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SOME PROBABILITY INEQUALITIES FOR ORDERED MTP_2 RANDOM VARIABLES: A PROOF OF THE SIMES CONJECTURE

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Some new probability inequalities involving the ordered components of an MTP_2 random vector are derived, which provide an analytical proof of an important conjecture in the field of multiple hypothesis testing. This conjecture has been mostly validated so far using simulation.

1. Introduction. This paper is motivated by the need to have a theoretical basis for a probability inequality which is fundamental to a number of recently devised multiple hypothesis testing procedures and has been mostly validated using simulation. This inequality, which concerns ordered values of a set of dependent continuous random variables, was originally used by Simes (1986) to improve upon the classical Bonferroni method.

Suppose that there are n hypotheses $H_1, \dots, H_n$ with the corresponding observed p-values $p_1, \dots, p_n$. Then, for the purpose of testing the overall hypothesis $H_0 = \bigcap_{i=1}^n H_i$ at a specified significance level α in terms of these p-values, Simes (1986) proposed a modification of the Bonferroni procedure by suggesting the rejection of H_0 if $p_{(i)} \le i\alpha/n$ for at least one i, where $p_{(1)} \le \cdots \le p_{(n)}$ are the ordered values of $p_1, \dots, p_n$.

Clearly, this method is more powerful than the Bonferroni method which rejects H_0 if $p_{(1)} \le \alpha/n$, and, as Simes noted using simulation, this power improvement is quite significant when the underlying test statistics are highly positively correlated. Having had strong empirical evidence of its superiority over the Bonferroni method, researchers have begun using it in improving tests that rely on the Bonferroni method. Hochberg (1988), Hochberg and Rom (1995), Hommel (1988, 1989), and Rom (1990) adopted this method in devising some step-up multiple testing procedures for making statements about the individual hypotheses H_i once H_0 is rejected. Being based on the Simes method, these procedures are more powerful than Holm's (1979) step-down procedure relying on the Bonferroni method and control the Type I error rate in situations where the Simes method does. But, except for some special cases, the conservativeness of the Simes method is yet to be theoretically verified. Simes (1986) proved that his method is conservative; that is, with $P_{(i)}$ representing the random variable corresponding to $p_{(i)}$, $i = 1, \dots, n$, the following inequality:

$$P_{H_0}\{P_{(i)} \ge i\alpha/n,\ i = 1, \dots, n\} \ge 1 - \alpha \tag{1.1}$$

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holds if under H_0 the underlying test statistics are iid with a continuous distribution, and conjectured, based on extensive simulation, that this may hold for dependent statistics with a variety of multivariate distributions with common marginals.

In attempting to prove this conjecture, Sarkar and Chang (1997) considered a class of exchangeable positively dependent multivariate distributions, and showed that this inequality is indeed true for these distributions. This is a multivariate generalization of Hochberg and Rom (1995) and Samuel-Cahn (1996) who considered only the bivariate case. Hochberg and Rom (1995) proved the conjecture for bivariate distributions with the totally positive of order two (TP_2) property (to be defined later), while Samuel-Cahn (1996) considered the bivariate normal distribution with positive correlation and the absolute-valued bivariate normal distribution. Although Samuel-Cahn's results follow from that of Hochberg and Rom, because the distributions considered by her are TP_2, she actually derived more general results for these distributions, providing certain montonicity properties of the Type I error rate of the Simes test with respect to the correlation. It has also been noted by these authors that the conjecture is not true for negatively dependent test statistics.

The class considered by Sarkar and Chang (1997) contains many continuous multivariate distributions commonly encountered in multiple hypothesis testing situations, for example, the equicorrelated and the absolute-valued equicorrelated multivariate normals, certain types of multivariate t, F, and gamma distributions, and includes the ones Simes (1986) used in his simulation study. Nevertheless, there are still a variety of other useful multivariate distributions for which the conjecture has yet to be proved. The main goal of this paper is to establish theoretically the conjecture for a much larger class of multivariate distributions.

