

SOME PROBLEMS IN THE FORMULATION OF THE EQUATIONS FOR GAS/LIQUID FLOWS

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This paper is concerned with the formulation of hydrodynamic equations for two phase flows. In particular stratified flow and bubbly flow are considered. Emphasis is laid on those aspects which are associated with relative translational motion. The occurrence of complex characteristics in separated flow models, the equation for relative motion in bubbly flow, the effect of void fraction on virtual mass, are among the discussed topics. Where possible, a comparison with experimental observation is included.

1. Introduction

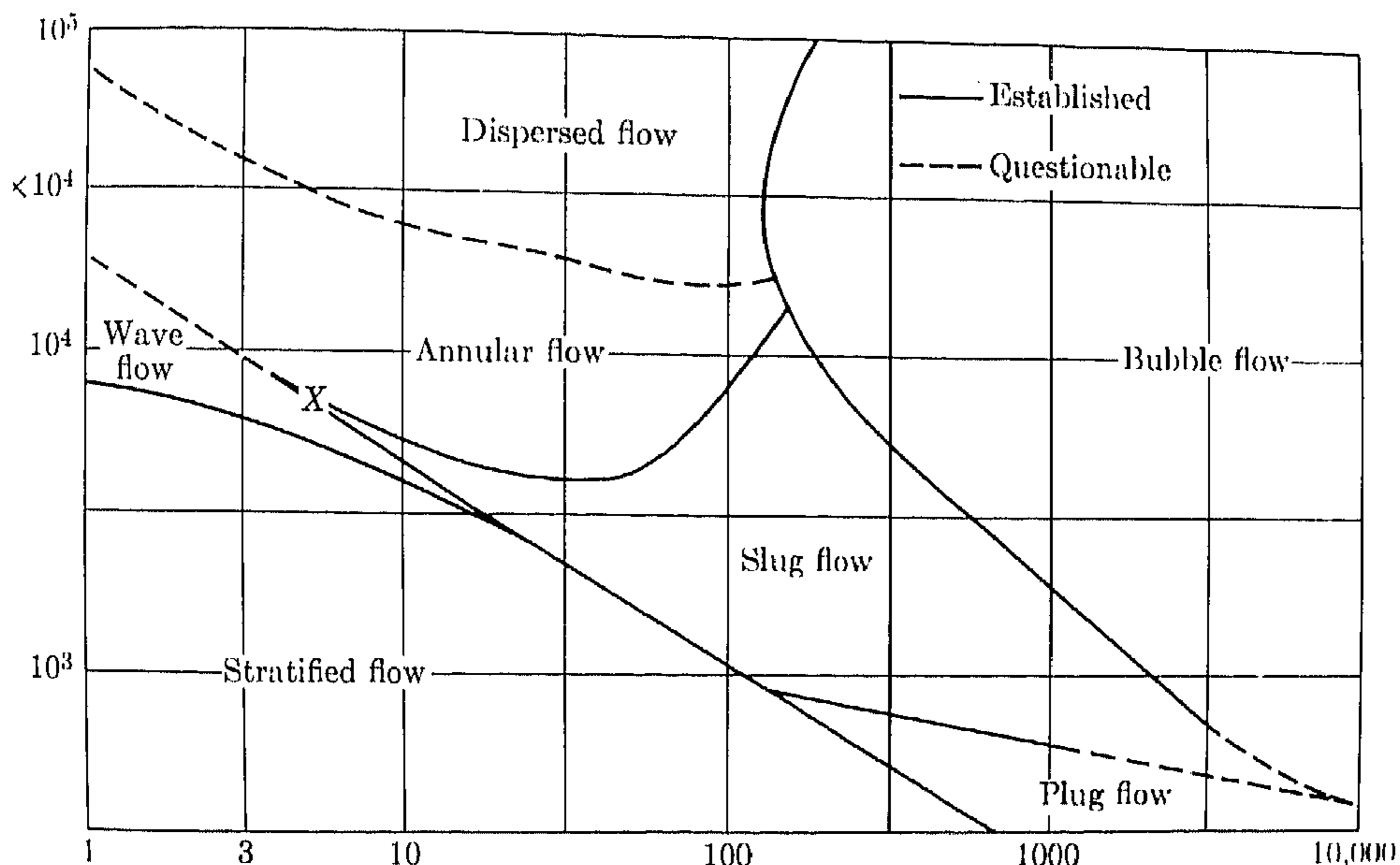


Figure 1: Plot for horizontal two phase flow. The horizontal coordinate is representative for the ratio between mass velocities and the vertical coordinate for the gas velocity. Since the function of the plot here is purely qualitative, details are omitted, but can be found in Brodkey [1]

