a smooth separated algebraic k-scheme with non-empty connected fibers, and U an open subscheme of X. Then the following conditions are equivalent*:

(i) *U is dense in X.*

(ii) *There exists an fppf extension R of k such that $U(R) \neq \varnothing$.*

(iii) *$U(K) \neq \varnothing$ for any algebraically closed field $K \in k$-**alg**.*

0.3. Let M be a k-module. We define k-functors M_a, $\boldsymbol{P}(M)$, and M_m by setting

$$M_a(R) := M_R := M \otimes_k R\,,$$

$$\boldsymbol{P}(M)(R) := \{L \mid \text{direct factor of } M_R \text{ and invertible as an } R\text{-module}\}\,,$$

$$M_m(R) := \{x \in M_R \mid \lambda(x) = 1,\ \exists \lambda \in {}^t(M_R)\}\,,$$

for $R \in k$-**alg**, where ${}^t(M_R)$ stands for the R-module dual to M_R. Denote by $p_M\colon M_m \to \boldsymbol{P}(M)$ the morphism of k-functors sending $x \in M_m(R)$, $R \in k$-**alg**, to $p_M(x) := R \cdot x$, the R-submodule of M_R spanned by x. If M is finitely generated and projective, these k-functors are all k-schemes and we have, in the terminology of EGA, $M_a = \operatorname{Spec} \boldsymbol{S}({}^tM)$ and $\boldsymbol{P}(M) = \operatorname{Proj} \boldsymbol{S}({}^tM)$ (cf. [EGA II, 4.2.3]). For example, we have $k_a = \boldsymbol{O}_k$ and $k_m = \mu_k$.

0.4. Let M, N be k-modules. By a *polynomial law* on the couple (M, N), we understand a morphism of k-functors $M_a \to N_a$ (cf. [Roby, p. 219]). Let $f\colon M_a \to N_a$ be a polynomial law and p an integer ≥ 0. We say that f is *homogeneous of degree p* if $f(tx) = t^p f(x)$ for all $t \in R$, $x \in M_R$, $R \in k$-**alg**. Denote by $\mathcal{O}(M, N)$ (resp. by $\mathcal{O}^p(M, N)$) the k-module of the polynomial laws (resp. those which are homogeneous of degree p) on (M, N). For $N = k$, we write $\mathcal{O}(M) := \mathcal{O}(M, k)$ and $\mathcal{O}^p(M) := \mathcal{O}^p(M, k)$. Denote by N^M the k-module of the maps from the underlying set of M to that of N. We say that a map $Q \in N^M$ is *quadratic* if $Q(tx) = t^2 Q(x)$ for $t \in k$ and $x \in M$, and if the map $M \times M \to N$ sending (x, y) to $Q(x+y) - Q(x) - Q(y)$ is bilinear. In this case, we write

$$Q(x, y) := Q(x+y) - Q(x) - Q(y).$$

By definition, we have a natural map from $\mathcal{O}^p(M, N)$ to N^M, which is not injective in general. However, this is the case if $p \leq 2$. More precisely, $\mathcal{O}^0(M, N)$ (resp. $\mathcal{O}^1(M, N)$, $\mathcal{O}^2(M, N)$) is identified with the *constant* (resp. *linear*, *quadratic*) maps from M to N. We refer to [Roby] and [Bou, IV, §5, Exercises] for details. For this reason, an element of $\mathcal{O}^2(M, N)$ is also called a *quadratic map*. Similarly, an element of $\mathcal{O}^2(M)$ is called a *quadratic form*. By a *cubic* (resp. *quartic*, ...) *form* on M, we understand an element of $\mathcal{O}^p(M)$ for $p = 3$ (resp. $p = 4, \ldots$).

0.5. Let M, N be k-modules and f a polynomial law on (M, N). (cf. 0.4). For any $x, y \in M_R$, $R \in k$-**alg**, we set

$$f(x + \varepsilon y) =: f(x) + \varepsilon \partial_y f(x) \in N_{R[\varepsilon]},$$

where $R[\varepsilon]$ is the ring of *dual numbers*, to obtain the polynomial law $\partial_y f \in \mathcal{O}(M_R, N_R)$. This definition may be read as follows: the *tangent bundle* T_{M_a} of the k-functor M_a can be identified with $M_a \times M_a$ by means of $T_{M_a}(R) := M_{R[\varepsilon]} \ni a + \varepsilon b \mapsto (a, b) \in M_R \times M_R$. f is a morphism $M_a \to N_a$ (cf. 0.4), from which the morphism $T_f\colon T_{M_a} \to T_{N_a}$ is induced. Then we have

$$T_f(x, y) = (f(x), \partial_y f(x))$$

for all $x, y \in M_R$, $R \in k$-**alg**.

1. Basic Jordan identities.

1.1. Consider a quadruple $(J; N, \sharp, T)$, where J is a finitely generated projective k-module, N a cubic form (cf. 0.4) on J, $?^\sharp\colon x \mapsto x^\sharp$ a quadratic map (cf. 0.4) from J to J, and T a symmetric k-bilinear form on J, satisfying the following two identities (CJ1), (CJ2) and the condition $(*)$: for all $x, y \in J_R$, $R \in k$-**alg**, we have

(CJ1) $$x^{\sharp\sharp} = N(x)x,$$

(CJ2) $$\partial_y N(x) = T(x^\sharp, y),$$

and

$(*)$ There exist $c_1, c_2 \in J$ and a linear form λ on J such that

$$N(c_1) \in k^* \quad \text{and} \quad \lambda(c_2^\sharp) = 1.$$

The identities (CJ1) and (CJ2) should be read as commutative diagrams

(CJ1bis)
$$\begin{array}{ccc} J_a & \xrightarrow{\ \#\ } & J_a \\ {\scriptstyle \#}\uparrow & & \uparrow{\scriptstyle \varphi} \\ J_a & \xrightarrow{(N,\,\mathrm{Id})} & \boldsymbol{O}_k \times J_a \end{array}$$

and

(CJ2bis)
$$\begin{array}{ccc} T_{J_a} & \xrightarrow{\ T_N\ } & T_{\boldsymbol{O}_k} \\ {\scriptstyle \mathrm{can.}}\uparrow\wr & & \wr\uparrow{\scriptstyle \mathrm{can.}} \\ J_a \times J_a & \xrightarrow{(N\circ \mathrm{pr}_1,\,\psi)} & \boldsymbol{O}_k \times \boldsymbol{O}_k \end{array}$$

of *k-schemes*, where $\varphi\colon \boldsymbol{O}_k \times J_a \to J_a$ is the scalar multiplication and $\psi\colon J_a \times J_a \to \boldsymbol{O}_k$ is the morphism sending (x, y) to $T(x^\#, y)$ (cf. 0.5). In the following, we fix such a quadruple $(J; N, \#, T)$ and set

$$x \times y := (x+y)^\# - x^\# - y^\# , \tag{1}$$

$$Q(x)y := T(x, y)x - x^\# \times y , \tag{2}$$

$$N(x, y) := 1 - T(x, y) + T(x^\#, y^\#) - N(x)N(y) , \tag{3}$$

$$P(x, y) := x - x^\# \times y + N(x)y^\# , \tag{4}$$

for all $x, y \in J_R$, $R \in k$-**alg**, to obtain a bilinear product $\times$ in J and polynomial laws $Q \in \mathcal{O}^2(J, \mathrm{End}(J))$, $N(\ ,\ ) \in \mathcal{O}(J \times J)$, $P(\ ,\ ) \in \mathcal{O}(J \times J, J)$ (cf. 0.4). By definition, $(x, y, z) \mapsto Q(x, z)y$ (cf. 0.4) is a trilinear product in J. Denote any scalar extension of it by $\{\ \ \}$ and let $D(x, y)z := \{xyz\}$. Hence we have

$$D(x, y)z := \{xyz\} := Q(x, z)y = T(x, y)z + T(y, z)x - (z \times x) \times y , \tag{5}$$

for all $x, y, z \in J_R$. Finally we set

$$B(x, y)z := z - \{xyz\} + Q(x)Q(y)z , \tag{6}$$

to obtain a polynomial law $B \in \mathcal{O}(J \times J, \mathrm{End}(J))$.