The aforementioned results seem to suggest that the Simes conjecture probably holds only for positively dependent test statistics. A general class of positively dependent multivariate distributions is characterized by the multivariate totally positive of order two (MTP_2) condition, a natural multivariate extension of the TP_2 condition [Karlin and Rinott (1980)]. This is what we consider in this paper, and we prove that the conjecture is true for these distributions. There are some multivariate distributions arising in multiple testing situations which are not MTP_2 but are certain scale mixtures of MTP_2 distributions. An important subclass of these distributions contains the central multivariate t of Dunnett and Sobel (1954) type with the associated correlation matrix having a common and nonnegative correlation, and also contains the absolute-valued multivariate t of this type with any common correlation. We establish the conjecture also for that subgroup, thereby giving a theoretical proof of the Simes inequality for these multivariate t and absolute-valued multivariate t distributions, which was numerically verified in Sarkar and Chang (1997). Karlin and Rinott (1980) also introduced the strongly multivariate reverse rule of order two ($S - MRR_2$) condition that defines negatively dependent multivariate distributions. We believe that the conjecture does not hold in general for such distributions, although we have not been able to establish this yet and verify it only for some specific distributions of this type.

An n-dimensional random vector $\mathbf{X} = (X_1, \dots, X_n)'$ is said to have an MTP_2 (TP_2 when $n = 2$) distribution if the corresponding probability density, $f(\mathbf{x})$, satisfies the following condition:

$$f(\mathbf{x} \vee \mathbf{y}) f(\mathbf{x} \wedge \mathbf{y}) \geq f(\mathbf{x}) f(\mathbf{y}) \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathscr{R}^n,$$

where, with $\mathbf{x} = (x_1, \dots, x_n)'$ and $\mathbf{y} = (y_1, \dots, y_n)'$, $\mathbf{x} \vee \mathbf{y} = (\max(x_1, y_1), \dots, \max(x_n, y_n))$ and $\mathbf{x} \wedge \mathbf{y} = (\min(x_1, y_1), \dots, \min(x_n, y_n))$. This condition is satisfied by a large family of multivariate distributions in addition to those in Sarkar and Chang (1997), such as the multivariate normal with nonnegative correlations and the absolute-valued multivariate normal with some specific covariance structures, and is fundamental to many probability inequalities that have important applications to multivariate analysis, simultaneous statistical inference, and approximating probabilities [Karlin and Rinott (1980, 1981), Perlman and Olkin (1980), and Glaz and Johnson (1984)]. Toward proving the conjecture, we derive some new probability inequalities for the ordered components of an MTP_2 random vector. Denoting the ith marginal cdf of $\mathbf{X}$ by F_i, $i = 1, \dots, n$, and the ordered components by $X_{(1)} \leq \cdots \leq X_{(n)}$, we show, in particular, that for these distributions the probabilities $P\{X_{(1)} \geq a_1, \dots, X_{(n)} \geq a_n\}$ and $P\{X_{(1)} \leq b_1, \dots, X_{(n)} \leq b_n\}$ are greater than or equal to $1 - (1/n) \sum_{i=1}^n F_i(a_n)$ and $(1/n) \sum_{i=1}^n F_i(b_1)$, respectively, if the constant's a_i's and b_i's are chosen in some particular ways from the marginals. Similar inequalities hold also for the subgroup of scale mixtures of MTP_2 distributions mentioned above. It is important to point out that the inequalities derived in this paper are different from those known in the literature for random variables which are ordered as well as MTP_2; for example, the ordered characteristic roots of random Wishart matrices, the ordered components of an exchangeable MTP_2 random vector [Dykstra and Hewett (1978), Karlin and Rinott (1980), Sarkar and Smith (1986)].

2. An identity. We present in this section an identity involving the joint probability distribution of the ordered components of any random vector $\mathbf{Y} = (Y_1, \dots, Y_n)$, not necessarily MTP_2. This is the key to the proofs of the probability inequalities presented in the next section, which will lead to a theoretical verification of the Simes conjecture.