Gas/liquid flows occur in technology in a wild variety of appearance. Many attempts have been made to classify or categorise these. An example is shown in Figure 1 taken from the book by Brodkey [1]. Regimes like bubbly flow, annular flow, stratified flow and so on, are shown in dependence of the ratio between the mass flows of the two phases. This type of maps contains areas where the territory is unknown or almost unknown.

Also these maps cannot always be trusted. They suggest that, once the average mass or volume velocities are known, the topology of the flow can be predicted. This is not always true. A counter example is furnished by the experiments of Witte[2] where stratified flow and bubbly flow (see Figure 2) occur at the two sides of a shock wave. Obviously the mass flow ratio is the same at both sides, but the topology is not.

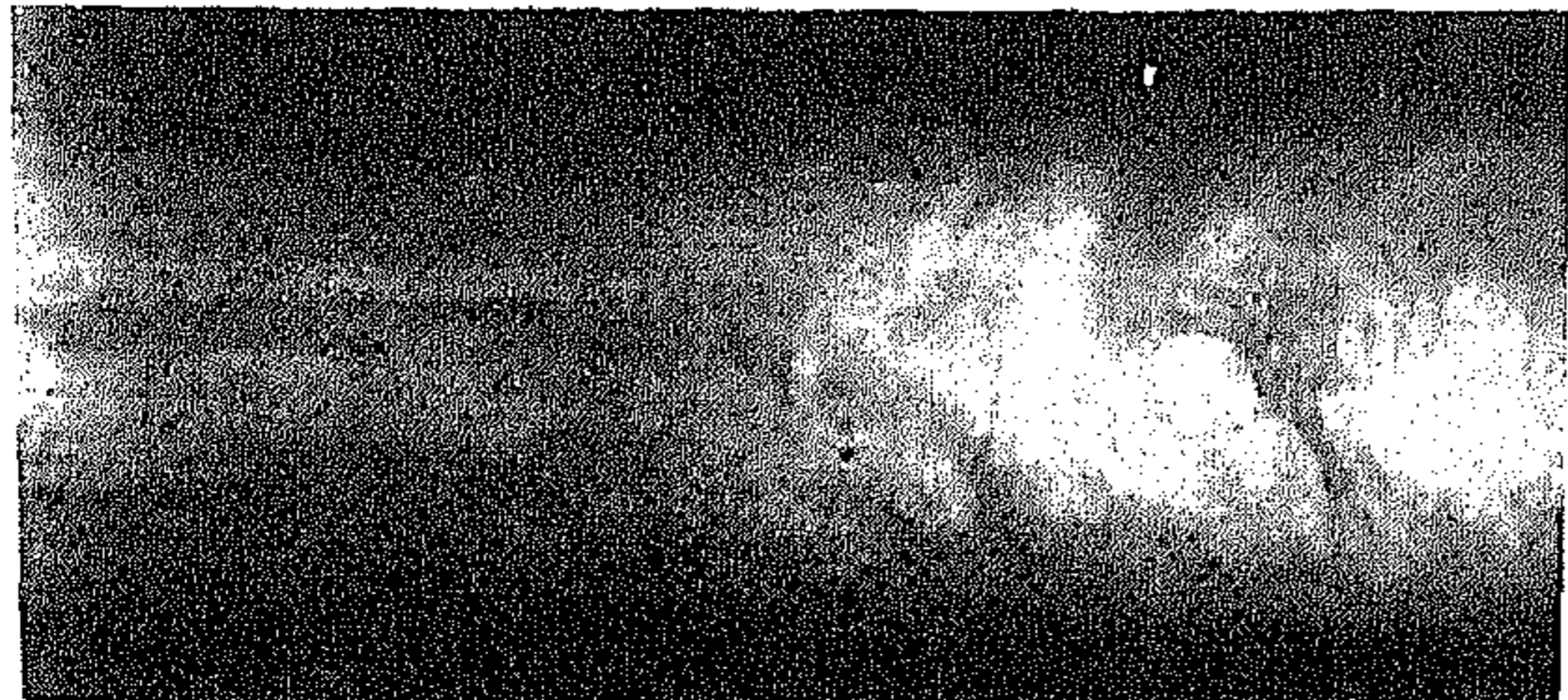


Figure 2: A shock wave in a gas-liquid flow with stratified flow ahead of the shock (at left in the Fig./ and bubbly flow behind the shock (at the right in the Figure). From Witte [2].

This lecture will be concerned with a number of problems that arise when one tries to make mathematical models which are able to predict the development of gas/liquid flows. Various models have been proposed in the course of time but although a lot of work has been done, especially over the last decade, the situation may still adequately be described by what Batchelor [3] wrote in 1972 about two phase systems "There are technical problems in abundance, intriguing observations, puzzling phenomena and some scraps of theory about specific features, but not yet the kind of secure foundations and body of theory which turn a collection of particular problems into a subject". Ronald Panton [4] remarks, perhaps less scholarly phrased but certainly not less clearly "I have always thought that two phase was a lot like the field of turbulence, only worse". However gas/liquid flow occurs in many a technical system, as for example a nuclear reactor. Engineers and physicists who have to think about the design and safety of nuclear reactors want to know about the dynamic and thermal behaviour of gas/liquid flow. And when secure foundations and a firm body of theory are not available they have to work with whatever scrap of theory or other information they can lay hands on. It is reasonable to assume that given the variety of topologies in gas/liquid flows, in one representation physical processes will be dominant which are less dominant in others. Equations allowing for the dominating mechanisms are therefore expected to be different for the different regimes shown in Figure 1. The nuclear engineer would nevertheless like to be provided with a general set of equations applicable to any type of flow. This for two reasons : first he does not always know from the outset what the topology in a particular situation is or will be, and further, even if he knows his topology may, for example by mass transfer between the phases, change from one location in a circuit to another.