1.2. Let $R \in k$-**alg**, and $x, y, z, u, v \in J_R$. Since N is a cubic form, there exists an R-linear form $\tilde{N}$ on the degree 3 component $\Gamma_3(J_R)$ of the Γ-algebra $\Gamma(J_R)$ of J_R such that $N(x) = \langle \gamma_3(x), \tilde{N} \rangle$ (cf. [Bou, IV, §5, exerc. 10)]). $\gamma_p\colon J_R \to \Gamma(J_R)$ $(p \geq 0)$ satisfy

$$\gamma_3(x+y) = \gamma_3(x) + \gamma_2(x)\gamma_1(y) + \gamma_1(x)\gamma_2(y) + \gamma_3(y) ,$$

$$\gamma_2(x+y)\gamma_1(z) = \gamma_2(x)\gamma_1(z) + \gamma_2(y)\gamma_1(z) + \gamma_1(x)\gamma_1(y)\gamma_1(z) ,$$

$$\gamma_2(x)\gamma_1(x) = 3\gamma_3(x) ,$$

(cf. [Bou, IV, §5, exerc. 2)]), and we have $T(u^\#, v) = \partial_v N(u) = \langle \gamma_2(u)\gamma_1(v), \tilde{N} \rangle$ by (CJ2). Hence, applying $\langle ?, \tilde{N} \rangle$ to the above identities, we get

(CJ3) $$N(x+y)=N(x)+T(x^{\#}, y)+T(x, y^{\#})+N(y)\,,$$

(CJ4) $$T(x\times y, z)=N(x, y, z)=T(x, y\times z)\,,$$

(CJ5) $$T(x^{\#}, x)=3N(x)\,,$$

where $N(x, y, z):=\langle\gamma_1(x)\gamma_1(y)\gamma_1(z), \tilde{N}\rangle=N(x+y+z)-N(x+y)-N(y+z)-N(z+x)+N(x)+N(y)+N(z)$. Since $N(x, y, z)$ is symmetric, the latter equality of (CJ4) follows from the former. Next, taking the scalar extension $R\to R[t]$ to the polynomial ring in one variable t, replacing x by $x+ty$ in (CJ1), expanding the result by using 1.1(1) and (CJ3), and comparing the terms in t, t^2, we get

(CJ6) $$x^{\#}\times(x\times y)=N(x)y+T(x^{\#},y)x\,,$$

(CJ7) $$x^{\#}\times y^{\#}+(x\times y)^{\#}=T(x^{\#}, y)y+T(x, y^{\#})x\,.$$

Linearization of (CJ7) with respect to y yields

(CJ8) $$x^{\#}\times(y\times z)+(x\times y)\times(x\times z)=T(x^{\#}, y)z+T(x^{\#}, z)y+T(x, y\times z)x\,.$$

Applying $T(?, z)$ to (CJ6) with (CJ4) in mind, we get

(CJ9) $$N(x^{\#}, x\times y, z)=N(x)T(y, z)+T(x^{\#}, y)T(x, z)\,.$$

If we interchange y and z in (CJ9), and calculate the left-hand side using (CJ4) and the symmetry of $N(\ ,\ ,\ )$, then the result is

(CJ10) $$N(x, x^{\#}\times y, z)=N(x)T(y, z)+T(x^{\#}, z)T(x, y)\,.$$

Applying $T(x^{\#}, ?)$ to (CJ7) with (CJ4), (CJ5), and (CJ1) in mind, we get

(CJ11) $$T(x^{\#}, (x\times y)^{\#})=T(x^{\#}, y)^2+N(x)T(x, y^{\#})\,.$$

On the other hand, we have

(1) $$T(Q(x)u, v)=T(x, u)T(x, v)-T(x^{\#}, u\times v)\,,$$

by 1.1(2) and (CJ4), which gives $T(Q(x)u, v)=T(Q(x)v, u)$, since the right-hand side of (1) is symmetric in u and v. However T is also symmetric by assumption. This gives

(2) $$T(Q(x)u, v)=T(u, Q(x)v)\,.$$

Similarly, we have $T(D(x, y)u, v)=T(x, y)T(u, v)+T(y, u)T(x, v)-T(u\times x, v\times y)$ by 1.1(5) and (CJ4), which gives

(3) $$T(D(x, y)u, v)=T(u, D(y, x)v)\,.$$

Finally, using (2), (3), and 1.1(6), we get

(4) $$T(B(x, y)u, v)=T(u, B(y, x)v)\,.$$

1.3. Now we shall use the assumption $(*)$ in 1.1. Recall that the k-functor $J_a\colon R\mapsto J_R$ is a smooth separated algebraic k-scheme with non-empty connected fibers (cf. 0.3). Moreover we have:

(a) The inverse image of $J_m \subset J_a$ under the morphism $?^{\#}: J_a \to J_a$ and the principal open subscheme defined by the section $N \in \mathcal{O}(J)$ are both *dense* in J_a.

(b) The morphism $?^{\#}: J_a \to J_a$ is *scheme-theoretically dominant* (cf. [EGA I, 5.4.2]).

Indeed, (a) follows from 0.2, and (b) amounts to saying that the corresponding ring homomorphism $\mathcal{O}(J) \to \mathcal{O}(J)$, say φ, is *injective* (cf. [EGA I, 5.4.1]). This can be verified as follows: since $?^{\#}$ is quadratic, we have $\varphi(\mathcal{O}^p(J)) \subset \mathcal{O}^{2p}(J)$ for all $p \geq 0$. This implies that ker φ is a *homogeneous* ideal. Thus it suffices to show that any *homogeneous* element f of ker φ is zero. Indeed, choosing $p \geq 0$ so that $f \in \mathcal{O}^p(J)$, we have $0 = \varphi(f)(x^{\#}) = f(x^{\#\#}) = f(N(x)x)$ (by (CJ1)) $= N(x)^p f(x)$. Hence $f: J_a \to \boldsymbol{O}_k$ vanishes on the principal open subscheme of J_a defined by the section $N \in \mathcal{O}(J)$. This implies $f=0$, in view of (a).

Using (a), we get

(CJ12) $$N(x^{\#}) = N(x)^2 ,$$

(CJ13) $$x \times (x^{\#} \times y) = N(x)y + T(x, y)x^{\#} ,$$

for all $x, y \in J_R$, $R \in k$-**alg**. Indeed, we have $N(x^{\#})x^{\#} = x^{\#\#\#} = N(x)^2 x^{\#}$ by (CJ1). Hence the morphisms $N \circ \#: J_a \to \boldsymbol{O}_k$ and $N^2: J_a \to \boldsymbol{O}_k$ coincide on the invese image of $J_m \subset J_a$ under the morphism $?^{\#}: J_a \to J_a$. In view of (a), this implies $N \circ \# = N^2$, namely (CJ12). As for (CJ13), we fix y and consider the morphisms $f: J_a \otimes_k R \to J_a \otimes_k R$ sending x to $x \times (x^{\#} \times y)$ and $g: J_a \otimes_k R \to J_a \otimes_k R$ sending x to $N(x)y + T(x, y)x^{\#}$. Replacing x by $x^{\#}$ in (CJ6) and using (CJ1) and (CJ12), we get $N(x)x \times (x^{\#} \times y) = N(x)^2 y + N(x)T(x, y)x^{\#}$, namely $N(x)f(x) = N(x)g(x)$. Hence the morphisms f and g coincide on the $\otimes_k R$ of the principal open subscheme of J_a defined by the section $N \in \mathcal{O}(J)$. In view of (a), this implies $f = g$, namely (CJ13).

1.4 THEOREM (a modification of McCrimmon [Mc, Th. 1]). *The data* $(V^{\pm}, Q_{\pm})$ *with* $V^+ = V^- := J$, $Q_+ = Q_- := Q$ *is a Jordan pair over* k, *which has an invertible element.*

1.5. The proof of the theorem requires long calculations. Here we indicate its outlines with some additional identities for later use. We first recall that (cf. [LJP, 1.2]) a *Jordan pair* over k is a pair of k-modules (V^+, V^-) together with a pair (Q_+, Q_-) of quadratic maps $Q_\sigma: V^\sigma \to \mathrm{Hom}(V^{-\sigma}, V^\sigma)$, $\sigma = \pm$, satisfying

(JP1) $$D_\sigma(x, y)Q_\sigma(x) = Q_\sigma(x)D_{-\sigma}(y, x) ,$$

(JP2) $$D_\sigma(Q_\sigma(x)y, y) = D_\sigma(x, Q_{-\sigma}(y)x) ,$$

(JP3) $$Q_\sigma(Q_\sigma(x)y) = Q_\sigma(x)Q_{-\sigma}(y)Q_\sigma(x) ,$$

for all $\sigma = \pm$, $x \in V_R^\sigma$, $y \in V_R^{-\sigma}$, $R \in k$-**alg**. Here we set $D_\sigma(x, y)z := Q_\sigma(x, z)y$. An element x in V^σ is said to be *invertible* if the linear map $Q_\sigma(x): V^{-\sigma} \to V^\sigma$ is invertible (cf. [LJP, 1.10]). Returning to the situation in 1.1, let $R \in k$-**alg**, and $x, y, z \in J_R$. By direct calculations using 1.2, we have $Q(x)Q(x^\#)=N(x)^2\,\mathrm{Id}$ and $Q(x^\#)Q(x)=N(x^\#)\mathrm{Id}$, which become

$$\text{(CJ14)}\qquad Q(x)Q(x^\#)=Q(x^\#)Q(x)=N(x)^2\,\mathrm{Id}\,,$$

by (CJ12). Hence $Q(c_1)\in\mathrm{End}(J)$ is invertible for c_1 in 1.1(∗). Thus it remains to check the defining identities of Jordan pairs. Start with taking the scalar extension $R\to R[t]$ to the polynomial ring in one variable t, replace x by $x+tz$ in (CJ13), expand the result by using 1.1(1) and (CJ3), and compare the terms in t. Then we get

$$\text{(CJ15)}\qquad x\times((x\times z)\times y)+z\times(x^\#\times y)=T(x^\#,z)y+T(x,y)x\times z+T(y,z)x^\#\,.$$

Moreover, by direct calculations using 1.1(1), (2), and (CJ13, 1, 7), we have

$$\text{(CJ16)}\qquad (Q(x)y)^\#=Q(x^\#)y^\#\,.$$

We can now verify (JP1), (JP2), and (JP3) by straightforward calculations using (CJ6, 3), (CJ15, 8, 4), and (CJ16, 6, 15, 10), respectively. Thus the proof of the theorem is complete. Let us introduce some more identities. Add $2N(x)y$ to (CJ6) (resp. (CJ13)), use (CJ5), and subtract $x^\#\times(x\times y)$ (resp. $x\times(x^\#\times y)$). Then we get

$$2N(x)y=T(x^\#,x)y+T(x^\#,y)x-x^\#\times(x\times y)$$

$$(\text{resp. } 2N(x)y=T(x^\#,x)y+T(x,y)x^\#-x\times(x^\#\times y))\,,$$

which in operator forms become

$$\text{(CJ17)}\qquad D(x,x^\#)=D(x^\#,x)=2N(x)\mathrm{Id}\,,$$

whose linearization yields

$$\text{(CJ18)}\qquad D(x,x\times y)+D(y,x^\#)=D(x\times y,z)+D(x^\#,y)=2T(x^\#,y)\mathrm{Id}\,.$$

1.6. From now on, we apply the notion of Jordan pair (cf. [LJP]) to $(V^\pm,Q_\pm)$ with $V^+=V^-:=J$, $Q_+=Q_-:=Q$. Recall that an element x of J is said to be *invertible* if $Q(x)\in\mathrm{End}(J)$ is invertible (cf. [LJP, 1.10]). In this case, $x^{-1}:=Q(x)^{-1}x$ is the *inverse* of x (cf. [LJP, 1.10]). If $N(x)\in k^*$, then x is invertible by (CJ14), and we have $x^{-1}=N(x)^{-2}Q(x^\#)x=N(x)^{-1}x^\#$ by 1.1(1), (2), and (CJ1, 5).