LEMMA 2.1. *Let* $Y_{(1)} \leq \cdots \leq Y_{(n)}$ *be the ordered components of* $\mathbf{Y} = (Y_1, \dots, Y_n)'$. *Then,*

$$\begin{aligned} P\{Y_{(1)} \geq a_1, \dots, Y_{(n)} \geq a_n\} &= 1 - \frac{1}{n} \sum_{i=1}^n F_i(a_n) \\ &\quad + \sum_{i=1}^n \sum_{j=1}^{n-1} E\biggl[\biggl\{\frac{I(Y_i \leq a_{j+1})}{j+1} - \frac{I(Y_i \leq a_j)}{j}\biggr\} \\ &\qquad \times P\bigl\{\mathbf{Y}_{(j)}^{(-i)} \geq a_{j+1}, \dots, \mathbf{Y}_{(n-1)}^{(-i)} \geq a_n \big| Y_i\bigr\}\biggr], \end{aligned} \tag{2.1}$$

where, for each $i = 1, \dots, n$, $\mathbf{Y}^{(-i)}_{(1)} \le \cdots \le \mathbf{Y}^{(-i)}_{(n-1)}$ *denote the ordered components of the* $(n-1)$*-dimensional random vector* $\mathbf{Y}^{(-i)}$ *obtained by eliminating* Y_i *from* $\mathbf{Y}$.

PROOF. We start with the following identity:

$$P\{Y_{(1)} \ge a_1, \dots, Y_{(n)} \ge a_n\} = 1 - \sum_{j=1}^{n} P\{Y_{(j)} < a_j, Y_{(j+1)} \ge a_{j+1}, \dots, Y_{(n)} \ge a_n\}. \tag{2.2}$$

This identity follows from the following probability theory result: $P(\bigcap_{j=1}^n \bar{A}_j) = 1 - P(\bigcup_{j=1}^n A_j) = 1 - \sum_{j=1}^n P(A_j \bar{A}_{j+1}, \dots, \bar{A}_n)$, for any set of n events $A_1, \dots, A_n$, with $\bar{A}_1, \dots, \bar{A}_n$ being the corresponding complements. The next step is to show that the right-hand side of (2.2) is exactly equal to that of (2.1). For this, we will first prove that, for $j = 1, \dots, n$,

$$P\{Y_{(j)} < a_j, Y_{(j+1)} \ge a_{j+1}, \dots, \mathbf{Y}^{(-i)}_{(n-1)} \ge a_n\} = \frac{1}{j} \sum_{i=1}^{n} P\{Y_i < a_j, \mathbf{Y}^{(-i)}_{(j-1)} < a_j, \mathbf{Y}^{(-i)}_{(j)} \ge a_{j+1}, \dots, \mathbf{Y}^{(-i)}_{(n-1)} \ge a_n\}. \tag{2.3}$$

To this end, for any subset $\{i_1, \dots, i_j\} \subset \{1, \dots, n\}$ of j indices, let $A_{i_1 \cdots i_j}$ denote the event that $\max(Y_{i_1}, \dots, Y_{i_j}) < a_j$ and the ordered components of the $(n-j)$ random variables left after ignoring $S_j = (Y_{i_1}, \dots, Y_{i_j})$, say $\mathbf{Y}^{\bar{S}_j}_{(1)} \le \cdots \le \mathbf{Y}^{\bar{S}_j}_{(n-j)}$, satisfy $\mathbf{Y}^{\bar{S}_j}_{(1)} \ge a_{j+1}, \dots, \mathbf{Y}^{\bar{S}_j}_{(n-j)} \ge a_n$. Then, the probability in the left-hand side of (2.3) is

$$\sum_{1 \le i_1 < \cdots < i_j \le n} P\{A_{i_1 \cdots i_j}\}. \tag{2.4}$$

Since $P\{A_{i_1 \cdots i_j}\}$ is same for all the permutations of $\{i_1, \dots, i_j\}$, the probability in (2.4) can be written as

$$\frac{1}{j} \sum_{i=1}^{n} \sum\nolimits^{*} P\{A_{i i_1 \cdots i_{j-1}}\},$$

with $\sum^{*}$ being the summation over $i_1, \dots, i_{j-1}$ such that $1 \le i_1 < \cdots < i_{j-1} \le n-1$ and $i_k \ne i$ for $k = 1, \dots, j-1$, which is the right-hand side of (2.3). Thus, (2.3) is proved.