The theoretical models which have been used and are still used are to a large extent based on annular flow (for example [5]), stratified flow, generally speaking on separated flows. One hopes that results developed, for example with stratified flow in mind, will be valuable for other topologies as well. A classical example is the Lockhart- Martinelli relation for the frictional pressure drop in gas/liquid flow through pipes and ducts.

There is at the present time not yet enough experience with the various mathematical models and certainly there are not enough experiments done to warrant a critical survey, including comparison with observation. Instead I propose to discuss a number of problems and difficulties which turn up when one considers even the most simplified versions of equations for gas/liquid flows in the separated flow model. As it will appear, the unifying theme in these difficulties is relative motion between the two phases. First of all there is a difficulty in understanding what is precisely meant by "separated flow model". Physically the two phases are separated by an interface, which is quite evident. The general understanding is that when talking about "separated flow model" one thinks about a model in which conservation equations for mass, momentum and energy are written down for each of the two phases separately. Another, perhaps more appropriate, term is "two fluid model". It is however not obvious at all whether a system of equations can be formulated for all kinds of topologies. I will restrict to two extremes : stratified or annular flow on one hand and bubbly flow on the other. In the first representation the region occupied by the gas is singly connected, in the second one this region is multiple connected. The equations which are used most widely used in practice, see e.g. refs. [4], [6] and [7], are certainly more reminiscent of stratified flow than of bubbly flow. In the subsequent sections we will discuss some of the fluid dynamics in these configurations, first when the two phases are not in appreciable relative motion later on when they are.

2. Sound propagation through gas/liquid flow at rest.

Bubbly flow.

Assuming that locally the pressure in the gas and in the liquid is the same, that thermodynamic changes in the bubbles are isothermal (isentropic) and that locally the bubbles move with the liquid, the sound velocity is (See e.g. the review in [8])

$$c_o^2 = \frac{(\gamma) p}{\rho_1 \alpha (1-\alpha)}, \quad (2.1)$$

where p denotes pressure ρ_1 the liquid density and α the concentration of gas by volume. This expression allows a simple interpretation. In Figure 3^a is shown a volume V of a bubbly suspension which we want to increase by an amount ΔV .