PROPOSITION. *An element x of J is invertible if and only if the scalar $N(x)$ is invertible; if that is the case, we have*

$$(1)\qquad x^{-1}=N(x)^{-1}x^\#\,,$$

and, for any $y\in J$,

$$(2)\qquad N(x,y)=N(x)N(x^{-1}-y)\,.$$

Indeed, we have $N(x^\#-N(x)y)=N(x)^2N(x,y)$ by (CJ3, 12, 1). Hence (2) follows from (1). It remains to prove the implication: x invertible $\Rightarrow$ $N(x)\in k^*$. This is a consequence of (CJ20) in the following lemma, since there exist $c_1, y \in J$ such that $Q(x)y = c_1$, $N(c_1) \in k^*$ (cf. 1.1($*$)).

1.7 LEMMA. *For all $R \in k$-**alg**, and $x, y, z \in J_R$, we have*

(CJ19) $$N(x \times y) = T(x^\#, y)T(x, y^\#) - N(x)N(y)\,,$$

(CJ20) $$N(Q(x)y) = N(x)^2 N(y)\,,$$

(CJ21) $$N(B(x, y)z) = N(x, y)^2 N(z)\,.$$

PROOF. Taking the scalar extension $R \to R[t]$ to the polynomial ring in one variable t, replacing x by $x + ty$ in (CJ12), expanding the result, and comparing the terms in t^3, we get $N(x \times y) + N(x^\#, x \times y, y^\#) = 2N(x)N(y) + T(x^\#, y)T(x, y^\#)$, which becomes (CJ19) by (CJ9) and (CJ5), (CJ20) follows from the expansion of the left-hand side using 1.1(2), (CJ3), and (CJ19). For (CJ21) we may assume $N(x)$ to be invertible, since the principal open subscheme defined by the section $(x, y) \mapsto N(x)$ is dense in $J_a \times J_a$ (cf. 0.2). Then x is invertible by the remark at the beginning of 1.6 (which is independent of the proposition) and we have $B(x, y) = Q(x)Q(x^{-1} - y)$ by [LJP, 2.12]. Thus the assertion follows from (CJ20) and 1.6(2).

1.8. Recall that a pair (x, y) of elements of J is said to be *quasi-invertible* if $B(x, y) \in \mathrm{End}(J)$ is invertible (cf. [LJP, 3.2]). In this case, $x^y := B(x, y)^{-1}(x - Q(x)y)$ is the *quasi-inverse* of (x, y) (cf. [LJP, 3.2]).

COROLLARY. *A pair (x, y) of elements of J is quasi-invertible if and only if the scalar $N(x, y)$ is invertible; if that is the case, we have*

(CJ22) $$x^y = N(x, y)^{-1} P(x, y)\,,$$

(CJ23) $$(x^y)^\# = N(x, y)^{-1}(x^\# - N(x)y)\,,$$

(CJ24) $$N(x^y) = N(x, y)^{-1} N(x)\,,$$

and, for any $z, w \in J$,

(CJ25) $$(B(x, y)z)^\# = N(x, y)^2 B(y, x)^{-1} z^\#\,,$$

(CJ26) $$(B(x, y)z) \times (B(x, y)w) = N(x, y)^2 B(y, x)^{-1}(z \times w)\,.$$

The quasi-invertibility of (x, y) implies the invertibility of $N(x, y)$ by (CJ21), since there exist $c_1, z \in J$ such that $B(x, y)z = c_1$, $N(c_1) \in k^*$. Conversely if $N(x, y)$ is invertible, then we have $B(x, y)z = x - Q(x)y$ and $B(x, y)Q(z)y = Q(x)y$ for $z := N(x, y)^{-1}P(x, y)$ by the following 1.9(3), (4). This implies the quasi-invertibility of (x, y) together with (CJ22) by [LJP, 3.2 (ii)]. We have

(CJ23bis) $$P(x, y)^\# = N(x, y)(x^\# - N(x)y)\,,$$

by direct calculation using (CJ1, 6, 7, 13). Hence (CJ23) follows from (CJ22). Since

(CJ26) is a linearization of (CJ25), it remains to verify (CJ24) and (CJ25). We may assume x to be invertible, since the principal open subscheme defined by the section $(x, y) \mapsto N(x)$ is dense in $J_a \times J_a$ (cf. 0.2). Then, we have $x^y = (x^{-1} - y)^{-1}$, $B(x, y) = Q(x)Q(x^{-1} - y)$, and $B(y, x) = Q(x^{-1} - y)Q(x)$ (cf. [LJP, 2.12, 3.13]). Hence (CJ24) follows from 1.6(2), and (CJ25) can be verified as follows:

$$\begin{aligned}(B(x, y)z)^\# &= (Q(x)Q(x^{-1} - y)z)^\# \\ &= Q(x^\#)Q((x^{-1} - y)^\#)z^\# \quad \text{(by (CJ16))} \\ &= N(x)^2 N(x^{-1} - y)^2 Q(x)^{-1} Q(x^{-1} - y)^{-1} z^\# \quad \text{(by (CJ14))} \\ &= N(x, y)^2 B(y, x)^{-1} z^\# \quad \text{(by 1.6(2))}\,.\end{aligned}$$

1.9 Lemma. *For any $x, y, z \in J_R$, $R \in k$-**alg**, we have*

(1) $$B(x, y)y^\# = y^\# - N(y)x - N(y)(x - Q(x)y)\,,$$

(2) $$B(x, y)(z \times y) = z \times P(y, x) + T(z, x)(y^\# - N(y)x) - T(z, y^\#)(x - Q(x)y)\,,$$

(3) $$B(x, y)P(x, y) = x - Q(x)y\,,$$

(4) $$B(x, y)Q(P(x, y)) = N(x, y)^2 Q(x)y\,,$$

(5) $$B(x, y)(z - (z \times x) \times y + T(z, x^\#)y^\#) = N(x, y)z - T(z, P(y, x))(x - Q(x)y)\,.$$

This lemma was used in the proof of 1.8 (also will be used in 2.7). All the formulas can be proved independently of 1.8 by direct calculation.

1.10 Lemma. *For any $x, y, z \in J_R$, $t \in R$, $R \in k$-**alg** such that (x, y) is quasi-invertible, we have*

(1) $$N(tx, z) = N(x, tz)\,,$$

(2) $$P(tx, z) = P(x, tz)\,,$$

(3) $$N(x, y)N(x^y, z) = N(x, y + z)\,,$$

(4) $$N(x, y)P(x^y, z) = P(x, y + z)\,.$$

Proof. (1), (2): Direct consequences of the definitions 1.1(3), (4).

(3): We may assume x to be invertible, since the principal open subscheme defined by the section $(x, y, z) \mapsto N(x)$ is dense in $J_a \times J_a \times J_a$. Then, by 1.6(2) and (CJ24), we have $N(x^y, z) = N(x^y)N((x^y)^{-1} - z) = N(x, y)^{-1}N(x)N((x^{-1} - y) - z) = N(x, y)^{-1}N(x)N(x^{-1} - (y + z)) = N(x, y)^{-1}N(x, y + z)$.

(4): We may assume $(x, y + z)$ to be quasi-invertible, since the principal open subscheme defined by the section $(x, y, z) \mapsto N(x, y + z)$ is dense in $J_a \times J_a \times J_a$. Then, by (CJ22), (3), and [LJP, 3.7 (a)], we have $P(x, y + z) = N(x, y + z)x^{y + z} = N(x, y)N(x^y, z)(x^y)^z = N(x, y)P(x^y, z)$.

2. Representation.

2.0. In this section, let $(J; N, \sharp, T)$ be a quadruple as in 1.1, and we use the following notation:

(J, J): the associated Jordan pair (cf. 1.4), i.e., the Jordan pair $V=(V^{\pm}, Q_{\pm})$ with $V^{+}=V^{-}:=J$, $Q_{+}=Q_{-}:=Q$, $Q(x)y:=T(x, y)x-x^{\sharp}\times y$ (cf. 1.1(2)).

$\mathscr{W}$: the scheme of quasi-invertible pairs in (J, J). This is precisely the principal open subscheme of $J_a \times J_a$ defined by the section $(x, y)\mapsto N(x, y)$ (cf. 1.8), which is dense in $J_a \times J_a$.