Now, note that the right-hand side of (2.3) can be written as

$$
\begin{aligned}
(2.5)\qquad &\frac{1}{j}\sum_{i=1}^{n} P\{Y_i < a_j, \mathbf{Y}_{(j)}^{(-i)} \ge a_{j+1}, \dots, \mathbf{Y}_{(n-1)}^{(-i)} \ge a_n\}\\
&\quad - \frac{1}{j}\sum_{i=1}^{n} P\{Y_i < a_j, \mathbf{Y}_{(j-1)}^{(-i)} \ge a_j, \dots, \mathbf{Y}_{(n-1)}^{(-i)} \ge a_n\}
\end{aligned}
$$

for $j = 1, \dots, n$. Of course, when $j = 1$, the second term in (2.5) does not appear and the first term is

$$\sum_{i=1}^{p} P\{Y_i < a_1, \mathbf{Y}_{(1)}^{(-i)} \ge a_2, \dots, \mathbf{Y}_{(n-1)}^{(-i)} \ge a_n\};$$

whereas, when $j = n$, (2.5) is

$$\frac{1}{n}\sum_{i=1}^{n} P\{Y_i < a_n\} - \frac{1}{n}\sum_{i=1}^{n} P\{Y_i < a_n, \mathbf{Y}_{(n-1)}^{(-i)} \ge a_n\}.$$

Summing the terms in (2.5) from $j = 2$ to $n - 1$, we get

$$
\begin{aligned}
&\sum_{j=2}^{n-1} P\{Y_{(j)} < a_j, Y_{(j+1)} \ge a_{j+1}, \dots, \mathbf{Y}_{(n-1)}^{(-i)} \ge a_n\}\\
&\quad = \sum_{i=1}^{n}\sum_{j=2}^{n-1} \frac{1}{j} P\{Y_i < a_j, \mathbf{Y}_{(j)}^{(-i)} \ge a_{j+1}, \dots, \mathbf{Y}_{(n-1)}^{(-i)} \ge a_n\}\\
&\qquad - \sum_{i=1}^{n}\sum_{j=1}^{n-2} \frac{1}{j+1} P\{Y_i < a_{j+1}, \mathbf{Y}_{(j)}^{(-i)} \ge a_{j+1}, \dots, \mathbf{Y}_{(n-1)}^{(-i)} \ge a_n\}\\
&\quad = \sum_{i=1}^{n}\sum_{j=1}^{n-1} E\biggl[\biggl\{\frac{I(Y_i < a_j)}{j} - \frac{I(Y_i < a_{j+1})}{j+1}\biggr\}\\
&\qquad\qquad \times P\{\mathbf{Y}_{(j)}^{(-i)} \ge a_{j+1}, \dots, \mathbf{Y}_{(n-1)}^{(-i)} \ge a_n | Y_i\}\biggr]\\
&\qquad + \frac{1}{n}\sum_{i=1}^{n} P\{Y_i < a_n, \mathbf{Y}_{(n-1)}^{(-i)} \ge a_n\}\\
&\qquad - \sum_{i=1}^{n} P\{Y_i < a_1, \mathbf{Y}_{(1)}^{(-i)} \ge a_2, \dots, \mathbf{Y}_{(n-1)}^{(-i)} \ge a_n\}.
\end{aligned}
$$

If we now sum all the terms in (2.5) from $j = 1$ to n and subtract it from 1 to get the right-hand side of (2.2), it will be exactly what is given in the right-hand side of (2.1). Hence, the lemma is proved. $\square$