Conservation of mass produces a $\Delta\rho$, which is

$$\Delta\rho = - \frac{\rho_1(1-\alpha)\Delta V}{V} \quad (2.2)$$

Taking the liquid as incompressible, the change in volume is entirely due to expansion of the gasbubbles. Let the number density of bubbles with volume τ_i be indicated with n_i . Then, for isothermal changes,

$$p \Sigma n_i \Delta \tau_i + \Delta p \Sigma n_i \tau_i = 0$$

or

$$\Delta p = - \frac{p \Sigma n_i \Delta \tau_i}{\Sigma n_i \tau_i} = - \frac{p \Delta V}{\alpha V},$$

which gives together with (2.2) the expression (2.1) for the square of the speed of sound. Writing c_0^2 as $dp/d\rho$ we obtain from (2.1) by integration

$$\frac{p\alpha}{\rho_1(1-\alpha)} = \text{const}, \quad (2.3)$$

which can be regarded as an equation of state and which expresses the fact that, because the bubbles move with the liquid, the mass of gas in a mass of the suspension remains constant. More complicated calculations are needed to find the velocity of sound when relative motion between bubbles and liquid is not inhibited.

For not too large α the sound velocity is, with a purely inertial interaction between bubbles and liquid, given by (Crespo [9], see also [8])

$$c_f^2 = \frac{(1+2\alpha)p}{\rho_1\alpha(1-\alpha)}. \quad (2.4)$$

This is for spherical bubbles. For other shapes the figure 2 in the nominator, which is associated with the virtual mass, is different.