Recall that the *automorphism group* $\mathrm{Aut}(V)$ of a Jordan pair $V=(V^{\pm}, Q_{\pm})$ is the group of all $(h_+, h_-)\in GL(V^+)\times GL(V^-)$ such that $h_\sigma Q_\sigma(x)=Q(h_\sigma(x))h_{-\sigma}$ for $\sigma=\pm$, $x\in V_R^{\sigma}$, $R\in k$-**alg** (cf. [LJP, 1.3]). The k-group functor $R\mapsto \mathrm{Aut}(V_R)$ is denoted by $\mathbf{Aut}(V)$, which is an affine algebraic k-group scheme (cf. [LAG, 2.3]).

2.1. Consider the k-group scheme $\mu_k \times \boldsymbol{GL}(J)^2$ and denote any R-valued point h of it in the form

$$h=:(\chi(h), h_+, h_-)\,, \tag{1}$$

where $\chi(h)\in R^*$ and $h_+, h_-\in GL(J_R)$. Denote by H the subgroup scheme of $\mu_k\times \boldsymbol{GL}(J)^2$ whose R-valued points is the group of h's satisfying

$$T(h_+x, h_-y)=T(x, y)\,, \tag{H1}$$

$$(h_+x)^{\sharp}=\chi(h)^{-1}h_-x^{\sharp}\,,\quad (h_-x)^{\sharp}=\chi(h)h_+x^{\sharp}\,, \tag{H2}$$

$$N(h_+^{-1}x)=N(h_-x)=\chi(h)N(x)\,, \tag{H3}$$

for all $x, y\in J_S$, $S\in R$-**alg**. Note that we have

$$\begin{cases} (h_+x)^{\sharp}=\chi(h)^{-1}h_-x^{\sharp}\,, & (h_+x)\times(h_+y)=\chi(h)^{-1}h_-(x\times y)\,,\\ (h_-x)^{\sharp}=\chi(h)h_+x^{\sharp}\,, & (h_-x)\times(h_-y)=\chi(h)h_+(x\times y)\,, \end{cases} \tag{H2bis}$$

which is the linearization of (H2). Note also that the inclusion $H\to\mu_k\times\boldsymbol{GL}(J)^2$ is a *finitely presented closed immersion*. Indeed, our definition amounts to saying that the diagram

$$\begin{array}{ccc} H & \xrightarrow{\text{incl.}} & \mu_k\times\boldsymbol{GL}(J)^2 \\ \downarrow & & \downarrow{\scriptstyle \boldsymbol{d}} \\ \operatorname{Spec} k & \xrightarrow{\boldsymbol{s}} & E_a \end{array}$$

is *Cartesian*, where $E:={}^t(J\otimes J)\times\mathcal{O}^2(J, J)^2\times\mathcal{O}^3(J)^2$, $\boldsymbol{s}:=$the section corresponding to $(T, \sharp, \sharp, N, N)\in E=E_a(k)$, and $\boldsymbol{d}(\lambda, h_+, h_-):=(T\circ(h_+\otimes h_-), \lambda h_-^{-1}\circ\sharp\circ h_+, \lambda^{-1}h_+^{-1}\circ\sharp\circ h_-$, $\lambda^{-1}N\circ h_+^{-1}, \lambda^{-1}N\circ h_-)$ for $\lambda\in R^*$, $h_+, h_-\in GL(J_R)$, $R\in k$-**alg**. However the section $\boldsymbol{s}$ is a finitely presented closed immersion, since E is a finitely generated projective k-module.

In particular, H is an *affine algebraic* k-group scheme. If $h\in H(R)$, then

(2) $$h^\vee := (\chi(h)^{-1}, h_-, h_+)$$

also belongs to $H(R)$ and $h \mapsto h^\vee$ becomes an automorphism of H of period two. If $t \in R^*$, then

(3) $$z(t) := (t^{-3}, t\,\mathrm{Id}, t^{-1}\mathrm{Id})$$

belongs to $H(R)$ and, varying R, we get an inclusion $z: \mu_k \to H$, which factors through the center of H. Define $-1 \in H(k)$ to be $z(-1)$ and set

(4) $$-h := z(-1)h \in H(R)$$

for $h \in H(R)$, $R \in k$-**alg**. Then $h \mapsto -h$ becomes an automorphism of H of period two. Since we have $\chi(z(t)) = t^{-3}$ by (3), the character $\chi: H \to \mu_k$ is an epimorphism of k-sheaves. By 1.2(2), (CJ4), (CJ16), and 1.6(1),

(5) $$b(x) := (N(x), N(x)^{-1}Q(x), N(x)Q(x^{-1}))$$

belongs to $H(R)$ for invertible $x \in J_R$.

2.2. We see from (H2bis) and 2.0 that $\mathrm{pr}_2: H \to \boldsymbol{GL}(J)^2$ factors through $\mathbf{Aut}(J, J)$. Since there exists $x \in J$ such that $N(x) \in k^*$ (cf. 1.1($*$)), we see from (H3) that the morphism $H \to \mathbf{Aut}(J, J)$ sending h to (h_+, h_-) is a monomorphism. If $h \in H(R)$ and $(x, y) \in \mathscr{W}(R)$, then

(1) $$\rho(h) = (\rho_+(h), \rho_-(h)) := (\chi(h)h_+, \chi(h)^{-1}h_-)$$

as well as (h_+, h_-) belongs to $\mathbf{Aut}(J, J)(R)$, while

(2) $$b(x, y) := (N(x, y), N(x, y)^{-1}B(x, y), N(x, y)B(y, x)^{-1})$$

belongs to $H(R)$ by 1.2(4), (CJ25), and (CJ21). Varying R, we get a homomorphism $\rho: H \to \mathbf{Aut}(J, J)$ of k-groups and a morphism $b: \mathscr{W} \to H$ of k-schemes. Note that we have

(3) $$\rho_+(h^\vee) = \rho_-(h), \quad \rho_-(h^\vee) = \rho_+(h),$$

(4) $$b(x, y)^\vee = b(y, x)^{-1},$$

by definition and 2.1(2).

2.3 Lemma. *$((J, J), H, \rho, b)$ is a Jordan system and the kernel of $\rho: H \to$ $\mathbf{Aut}(J, J)$ is the functor-image of ${}_2\mu_k \subset \mu_k$ under $z: \mu_k \to H$.*

Proof. We first recall that (cf. [LAG, 5.1]) a *Jordan system* over k is a quadruple (V, H, ρ, b) where (1) $V = (V^\pm, Q_\pm)$ is a Jordan pair with $V^\pm$ finitely generated projective k-modules, (2) H is a separated k-group sheaf, (3) $\rho = (\rho_+, \rho_-)$ is a homomorphism $H \to \mathbf{Aut}(V)$ of k-groups, (4) b is a morphism $\mathscr{W} \to H$, with $\mathscr{W}$ the scheme of quasi-invertible pairs of V, satisfying

(JS1) $$\rho(b(x, y)) = (B_+(x, y), B_-(y, x)^{-1}),$$

(JS2) $$hb(x, y)h^{-1} = b(\rho_+(h)x, \rho_-(h)y),$$

(JS3) $$b(tx, t^{-1}y) = b(x, y),$$

(JS4) $$b(x, y)b(x^y, w) = b(x, y+w),$$

(JS5) $$b(z, y^x)b(x, y) = b(z+x, y),$$

for all $R \in k$-**alg**, $t \in R^*$, $h \in H(R)$, and $x, z \in V_R^+$, y, $w \in V_R^-$ such that (x, y), $(x, y+w)$, $(x+z, y) \in \mathscr{W}(R)$. Here we set, in the notation of 1.5, $B_\sigma(u, v) := \mathrm{Id} - D_\sigma(u, v) + Q_\sigma(u)Q_{-\sigma}(v)$. In our situation, (JS1) follows from the definitions (1) and (2) in 2.2, and (JS2–5) from [LJP, 3.9] and 1.10. As for the last assertion, we have $\rho(\mathbf{z}(t)) = (t^{-2}\mathrm{Id}, t^2\mathrm{Id})$ by 2.1(3) and 2.2(1). Hence $\rho\mathbf{z}$ is trivial on ${}_2\mu_k$ (cf. 0.1). Conversely if $\rho(h) = 1$, then we have $h_+ = t^{-1}\mathrm{Id}$ and $h_- = t\mathrm{Id}$ with $t := \chi(h)$, from which we get $t^2 = 1$ by (H3). This shows $h = \mathbf{z}(t)$ by 2.1(3).

2.4. Denote by (G, ψ) the elementary system associated to the Jordan system $((J, J), H, \rho, b)$ in 2.2 (cf. [LAG, 5.2]). By definition, G is a separated k-group sheaf, ψ is an action $\mu_k \times G \to G$ of μ_k on G, and we have a diagram

$$J_a \overset{\exp_+,\ \exp_-}{\rightrightarrows} G \longleftarrow H$$

of k-group sheaves whose arrows are all monomorphic (cf. [LAG, 3.1, 3.3]). Hence, H can be identified with its image, which coincides with the subgroup sheaf G^ψ, the set of fixed points of G under ψ (cf. [LAG, 4.9]). Denote by U^σ ($\sigma = \pm$) the functor-image of $\exp_\sigma$. Then, H normalizes U^σ and the multiplication $U^+ \times U^- \times H \times U^+ \to G$ is an epimorphism of k-sheaves (cf. [LAG, 3.6, 3.8]). The multiplication $U^- \times H \times U^+ \to G$ is an open immersion (cf. [LAG, 3.4]) whose functor-image Ω is dense in G (cf. [LAG, 3.8]). We have

$$\exp_+(x)\exp_-(y) = \exp_-(y^x)b(x, y)\exp_+(x^y)$$

for $(x, y) \in \mathscr{W}(R)$, $R \in k$-**alg**, and $\mathscr{W} \subset J_a \times J_a$ coincides with the inverse image of $\Omega \subset G$ under the morphism $J_a \times J_a \to G$ sending (x, y) to $\exp_+(x)\exp_-(y)$ (cf. [LAG, 4.1]).