REMARK 2.1. By applying Lemma 2.1 to $-\mathbf{Y}$, we see that, for any fixed $b_1 \leq \cdots \leq b_n$,

$$
\begin{aligned}
P\{Y_{(1)} &\leq b_1, \ldots, Y_{(n)} \leq b_n\} \\
&= P\{-Y_{(n)} \geq -b_n, \ldots, -Y_{(1)} \geq -b_1\} \\
&= 1 - \frac{1}{n}\sum_{i=1}^{n} P\{-Y_i \leq -b_1\} \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n-1} E\biggl[\biggl\{\frac{I(-Y_i \leq -b_{n-j})}{j+1} - \frac{I(-Y_i \leq -b_{n-j+1})}{j}\biggr\} \\
&\qquad\qquad \times P\bigl\{-\mathbf{Y}_{(1)}^{(-i)} \geq -b_1, \ldots, -\mathbf{Y}_{(n-j)}^{(-i)} \geq -b_{n-j} \big| -Y_i\bigr\}\biggr] \\
&= \frac{1}{n}\sum_{i=1}^{n} F_i(b_1) + \sum_{i=1}^{n}\sum_{j=1}^{n-1} E\biggl[\biggl\{\frac{I(Y_i \geq b_{n-j})}{j+1} - \frac{I(Y_i \geq b_{n-j+1})}{j}\biggr\} \\
&\qquad\qquad \times P\bigl\{\mathbf{Y}_{(1)}^{(-i)} \leq b_1, \ldots, \mathbf{Y}_{(n-j)}^{(-i)} \leq b_{n-j} \big| Y_i\bigr\}\biggr],
\end{aligned}
\tag{2.6}
$$

If Y_i's are iid with a common cdf $F(y)$, then the identities (2.1) and (2.6) reduce to

$$
\begin{aligned}
P\{Y_{(1)} &\geq a_1, \ldots, Y_{(n)} \geq a_n\} \\
&= 1 - F(a_n) + n\sum_{j=1}^{n-1}\biggl[\frac{F(a_{j+1})}{j+1} - \frac{F(a_j)}{j}\biggr] \\
&\qquad \times P\{Y_{(j):n-1} \geq a_{j+1}, \ldots, Y_{(n-1):n-1} \geq a_n\}
\end{aligned}
$$

and

$$
\begin{aligned}
P\{Y_{(1)} &\leq b_1, \ldots, Y_{(n)} \leq b_n\} \\
&= F(b_1) + n\sum_{j=1}^{n-1}\biggl[\frac{\bar{F}(b_{n-j})}{j+1} - \frac{\bar{F}(b_{n-j+1})}{j}\biggr] \\
&\qquad \times P\{Y_{(1):n-1} \leq b_1, \ldots, Y_{(n-j):n-1} \leq b_{n-j}\},
\end{aligned}
$$

respectively, where $\bar{F}(x) = 1 - F(x)$, and $Y_{(1):n} \leq \cdots \leq Y_{(n):n}$ denote the ordered components of $(Y_1, \ldots, Y_n)$. These last two identities were obtained by Sarkar and Chang (1997).

3. Some probability inequalities and proof of the Simes conjecture. The principal theorem leading to a proof of the Simes conjecture is the following.

THEOREM 3.1. *Let $X_{(1)} \leq \cdots \leq X_{(n)}$ be the ordered components of an MTP$_2$ random vector $\mathbf{X} = (X_1, \ldots, X_n)$ and F_i be the marginal cdf of X_i. Then we have the following.*

(i) *For fixed* $a_1 \le \cdots \le a_n$,

$$P\{X_{(1)} \ge a_1, \ldots, X_{(n)} \ge a_n\} \ge 1 - \frac{1}{n}\sum_{i=1}^{n} F_i(a_n), \tag{3.1}$$

if $j^{-1}F_i(a_j)$ *is nondecreasing in* $j = 1, \ldots, n$ *for all* $i = 1, \ldots, n$;

(ii) *For fixed* $b_1 \le \cdots \le b_n$,

$$P\{X_{(1)} \le b_1, \ldots, X_{(n)} \le b_n\} \ge \frac{1}{n}\sum_{i=1}^{n} F_i(b_1), \tag{3.2}$$

if $j^{-1}\bar{F}_i(b_{n-j+1})$ *is nondecreasing in* $j = 1, \ldots, n$ *for all* $i = 1, \ldots, n$, *where* $\bar{F}(x) = 1 - F(x)$.

PROOF. From Lemma 2.1, we see that

$$\begin{aligned} &P\{X_{(1)} \ge a_1, \ldots, X_{(n)} \ge a_n\} - \left\{1 - \frac{1}{n}\sum_{i=1}^{n} F_i(a_n)\right\} \\ &\quad = \sum_{i=1}^{n}\sum_{j=1}^{n-1} E\left[\left\{\frac{I(X_i < a_{j+1})}{j+1} - \frac{I(X_i < a_j)}{j}\right\}\right] \\ &\qquad \times P\left\{\mathbf{X}_{(j)}^{(-i)} \ge a_{j+1}, \ldots, \mathbf{X}_{(n-1)}^{(-i)} \ge a_n \middle| X_i\right\}. \end{aligned} \tag{3.3}$$

Since $\mathbf{X}$ is MTP_2 and the indicator function of the set $\{\mathbf{X}_{(j)}^{(-i)} \ge a_{j+1}, \ldots, \mathbf{X}_{(n-1)}^{(-i)} \ge a_n\}$ is nondecreasing in $\mathbf{X}^{(-i)}$, the conditional probability of this set given X_i is a nondecreasing function of X_i [Theorem 4.1 of Karlin and Rinott (1980)]. Also, note that $(I(X_i < a_{j+1})/j + 1) - I(X_i < a_j)/j$ is less than 0 when $X_i < a_j$, and is greater than or equal to 0 when $X_i \ge a_j$. Therefore, the (i, j)th term in the double summation in (3.3) is greater than or equal to

$$\left\{\frac{F_i(a_{j+1})}{j+1} - \frac{F_i(a_j)}{j}\right\} P\left\{\mathbf{X}_{(j)}^{(-i)} \ge a_{j+1}, \ldots, \mathbf{X}_{(n-1)}^{(-i)} \ge a_n \middle| X_i = a_j\right\}, \tag{3.4}$$

for all $i = 1, \ldots, n$; $j = 1, \ldots, n-1$. This proves the first part of the theorem.

The second part of the theorem follows by applying the first part to $-X_{(n)} \le \cdots \le -X_{(1)}$, the ordered components of $-\mathbf{X}$ which is also MTP_2.

REMARK 3.1. Suppose that X_i's have a common marginal distribution F, and that for each $i = 1, \ldots, n$, P_i, the random p-value corresponding to H_i, is based on a left-tailed or right-tailed test based on X_i. Since P_i is defined as $F(X_i)$ for a left-tailed test and as $\bar{F}(X_i)$ for a right-tailed test, in terms of $P_{(i)}$ the inequalities (3.1) and (3.2) are equivalent to

$$P\{P_{(i)} \ge c_i, i = 1, \ldots, n\} \ge 1 - c_n, \tag{3.5}$$

where $0 < c_1 \le \cdots \le c_n < 1$ are such that c_i/i is nondecreasing in $i = 1, \ldots, n$. This, with $c_i = i\alpha/n$ for all i, proves the following.

PROPOSITION 3.1. *The Simes conjecture* (1.1) *holds for* MTP_2 *random variables with common marginals.*

A number of multivariate distributions arising in multiple testing situations are not MTP_2 but are certain scale mixtures of MTP_2 distributions; for instance, the multivariate t, multivariate Cauchy, and so on. Does the Simes conjecture hold for these distributions as well? For some specific distributions of this type, as described in the following corollary to Theorem 3.1, the answer is yes.