2.5. Consider the k-module

(1) $$M := k \oplus J \oplus k \oplus J =: \left\{ \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \middle| \alpha, \beta \in k,\ a, b \in J \right\},$$

and set

(2) $$\theta_0(h) \cdot \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} := \begin{pmatrix} \chi(h)^{-1}\alpha & h_+ a \\ h_- b & \chi(h)\beta \end{pmatrix},$$

$$(3)\qquad \theta_+(x)\cdot\begin{pmatrix}\alpha & a\\ b & \beta\end{pmatrix}:=\begin{pmatrix}\alpha & a+\alpha x\\ b+a\times x+\alpha x^{\#} & \beta+T(b,x)+T(a,x^{\#})+\alpha N(x)\end{pmatrix},$$

$$(4)\qquad \theta_-(y)\cdot\begin{pmatrix}\alpha & a\\ b & \beta\end{pmatrix}:=\begin{pmatrix}\alpha-T(a,y)+T(b,y^{\#})-\beta N(y) & a-b\times y+\beta y^{\#}\\ b-\beta y & \beta\end{pmatrix},$$

$$(5)\qquad \phi(t)\cdot\begin{pmatrix}\alpha & a\\ b & \beta\end{pmatrix}:=\begin{pmatrix}t^{-1}\alpha & a\\ tb & t^2\beta\end{pmatrix},$$

$$(6)\qquad \varepsilon\cdot\begin{pmatrix}\alpha & a\\ b & \beta\end{pmatrix}:=\begin{pmatrix}\beta & -b\\ a & -\alpha\end{pmatrix},$$

for all $R\in k$-**alg**, $h\in H(R)$, $t\in R^*$, $x, y, a, b\in J_R$, and $\alpha, \beta\in R$. Thus we have

$$(7)\qquad \varepsilon^2=-\mathrm{Id}\,,$$

$$(8)\qquad \varepsilon\theta_+(x)\varepsilon^{-1}=\theta_-(x)\,,$$

$$(9)\qquad \varepsilon\theta_0(h)\varepsilon^{-1}=\theta_0(h^{\vee})\,.$$

By (2), the endomorpshisms $\theta_0(h)$ of the R-module M_R are invertible and $h\mapsto\theta_0(h)$ is a homomorphism. Since $\theta_+(0)=\mathrm{Id}$ and $\theta_+(x)\theta_+(y)=\theta_+(x+y)$ by (3) and (CJ3, 4), it follows, in view of (7) and (8), that the endmorphisms $\theta_\sigma(x)$ $(\sigma=\pm)$ are also invertible and $x\mapsto\theta_\sigma(x)$ are homomorphisms. Varying R, we get a diagram

$$J_a \overset{\theta_+,\,\theta_-}{\rightrightarrows} \mathbf{GL}(M) \overset{\theta_0}{\longleftarrow} H$$

of k-group schemes. Note that we have

$$(10)\qquad (\mathrm{Int}\,\phi(t))\cdot(\theta_0(h))=\theta_0(h)\,,$$

$$(11)\qquad (\mathrm{Int}\,\phi(t))\cdot(\theta_+(x))=\theta_+(tx)\,,$$

$$(12)\qquad (\mathrm{Int}\,\phi(t))\cdot(\theta_-(y))=\theta_-(t^{-1}y)\,,$$

for $t\in R^*$, $h\in H(R)$, $x, y\in J_R$. In addition, we have

$$(13)\qquad \theta_0(-h)=-\theta_0(h)\,,$$

$$(14)\qquad \theta_+(x)\theta_-(x^{-1})\theta_+(x)=-\theta_0(b(x))\varepsilon\,,$$

for invertible $x\in J_R$, by (2), (3), (4), 2.1(4), (5), and straightforward calculation.

2.6 THEOREM. *There exists a unique homomorphism $\theta\colon G\to\mathbf{GL}(M)$ of k-group sheaves extending θ_0, θ_+, and θ_-; moreover, θ is a monomorphism.*

To prove the first assertion, it suffices to verify

$$(1)\qquad (\mathrm{Int}\,\theta_0(h))\cdot(\theta_+(x))=\theta_+(\rho_+(h).x)\,,$$

$$(2)\qquad (\mathrm{Int}\,\theta_0(h))\cdot(\theta_-(y))=\theta_-(\rho_-(h).y)\,,$$

(3) $$\theta_+(x)\theta_-(y)=\theta_-(y^x)\theta_0(b(x,y))\theta_+(x^y),$$

for all $R\in k$-**alg**, $h\in H(R)$, $(x,y)\in\mathscr{W}(R)$ (cf. [LAG, 4.14]). Direct calculation using (H1, 2bis, 3) shows (1) and, in view of 2.5(8), (9) and 2.2(3), we see that (2) follows from (1). We prove (3) in 2.8 after introducing some formulas. We now prove the last assertion of the theorem on which the following 2.7 and 2.8 do not depend. Let $g\in G(R)$, $R\in k$-**alg**, such that $\theta(g)=\mathrm{Id}$. Then there exists an fppf extension S of R and $x, y, z\in J_S$, $h\in H(S)$ such that

$$g_S=\exp_+(x)\exp_-(y)\exp_+(z)h$$

(cf. 2.4). Hence we have $\mathrm{Id}=\theta(g_S)=\theta_+(x)\theta_-(y)\theta_+(z)\theta_0(h)$. In particular

$$\theta_+(x)\theta_-(y)\theta_+(z)\cdot\begin{pmatrix}0&0\\0&1\end{pmatrix}=\begin{pmatrix}-N(y)&y^\#-N(y)x\\-P(y,x)&N(y,x)\end{pmatrix}$$

and

$$\theta_0(h)^{-1}\cdot\begin{pmatrix}0&0\\0&1\end{pmatrix}=\begin{pmatrix}0&0\\0&\chi(h)^{-1}\end{pmatrix}$$

are equal and we get $N(y)=0$, $y^\#-N(y)x=0$, i.e., $y^\#=0$, and $P(x,y)=0$ (cf. 1.1(4)), i.e., $y=0$, successively. Hence $g_S=\exp_+(x+z)h$. Thus we have $\mathrm{Id}=\theta(g_S)=\theta_+(x+z)\theta_0(h)$. In particular

$$\theta_+(x+z)\cdot\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}1&x+z\\(x+z)^\#&1+N(x+z)\end{pmatrix}$$

and

$$\theta_0(h)^{-1}\cdot\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}\chi(h)&0\\0&\chi(h)^{-1}\end{pmatrix}$$

are equal and we get $x+z=0$. Hence $g_S=h$. Now $\mathrm{Id}=\theta(g_S)=\theta_0(h)$ implies $h=1$ by 2.5 (2) and, since $R\to S$ is fppf and G is a sheaf, $g_S=h=1$ implies $g=1$. This shows the last assertion.

2.7. For the proof of 2.6(3), we introduce some formulas. For any $m\in M$ with entries α, β, a, b (cf. 2.5(1)), we define polynomial laws $m^\delta\in\mathcal{O}(J)$, *and* $m^\vee\in\mathcal{O}(J,J)$ by setting

(1) $$\begin{pmatrix}\alpha&a\\b&\beta\end{pmatrix}^\delta(w):=\alpha-T(a,w)+T(b,w^\#)-\beta N(w),$$

(2) $$\begin{pmatrix}\alpha&a\\b&\beta\end{pmatrix}^\vee(w):=a-b\times w+\beta w^\#,$$

for all $w\in J_R$, $R\in k$-**alg** (cf. 0.4).

LEMMA. *The map $M \to \mathcal{O}(J) \times \mathcal{O}(J, J)$ sending m to $(m^\delta, m^\vee)$ is injective k-linear, and the following formulas hold for all $m \in M$, $h \in H(k)$, $x, y \in J$, $t \in k^*$ and $w \in J_R$, $R \in$ k-**alg** such that (w, x) is quasi-invertible*:

(3) $$(\theta_-(y) \cdot m)^\delta(w) = m^\delta(y + w)\,,$$

(4) $$(\theta_-(y) \cdot m)^\vee(w) = m^\vee(y + w)\,,$$

(5) $$(\phi(t) \cdot m)^\delta(w) = t^{-1} m^\delta(tw)\,,$$

(6) $$(\phi(t) \cdot m)^\vee(w) = m^\vee(tw)\,,$$

(7) $$(\theta_0(h) \cdot m)^\delta(w) = \chi(h)^{-1} m^\delta(\rho_-(h)^{-1} w)\,,$$

(8) $$(\theta_0(h) \cdot m)^\vee(w) = h_+ m^\vee(\rho_-(h)^{-1} w)\,,$$

(9) $$(\theta_+(x) \cdot m)^\delta(w) = N(x, w) m^\delta(w^x)\,,$$

(10) $$(\theta_+(x) \cdot m)^\vee(w) = (\theta_+(x) \cdot m)^\delta(w) x^w + N(x, w) B(x, w)^{-1} m^\vee(w^x)\,.$$