COROLLARY 3.1. *Let* $\mathbf{T} = (T_1, \dots, T_n) = Z^{-1}\mathbf{X}$, *where* $\mathbf{X}$ *is exchangeable, having a probability density of the following form*:

$$f(x_1, \dots, x_n) = \int \left\{ \prod_{i=1}^{n} g(x_i, y) \right\} h(y)\, dy,$$

and Z *is a positive-valued random variable independent of* $\mathbf{X}$. *Let* $g(x, y)$ *be such that*

$$g^*(x, y) = \int_0^\infty z g(zx, y) q(z)\, dz,$$

where $q(z)$ *is the probability density of* Z, *is* TP_2 *in* (x, y). *Then we have the following*:

(i) *for fixed* $a_1 \le \cdots \le a_n$, *all having the same sign*,

$$P\{T_{(1)} \ge a_1, \dots, T_{(n)} \ge a_n\} \ge 1 - P\{T_1 \le a_n\} \tag{3.6}$$

if $j^{-1} P\{T_1 \le a_j\}$ *is nondecreasing in* $j = 1, \dots, n$; *and*

(ii) *for fixed* $b_1 \le \cdots \le b_n$, *all having the same sign*,

$$P\{T_{(1)} \le b_1, \dots, T_{(n)} \le b_n\} \ge P\{T_1 \le b_1\} \tag{3.7}$$

if $j^{-1} P\{T_1 \ge b_{n-j+1}\}$ *is nondecreasing in* $j = 1, \dots, n$.

PROOF. Note that for fixed a_j's,

$$\begin{aligned} P\{T_{(1)} \ge a_1, \dots, T_{(n)} \ge a_n\} &= P\{X_{(1)} \ge a_1 Z, \dots, X_{(n)} \ge a_n Z\} \\ &= E\left[E\left\{ \prod_{i=1}^{n} I(a_i Z \le X_{(i)}) \right\} \middle| \mathbf{X} \right] \\ &\ge E\left\{ \prod_{i=1}^{n} P(a_i Z \le X_{(i)}) \middle| \mathbf{X} \right\} \\ &= P\left\{ X^*_{(1)} \ge a_1, \dots, X^*_{(n)} \ge a_n \right\}, \end{aligned} \tag{3.8}$$

where $\mathbf{X}^*$ has the density

$$f^*(x_1, \dots, x_n) = \int \left\{ \prod_{i=1}^{n} g^*(x_i, y) \right\} h(y)\, dy.$$

The inequality in (3.8) follows from the fact that, according to Kimball's inequality [Tong (1980)], the expectation of a product of positive-valued functions of a random variable, all nondecreasing or nonincreasing, is greater than or equal to the product of their expectations. Noting that the corresponding marginals of $\mathbf{T}$ and $\mathbf{X}^*$ are the same, and that $\mathbf{X}^*$ is MTP_2 because $g^*(x, y)$ is TP_2 [see, e.g., Karlin and Rinott (1980)], we see that the first part of the corollary follows from Theorem 3.1(i). Similarly, the second part of the corollary follows from Theorem 3.1(ii). □

REMARK 3.2. Suppose that $\mathbf{X}$ has the equicorrelated standard multivariate normal distribution with a nonnegative common correlation ρ. Since the X_i's with such a distribution have the following representation:

$$X_i = (1-\rho)^{1/2} Y_i + \rho^{1/2} Y_0,$$

where Y_i are iid $N(0,1)$ [Johnson and Kotz (1972)], the $g(x, y)$ for this distribution is

$$g(x, y) = \frac{1}{(1-\rho)^{1/2}} \phi\left(\frac{x - \rho^{1/2} y}{(1-\rho)^{1/2}}\right),$$