PROOF. $m \mapsto (m^\delta, m^\vee)$ is k-linear by the definitions (1) and (2). To show the injectivity, let $m^\delta = 0$ and $m^\vee = 0$ for $m \in M$ with entries α, β, a, b. Then we have, by (1) and (2), $\alpha = m^\delta(0) = 0$ and $a = m^\vee(0) = 0$. Moreover, if t is a variable over k and $w \in J$, then $\beta N(w) \in k$ is the coefficient of t^3 in $m^\delta(tw) \in k[t]$ and there exists $w \in J$ such that $N(w) \in k^*$ (cf. 1.1($*$)). This shows $\beta = 0$. There remain relations $T(b, w^\#) = 0$ and $b \times w = 0$, which yield $b = 0$, since $0 = (b \times w) \times w^\# = N(w)b + T(w^\#, b)w = N(w)b$ by (CJ6). Thus we get $m = 0$, which shows the injectivity. Let us show the latter half of the proposition. By the definitions (1), (2) and 2.5(4), we have

$$\theta_-(w) \cdot m = \begin{pmatrix} m^\delta(w) & m^\vee \\ b - \beta w & \beta \end{pmatrix}$$

(notation as in (1), (2)). Hence (3) and (4) follows from the fact that θ_- is a homomorphism (cf. 2.5). On the other hand, (5) and (6) follow from

$$\theta_-(w)\phi(t) \cdot m = \phi(t)\theta_-(tw) \cdot m = \begin{pmatrix} t^{-1} m^\delta(w) & m^\vee(tw) \\ t(b - \beta tw) & t^2 \beta \end{pmatrix}$$

(cf. 2.5(5), (12)). Moreover we have, by (1), (2) and 2.5(2),

$$(\theta_0(h) \cdot m)^\delta(w) = \chi(h)^{-1}\alpha - T(h_+ a, w) + T(h_- b, w^\#) - \chi(h)\beta N(w)\,,$$

$$(\theta_0(h) \cdot m)^\vee(w) = h_+ a - (h_- b) \times w + \chi(h)\beta w^\#\,,$$

from which (7) and (8) follow, in view of 2.1(H1), (H2bis), (H3). Finally, we have

(9bis) $$(\theta_+(x) \cdot m)^\delta(w) = \alpha N(x, w) - T(a, P(w, x)) + T(b, w^\# - N(w)x) - \beta N(w)$$

and

(10bis) $$(\theta_+(x) \cdot m)^\vee(w) = \alpha P(x, w) + a - (a \times x) \times w + T(a, x^\#) w^\# - b \times w + T(b, x) w^\# + \beta w^\#$$

by (1), (2), 2.5(3) and (CJ6). Thus (9) follows from (9bis) and (CJ22, 23, 24). (10) acted on by $B(x, w)$ becomes

$$B(x, w)(\theta_+(x)\cdot m)^\nu(w)=(\theta_+(x)\cdot m)^\delta(w)(x-Q(x)w)+N(x, w)m^\nu(w^x),$$

which we prove by acting $B(x, w)$ on (10bis) and by using 1.9(1), (2), (3), (5), (CJ22, 23, 24) and the above (9bis).

2.8 Verification of 2.6(3). Now we show the identity $\theta_-(y)\theta_+(x)=\theta_+(x^y)\theta_0(b(x, y)^{-1})\theta_-(y^x)$ which becomes 2.6(3) after we take inverses and replace (x, y) by $(-x, -y)$. After taking scalar extension and applying the first part of the lemma in 2.7, we are reduced to verifying the equalities of *polynomial laws*

(1) $$(\theta_-(y)\theta_+(x)\cdot m)^\delta=(\theta_+(x^y)\theta_0(b(x, y)^{-1})\theta_-(y^x)\cdot m)^\delta,$$

(2) $$(\theta_-(y)\theta_+(x)\cdot m)^\nu=(\theta_+(x^y)\theta_0(b(x, y)^{-1})\theta_-(y^x)\cdot m)^\nu,$$

for arbitrary $(x, y)\in\mathscr{W}(k)$ and $m\in M$. For this, it suffices to verify the equalities of the values at $w\in J_R$, $R\in k$-**alg** such that (w, x^y) is quasi-invertible, since such w's form a dense subscheme of J_a. Then the following calculations work:

$$\begin{aligned}
&(\theta_+(x^y)\theta_0(b(x, y)^{-1})\theta_-(y^x)\cdot m)^\delta(w)\\
&\quad=N(x^y, w)(\theta_0(b(x, y)^{-1})\theta_-(y^x)\cdot m)^\delta(w^{(x^y)}) \quad \text{(by 2.7(9))}\\
&\quad=N(x^y, w)N(x, y)(\theta_-(y^x)\cdot m)^\delta(B(y, x)^{-1}(w^{(x^y)})) \quad \text{(by 2.7(7))}\\
&\quad=N(x, y+w)m^\delta(y^x+B(y, x)^{-1}(w^{(x^y)})) \quad \text{(by 2.7(3), 1.9(3))}\\
&\quad=N(x, y+w)m^\delta((y+w)^x) \quad \text{(by [LJP, 3.7 (2)])}\\
&\quad=(\theta_+(x)\cdot m)^\delta(y+w) \quad \text{(by 2.7(9))}\\
&\quad=(\theta_-(y)\theta_+(x)\cdot m)^\delta(w) \quad \text{(by 2.7(3))},
\end{aligned}$$

from which (1) follows. Moreover

$$\begin{aligned}
&(\theta_+(x^y)\theta_0(b(x, y)^{-1})\theta_-(y^x)\cdot m)^\nu(w)\\
&\quad=(\theta_+(x^y)\theta_0(b(x, y)^{-1})\theta_-(y^x)\cdot m)^\delta(w)(x^y)^w\\
&\qquad+N(x^y, w)B(x^y, w)^{-1}(\theta_0(b(x, y)^{-1})\theta_-(y^x)\cdot m)^\nu(w^{(x^y)}) \quad \text{(by 2.7(10))}\\
&\quad=(\theta_-(y)\theta_+(x)\cdot m)^\delta(w)x^{y+w}\\
&\qquad+N(x^y, w)B(x^y, w)^{-1}N(x, y)B(x, y)^{-1}(\theta_-(y^x)\cdot m)^\nu(B(y, x)^{-1}w^{(x^y)})\\
&\qquad\quad\text{(by (1) above, 2.7(8), [LJP, 3.7 (1)])}\\
&\quad=(\theta_+(x)\cdot m)^\delta(y+w)x^{y+w}+N(x, y+w)B(x, y+w)^{-1}m^\nu((y+w)^x)\\
&\qquad\quad\text{(by 2.7(3), (4), 1.10(3), [LJP, 3.6 (JP33), 3.7 (2)])}\\
&\quad=(\theta_+(x)\cdot m)^\nu(y+w) \quad \text{(by 2.7(10))}\\
&\quad=(\theta_-(y)\theta_+(x)\cdot m)^\nu(w) \quad \text{(by 2.7(4))},
\end{aligned}$$

from which (2) follows. This completes the verification of 2.6(3).

3. Stabilizer.

3.0. We keep the notation in §2. The representation θ induces linear and projective representations

$$\mu_k \times G \to \mathbf{GL}(M)$$

and

$$G \to \mathbf{Aut}(\boldsymbol{P}(M)),$$

respectively. Note that, if $m \in M_m(k)$ (cf. 0.3), then $\mathrm{pr}_2 : \mu_k \times G \to G$ induces an *isomorphism*

$$\mathbf{Cent}_{\mu_k \times G}(m) \xrightarrow{\sim} \mathbf{Cent}_G(p_M(m)),$$

since M_m is a μ_k-torsor with structure morphism $p_M : M_m \to \boldsymbol{P}(M)$. We consider the element

$$u_0 := p_M(m_0) \in \boldsymbol{P}(M)(k), \quad \text{where} \quad m_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The purpose of this section is to determine the stabilizer of u_0 in G, which is canonically isomorphic to that of m_0 in $\mu_k \times G$.

3.1. We first introduce a notational convention. Recall that θ is a monomorphism (cf. 2.6) and the image of $G(k)$ under $\theta(k)$ contains $\varepsilon \in GL(M)$ (cf. 2.5(14)). We regard ε as an element of $G(k)$ via θ. Thus we have

(1) $$\varepsilon^2 = -1,$$

(2) $$\varepsilon h \varepsilon^{-1} = h^\vee,$$

(3) $$\exp_+(x)\exp_-(x^{-1})\exp_+(x) = -b(x)\varepsilon,$$

for $h \in H(R)$, and invertible $x \in J_R$, $R \in k$-**alg** (cf. 2.5(7), (9), (13), (14)).

3.2. Let $H' \subset H$ be the kernel of the character $h \mapsto \chi(h)^4$ of H (cf. 2.1). If $h = (\chi(h), h_+, h_-) \in H'(R)$, $R \in k$-**alg**, then

$$s(h) := (\lambda, \lambda^2 h_-, \lambda^2 h_+) \quad \text{with} \quad \lambda := \chi(h)$$

belongs to $H'(R)$. Indeed, since $\lambda^4 = 1$, we have $\boldsymbol{z}(\lambda^2) = (\lambda^2, \lambda^2 \mathrm{Id}, \lambda^2 \mathrm{Id})$ (cf. 2.1(3)), and hence $s(h) = \boldsymbol{z}(\lambda^2) h^\vee \in H(R)$ (cf. 2.1(2)). In particular, we have

(1) $$\chi(s(h)) = \chi(h),$$

from which $s(h) \in H'(R)$ follows. Thus we get an *automorphism* $s : h \mapsto s(h)$ of the k-group H' of period two. Let the constant k-group $(\boldsymbol{Z}/2\boldsymbol{Z})_k$ act on H' via s (cf. [D-G, II, §1, 3.3 a)]), and construct the semi-direct product $H' \times_s (\boldsymbol{Z}/2\boldsymbol{Z})_k$. Hence we have

(2) $$(h, f) \cdot (h', f') = (h s_f(h'), f * f'),$$

for all $h, h' \in H'(R)$, and $f, f' \in (\mathbf{Z}/2\mathbf{Z})_k(R)$, $R \in k$-**alg**, where we regard $(\mathbf{Z}/2\mathbf{Z})_k(R)$ as the group of idempotents in R with operation $f * f' := f + f' - 2ff'$ (cf. [D-G, III, §5, 2.4]), and $s_f(h') \in H'(R)$ is the element corresponding to $(h', s(h')) \in H'(R_{1-f}) \times H'(R_f)$ under the decomposition $R \simeq R_{1-f} \times R_f$ of R with respect to the idempotent f. Therefore, in view of (1),

$$(h, f) \mapsto \chi(h): H' \times_s (\mathbf{Z}/2\mathbf{Z})_k \longrightarrow {}_4\mu_k$$

is a character. Moreover, since the morphism $(\mathbf{Z}/2\mathbf{Z})_k \to {}_2\mu_k$ sending f to $1-2f$ is also a character, we can define a character $\chi': H' \times_s (\mathbf{Z}/2\mathbf{Z})_k \to {}_2\mu_k$ by setting

(3) $$\chi'(h, f) := \chi(h)^2(1-2f),$$

for $h \in H'(R)$ and $f \in (\mathbf{Z}/2\mathbf{Z})_k(R)$, $R \in k$-**alg**. Let $H'' \subset H' \times_s (\mathbf{Z}/2\mathbf{Z})_k$ be the kernel of χ'. Hence we have

(4) $$H''(R) = \{(h, f) \in H(R) \times R \mid f^2 = f, \chi(h)^2 = 1 - 2f\},$$

for all $R \in k$-**alg**. For any $(h, f) \in H''(R)$, $R \in k$-**alg**, define $\boldsymbol{f}(h, f) \in G(R)$ to be the element with components $(h, h\varepsilon) \in G(R_{1-f}) \times G(R_f)$ under the decomposition $R \simeq R_{1-f} \times R_f$. Varying R, we get a morphism

$$\boldsymbol{f}: H'' \to G$$

of k-sheaves.

3.3. Theorem. *$\boldsymbol{f}$ is a homomorphism of k-group sheaves and factors into the composite*

$$\boldsymbol{f}: H'' \xrightarrow{\sim} \mathbf{Cent}_G(u_0) \xrightarrow{\text{incl.}} G,$$

whose first arrow is an isomorphism.

3.4. First, we show that $\boldsymbol{f}$ is a homomorphsim. Consider (h, f), $(h', f') \in H'(R)$, $R \in k$-**alg**, and describe any element in $G(R)$ in terms of four components with respect to the decomposition

(1) $$R \simeq R_{(1-f)(1-f')} \times R_{(1-f)f'} \times R_{f(1-f')} \times R_{ff'}$$

of R. Then we have

$$\boldsymbol{f}(h, f) = (h, h, h\varepsilon, h\varepsilon) \quad \text{and} \quad \boldsymbol{f}(h', f') = (h', h'\varepsilon, h', h'\varepsilon)$$

by definition, so that we have

(2) $$\boldsymbol{f}(h, f)\boldsymbol{f}(h', f') = (hh', hh'\varepsilon, h(h')^\vee\varepsilon, -h(h')^\vee)$$

by 3.1(1), (2). On the other hand, we have

(3) $$\boldsymbol{f}((h, f)(h', f')) = (hs_f(h'), hs_f(h')\varepsilon, hs_f(h')\varepsilon, hs_f(h'))$$

by 3.2(2). However, by 3.2(4) and the formula $s(h)=z(\chi(h)^2)h^\vee$, the components of $s_f(h')\in H'(R)$ with respect to $R\simeq R_{1-f}\times R_f$ is $(h', s(h'))=(h', z(1-2f')h'^\vee)\in H'(R_{1-f})\times H'(R_f)$, and that of $1-2f'\in R$ with respect to $R\simeq R_{1-f'}\times R'_f$ is $(1, -1)$. Thus we have

$$s_f(h')=(h', h', (h')^\vee, -(h')^\vee) \tag{4}$$

with respect to (1). Hence, by (2), (3), and (4), we get $\boldsymbol{f}((h, f)(h', f'))=\boldsymbol{f}(h, f)\boldsymbol{f}(h', f')$.

3.5. Next, we show that $\boldsymbol{f}$ is a *monomorphism*. Consider $(h, f)\in H''(R)$, $R\in k$-**alg** such that $\boldsymbol{f}(h, f)=1_{G(R)}$. Then we have $h=1$ in $G(R_{1-f})$ and $h\varepsilon=1$ in $G(R_f)$. Hence we have $h=-\varepsilon$ in $G(R_f)$ (cf. 3.1(1)), which yields $h=1$ and $-\varepsilon=1$ in $G(R_f)$, since $H\cap U^+U^-U^+$ is trivial (cf. [LAG, 3.6 (c)]). In particular, we have $h=1$ in $H(R)$. Moreover, in view of 2.5(6), $-\varepsilon=1$ occurs only when $R_f=0$. Thus we have $f=0$.

3.6. For any $(h, f)\in H''(R)$, $R\in k$-**alg**, we have

$$\theta(\boldsymbol{f}(h, f))\cdot\begin{pmatrix}1&0\\0&1\end{pmatrix}=\chi(h)^{-1}\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

Indeed, since $\boldsymbol{f}(h, f)=(h, h\varepsilon)$ and $\chi(h)^2=1-2f=(1, -1)$ with respect to the decomposition $R\simeq R_{1-f}\times R_f$ (cf. 3.2(3), (4)), we have

$$\theta(\boldsymbol{f}(h, f))\cdot\begin{pmatrix}1&0\\0&1\end{pmatrix}_{R_{1-f}}=\theta_0(h)\cdot\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}\chi(h)^{-1}&0\\0&\chi(h)\end{pmatrix}=\chi(h)^{-1}\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

and

$$\theta(\boldsymbol{f}(h, f))\cdot\begin{pmatrix}1&0\\0&1\end{pmatrix}_{R_f}=\theta_0(h\varepsilon)\cdot\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}\chi(h)^{-1}&0\\0&-\chi(h)\end{pmatrix}=\chi(h)^{-1}\begin{pmatrix}1&0\\0&1\end{pmatrix},$$

from which the assertion follows. Thus the morphism $\boldsymbol{f}\colon H''\to G$ factors through $\mathbf{Cent}_G(u_0)$. The resulting morphism $\boldsymbol{f}'\colon H''\to\mathbf{Cent}_G(u_0)$ is a *monomorphism*, since so is $\boldsymbol{f}$ (cf. 3.5). To complete the proof of Theorem 3.3, it remains to show that $\boldsymbol{f}'$ is an *epimorphism* (cf. [D-G, III, §1, 2.1]). In view of 2.4, the question is reduced to the following lemma:

3.7 LEMMA. *Let $R\in k$-**alg**, $v\in R^*$, $x, y, z\in J_R$ and $h\in H(R)$ such that*

$$v\theta_+(x)\theta_-(y)\theta_+(z)\theta_0(h)\cdot\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}. \tag{1}$$

Then, there exists an idempotent $f\in R$ and an element $h'\in H(R)$ with the following properties:

(i) $\chi(h')^2=1-2f$,

(ii) $\chi(h')=v$,

(iii) *the components of $g:=\exp_+(x)\exp_-(y)\exp_+(z)h\in G(R)$ with respect to the decomposition $R\simeq R_{1-f}\times R_f$ are $(h', h'\varepsilon)\in G(R_{1-f})\times G(R_f)$.*

PROOF. We define α, $t \in R$ and $a \in J_R$, depending on (v, x, y, z, h), by

(2) $$t := \chi(h)^2\,,$$

(3) $$\alpha := N(z, y) - tN(y)\,,$$

(4) $$a := z - z^{\#} \times y + (N(z) + t)y^{\#}\,.$$

Then direct calculation shows that (1) is equivalent to the four conditions:

(5) $$\alpha = \chi(h)v^{-1}\,,$$

(6) $$\alpha^3 = t^2 N(y)(T(z, y) - z) - tN(z, y)(T(z, y) - 1)\,,$$

(7) $$y = Q(y)z\,,$$

(8) $$x = -\alpha^{-1}a\,.$$

By (2), (3) and (5), we have

(9) $$\chi(h)^{-1}vN(z, y) - \chi(h)vN(y) = 1\,.$$

Moreover, by (7), we have

(10) $$N(z, y)N(y) = 0\,.$$

Indeed, the left-hand side equals $N(y) - N(y)^2 N(z) - T(y, z)N(y) + T(y^{\#}, z^{\#})N(y)$ (cf. 1.1(4)). Acting $N(?)$ and $Q(y^{\#})$ on (7) with (CJ20, 1, 5, 14) in mind, we get $N(y) = N(y)^2 N(z)$ and $N(y)y^{\#} = N(y)^2 z$. Hence we have $T(y, z)N(y) = T(y, z)N(y)^2 N(z) = T(y, N(y)y^{\#})N(z) = 3N(y)^2 N(z) = 3N(y)$, and $T(y^{\#}, z^{\#})N(y) = T(N(y)^2 z, z^{\#}) = 3N(y)^2 N(z) = 3N(y)$ (cf. (CJ5)). Thus (10) holds. By (9) and (10), the element

$$f := -\chi(h)vN(y)$$

of R is an idempotent. Since y becomes invertible after the scalar extension $R \to R_f$ (cf. 1.6), we can define $h' \in H(R)$ to be the element with components $(h, -b(y^{-1})h^{\vee}) \in H(R_{1-f}) \times H(R_f)$ (cf. 2.1) with respect to the decomposition $R \simeq R_{1-f} \times R_f$. We claim that (f, h') is what we want. Namely:

(a) After the scalar extension $R \to R_{1-f}$, we have $\chi(h')^2 = 1 - 2f$, $\chi(h') = v$, and $g = h'$. Indeed, (y, z) becomes quasi-invertible by (9) and 1.7, and we have $B(y, z)y = B(y, z)Q(y)z = Q(y - Q(y)z) = 0$ by (7) and [LJP, 2.11 (JP23)], from which we get $y = 0$. Then we have $1 = \chi(h)v^{-1}$ (by (3), (5)), $1 = \chi(h)^2$ (by (2), (3), (6)), $x = -z$ (by (3), (4), (8)), and $g = \exp_+(x)\exp_-(y)\exp_+(z)h = h$. Thus the assertion follows, since we have $h' = h$ and $1 - 2f = (1 - 2f)(1 - f)/(1 - f) = 1$ after our scalar extension.