and $h(y) = \phi(y)$, where $\phi(\cdot)$ is the density of $N(0,1)$. Let $Z \sim \chi_\nu^2/\nu$. So, $g^*(x, y)$ here is the density of $(1-\rho)^{1/2} t'_\nu(\gamma)$, where $t'_\nu(\gamma)$ is the noncentral t with ν degrees of freedom and the noncentrality parameter $\gamma = (\rho^{1/2}/(1-\rho)^{1/2})y$. As Karlin [(1968), page 118] proved, the density of the noncentral t at x with the noncentrality parameter γ, say $f(x, \gamma)$, is TP_2 in (x, γ). Hence, $g^*(x, y)$ is TP_2 in (x, y) if $\rho \geq 0$. Thus, with $T_i = Z^{-1} X_i$, $i = 1, \ldots, n$ defined in terms of these X_i's and Z, that is, for the central multivariate t of Dunnett and Sobel (1954) type with the associated correlations being equal and nonnegative, the inequalities (3.6) and (3.7) hold if the a_i's and b_i's are chosen from the central univariate t distribution subject to the conditions stated there. Note that $\{T_{(1)} \geq a_1, \ldots, T_{(n)} \geq a_n\}$ is the acceptance region of the Simes test combining left-tailed tests based on the T_i's, whereas $\{T_{(1)} \leq b_1, \ldots, T_{(n)} \leq b_n\}$ is the acceptance region of the Simes test that involves right-tailed tests based on the T_i's. Hence, for the a_n to satisfy $P\{T_1 \leq a_n\} = \alpha$ and the a_i's to have the same sign, it is required that $0 < \alpha < \frac{1}{2}$. The same condition on α is required for the right-tailed tests.

Next, suppose that the X_i's have the squared equicorrelated multivariate normal distribution with any correlation ρ. Since this distribution depends on ρ only through its square, we may assume without any loss of generality that $\rho \geq 0$. Using the above representation for the equicorrelated multivariate normal with nonnegative ρ, one can see that here $g(x, y)$ is the density of $(1-\rho){\chi'_1}^2(\lambda)$, where ${\chi'_1}^2(\lambda)$ is the noncentral chi-squared random variable with 1 degree of freedom and the noncentrality parameter $\lambda = \rho y/(1-\rho)$, and $h(y)$ is the density of χ_1^2, the central chi-square with 1 degree of freedom. Hence, with $Z \sim \chi_\nu^2/\nu$, $g^*(x, y)$ is the density at x of $(1-\rho)$ times the noncentral F with 1 and ν degrees of freedom and the noncentrality parameter λ, which is TP_2 in (x, λ), and hence in (x, y). Hence, the inequalities (3.6) and (3.7), with

T_i's defined in terms of these X_i's and Z and the constants subject to the stated conditions involving the central F distribution with 1 and ν degrees of freedom, also hold.

Thus, we have also established the following proposition.

PROPOSITION 3.2. *For one-sided t tests with* $0 < \alpha < \frac{1}{2}$, *the Simes conjecture* (1.1) *holds when the underlying normals have equal and nonnegative correlation. This conjecture also holds for two-sided t tests when the underlying normals are equicorrelated with any common correlation.*

REMARK 3.3. It is clear from the proof of Theorem 3.1 that the inequalities in (3.1) and (3.2) will reverse; that is, the Simes conjecture will not hold if the distribution of $\mathbf{X}$ is such that the probability $P\{\mathbf{X}_{(j)}^{(-i)} \geq a_{j+1}, \ldots, \mathbf{X}_{(n-1)}^{(-i)} \geq a_n | X_i\}$ is strictly decreasing in X_i. For example, in the bivariate case, if $\mathbf{X} = (X_1, X_2)$ is negatively dependent in the sense of being strictly reverse rule of order two (RR_2) that is, if $(X_1, -X_2)$ is strictly TP_2, then the above property is true. In other words, the Simes conjecture does not hold for a bivariate distribution with the strict RR_2 property. A multivariate version of the strict RR_2 property, known as the strong multivariate reverse rule of order two ($\mathrm{S} - \mathrm{MRR}_2$), has been introduced by Karlin and Rinott (1980). Although it can be proved that the above probability is strictly decreasing in X_i for some specific distributions of this type, for example, the equicorrelated multivariate normal with a negative common correlation and the exchangeable Dirichlet distributions, it is, however, not known if this property is true in general for all such negatively dependent multivariate distributions. Thus, it is not known yet if the Simes conjecture is false in general for all $\mathrm{S} - \mathrm{MMR}_2$ distributions.

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