(b) After the scalar extension $R \to R_f$, we have $\chi(h')^2 = 1 - 2f$, $\chi(h') = v$, and $g = h'\varepsilon$. Indeed, y becomes invertible by 1.6, and we have $z = y^{-1}$ by (7), from which we get $N(z, y) = 0$, $T(z, y) = 3$, $z^{\#} \times y = 2z$, and $y^{\#} = N(z)^{-1}z$, in view of 1.6(2), (CJ5), and (CJ1). Then we have $v = -N(y)^{-1}\chi(h)^{-1}$ (by (2), (3), (5)), $N(y)^2\chi(h)^2 = -1$ (by (2), (3), (6)), $x = y^{-1}$ (by (2), (3), (4), (8)), and $g = \exp_+(x)\exp_-(y)\exp_+(z)h = \exp_+(x)\exp_-(x^{-1})\exp_+(x)h =$

$-b(x)\varepsilon h = -b(y^{-1})h^{\vee}\varepsilon$ (by 3.1(2), (3)). Thus the assertion follows, since we have $h' = -b(y^{-1})h^{\vee}$, $1-2f=(1-2f)f/f=-1$, $\chi(h') = -N(y)^{-1}\chi(h)^{-1}$ (by 2.1(1), (4)) after our scalar extension.

4. Freudenthal quartic and transitivity.

4.0. Recall that $(J; N, \#, T)$ is a quadruple as in 1.1, that G is the k-group sheaf defined in 2.4, and that M is the k-module $k \oplus J \oplus k \oplus J$ (cf. 2.5), on which G acts via the representation θ defined in 2.6. Recall also that we have two vector subgroups U^{σ} $(\sigma = \pm)$ of G together with isomorphisms $J_a \simeq U^{\sigma}$ (cf. 2.4), and that the composite $J_a \simeq U^{\sigma} \subset G \xrightarrow{\theta} \boldsymbol{GL}(M)$ coincides with $\theta_{\sigma}\colon J_a \to \boldsymbol{GL}(M)$ described in 2.5(3), (4). The purpose of this section is to show that the projective representation $G \to \mathbf{Aut}(\boldsymbol{P}(M))$ associated to θ admits a G-stable open subscheme $Y \subset \boldsymbol{P}(M)$ such that, under some condition on $(J; N, \#, T)$, the action of $G(K)$ on the set $Y(K)$ is *transitive* for any algebraically closed field $K \in k$-**alg** (cf. 4.4, Corollary 1). Y is the principal open subscheme defined by the "Freudenthal quartic" (cf. 4.1).

4.1. Consider the quartic form (cf. 0.4) $f \in \mathcal{O}^4(M)$ and the alternating form $\{\ ,\ \} \in {}^t(\bigwedge^2 M)$ such that

$$(1)\qquad f\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} := (T(a,b) - \alpha\beta)^2 + 4N(a)\beta + 4N(b)\alpha - 4T(a^{\#}, b^{\#}),$$

$$(2)\qquad \left\{ \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}, \begin{pmatrix} \alpha' & a' \\ b' & \beta' \end{pmatrix} \right\} := T(a, b') - T(b, a') + \beta\alpha' - \alpha\beta',$$

for all $\alpha, \alpha', \beta, \beta' \in R$, and $a, b, a', b' \in J_R$, $R \in k$-**alg**. By calculation, we see that G *stabilizes* f *and* $\{\ ,\ \}$. Denote by $(M_a)_f$ (resp. $D_+(f)$) the principal open subscheme of M_a (resp. $\boldsymbol{P}(M)$) defined by the section $f \in \mathcal{O}^4(M)$, and define subschemes $(M_a)_f^+$, $(M_a)_f^{++}$ of $(M_a)_f$ by

$$(M_a)_f^+(R) := \left\{ \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \in (M_a)_f(R) \,\middle|\, \alpha \in R^* \right\},$$

$$(M_a)_f^{++}(R) := \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \,\middle|\, \alpha, \beta \in R^* \right\},$$

for $R \in k$-**alg**. Since G stabilizes f, the subscheme $(M_a)_f$ is stable under G, and so is $D_+(f)$.

4.2 Proposition. *If $K \in k$-**alg** is an algebraically closed field, then we have*

$$(M_a)_f^+(K) = U^+(K)U^-(K) \cdot (M_a)_f^{++}(K).$$

This follows from the same argument as in [Igusa, pp. 427–428].

4.3. Consider the following condition on a quadruple $(J; N, \#, T)$:

(**) For any field $K \in k$-**alg** of characteristic different from two, the symmetric bilinear form $(x, y) \mapsto T(x, y)$ on J_K is non-degenerate.

PROPOSITION. *Under the assumption* $(**)$, *we have*

$$(M_a)_f(K)=U^-(K)\cdot(M_a)_f^+(K),$$

for any infinite field $K\in k$-**alg**.

PROOF. In view of 2.5(4) and 2.7(1), it suffices to show that for any $m\in(M_a)_f(K)$ there exists $w\in J_K$ such that $m^\delta(w)\in K^*$. Then we are reduced to showing that the *polynomial law* $m^\delta\in\mathcal{O}(J_K)$ (cf. 0.4) is not zero, since K is an infinite field (cf. [Bou, IV, §2, no 3, Cor. 2 of Prop. 9]). In general, we have

$$(1)\qquad \{m\in M\,|\,m^\delta=0\}=\left\{\begin{pmatrix}0 & a\\ b & 0\end{pmatrix}\middle|\, a, b\in J,\ T(a,?)=T(b,?)=0\in\mathcal{O}^1(J)\right\}.$$

Indeed, the left-hand side contains the right-hand side by the definition 2.7(1). To see the converse, let $m^\delta=0$ for $m\in M$ with entries α, β, a, b. Then, equating the homogeneous components of the polynomial m^δ to zero, we get $\alpha=0$, $T(a,?)=0\in\mathcal{O}^1(J)$, $T(b,?^\#)=0\in\mathcal{O}^2(J)$, and $\beta N=0\in\mathcal{O}^3(J)$. However this also implies $T(b,?)=0$ and $\beta=0$, since the morphism $?^\#$ is *scheme-theoretically dominant* (cf. 1.3(b)) and there exists $c_1\in J$ such that $N(c_1)\in k^*$ (cf. 1.1(*)). This shows (1). Now apply (1) after the scalar extension $k\to K$. If $\mathrm{char}(K)\neq 2$, the right-hand side of (1) is $\{0\}$ by our assumption $(**)$, and if $\mathrm{char}\,K=2$, we have $f(m)=0$ for all m in the right-hand side of (1). In all cases, we have $\{m\in M_K\,|\,m^\delta=0\}=\{m\in M_K\,|\,f(m)=0\}$, i.e., $m^\delta\in\mathcal{O}(J_K)$ is not zero if $m\in(M_a)_f(K)$.

4.4. Let us assume the condition $(**)$ in 4.3.

COROLLARY 1. *For any algebraically closed field* $K\in k$-**alg**, *the action of* $G(K)$ *on* $D_+(f)(K)$ *is transitive.*

In view of the canonical bijection $\{x\in M_K\,|\,f(x)\in K^*\}/K^*\ \tilde{\to}\ D_+(f)(K)$, this follows from:

COROLLARY 2. *For any algebraically closed field* $K\in k$-**alg**, *the set* $(M_a)_f(K)$ *is a single orbit under* $K^*\times G(K)$.

Indeed, if $m, m'\in(M_a)_f(K)$, there exists $t\in K^*$ such that $t^4=f(m)^{-1}f(m')$, since K is algebraically closed. For such t, we have $f(tm)=f(m')$, since f is a quartic form (cf. 0.4). Now the assertion follows from:

COROLLARY 3. *For any algebraically closed field* $K\in k$-**alg** *and* $i\in K^*$, *the set* $\{m\in M_K\,|\,f(m)=i\}$ *is a single orbit under* $G(K)$.

PROOF. By 4.2 and 4.3, we are reduced to verifying that two elements

$$m=\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\quad\text{and}\quad m'=\begin{pmatrix}\alpha' & 0\\ 0 & \beta'\end{pmatrix}$$

of M_K are conjugate under $G(K)$ if $(\alpha\beta)^2=(\alpha'\beta')^2$, or, in view of the action of ε, if $\alpha\beta=\alpha'\beta'$. Since $\chi\colon H\to\mu_k$ is an *epimorphism* of k-sheaves (cf. 2.1), there exists

$h \in H(K)$ such that $\chi(h) = \beta'^{-1}\beta$. For such h, we have $\theta_0(h) \cdot m = m'$ (cf. 2.5(2)).

This corollary was proved by Igusa in [Igusa, p. 428] in the case where $\mathrm{char}(K) \neq 2, 3$.

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