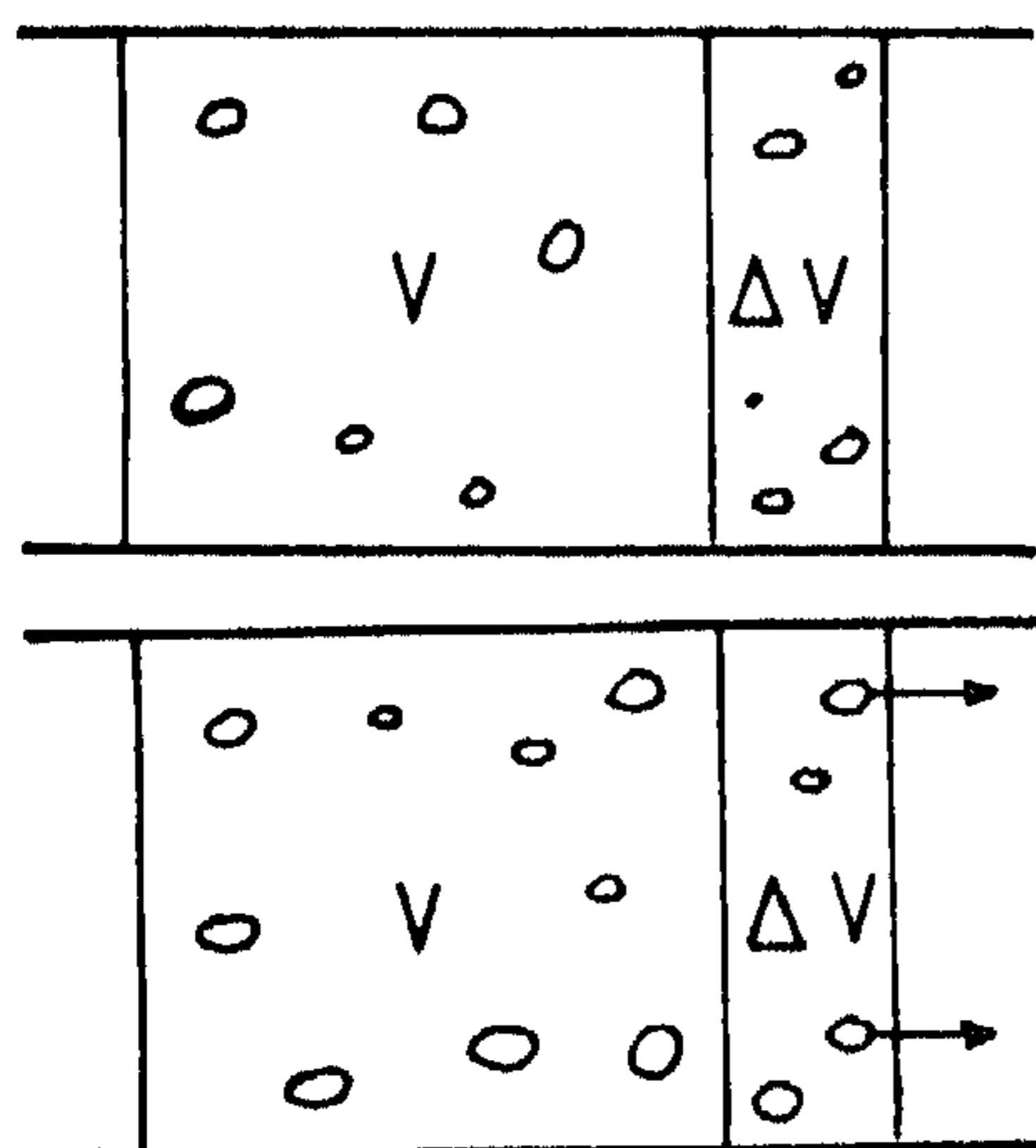


Figure 3.

a)

b)

A volume V of a bubbly liquid expands with ΔV . In a) bubbles move with the liquid, in b) they are free to move with respect to the liquid.

The difference with the expression (2.1) for c_0^2 may be explained by considering Figure 3b in which again a volume change ΔV is produced now however with relative motion.

The change in volume is now only partly due to changes in bubble volume, the remainder is caused by a net volume flow, into the considered volume,

$$\Delta V = V \Sigma n_i \Delta \tau_i - \Sigma n_i \tau_i (v_i - u) \cdot dA dt$$

Here v_i is the velocity of bubbles with number density n_i , dA is an surface element of the considered volume and the time wise integration is over the time that is involved in producing the change ΔV of V . When viscous friction is negligible we have, see section 5, $v_i = 3u$, approximately. The integral then is

$$- \Sigma 2 n_i \tau_i v_i \cdot u dV dt \sim -2\alpha \Delta V.$$

Therefore

$$\frac{\Delta V}{V} (1+2\alpha) = \Sigma n_i \Delta \tau_i = - \frac{\alpha \Delta p}{p}.$$

Combination with (2.2) gives (2.4). To produce a certain ΔV a larger Δp (in absolute magnitude) is needed now that bubbles may escape from the considered volume. The compressibility therefore has become smaller and the velocity of sound correspondingly higher. In most experiments carried out in the laboratory to measure the propagation speed of acoustic waves, α is of the order of a few percent. The difference between c_0 in (2.1) and c_f in (2.4) is under those circumstances within the experimental error. Most experimental data have been compared with c_0 and good agreement has been found [8] for frequencies well below the resonance frequency of the bubbles. For frequencies which are comparable with the resonance frequency the pressure inside the bubbles differs from the pressure in the liquid through the inertia of the relative radial motion which manifests itself macroscopically as wave dispersion. The existence of two sound speeds means that relaxation effects occur in wave propagation. The classical example is wave propagation through a gas in which a chemical reaction takes place. This type of flow is discussed in detail in the two volumes "Nonequilibrium Flow" edited by Wegener [10]. There are two sound speeds, one in which the reaction is in equilibrium as the wave passes, comparable to c_0 in our case, the other in which the reaction is frozen as the wave passes, comparable with c_f . The tendency to relax toward equilibrium characterized by a relaxation time T , say, has a diffusive effect on waves, the diffusion coefficient being equal to $T(c_f^2 - c_0^2)$. As a result of this diffusive action two new types of shock waves are possible next to the familiar one in which the reaction is absent. These are the so-called partly and fully dispersed shock in which over part of the shock and over the entire shock, respectively, there is a balance between nonlinear steepening on one hand and relaxation on the other. These shocks have been observed by Noordzij [11], see also [12] in bubbly mixtures confirming the existence of two sound speeds.

In Figure 4 these shocks are shown. The partly dispersed shock is indicated as B, the fully dispersed shock as C, whereas the type without relaxation effect is labelled with A.

A.

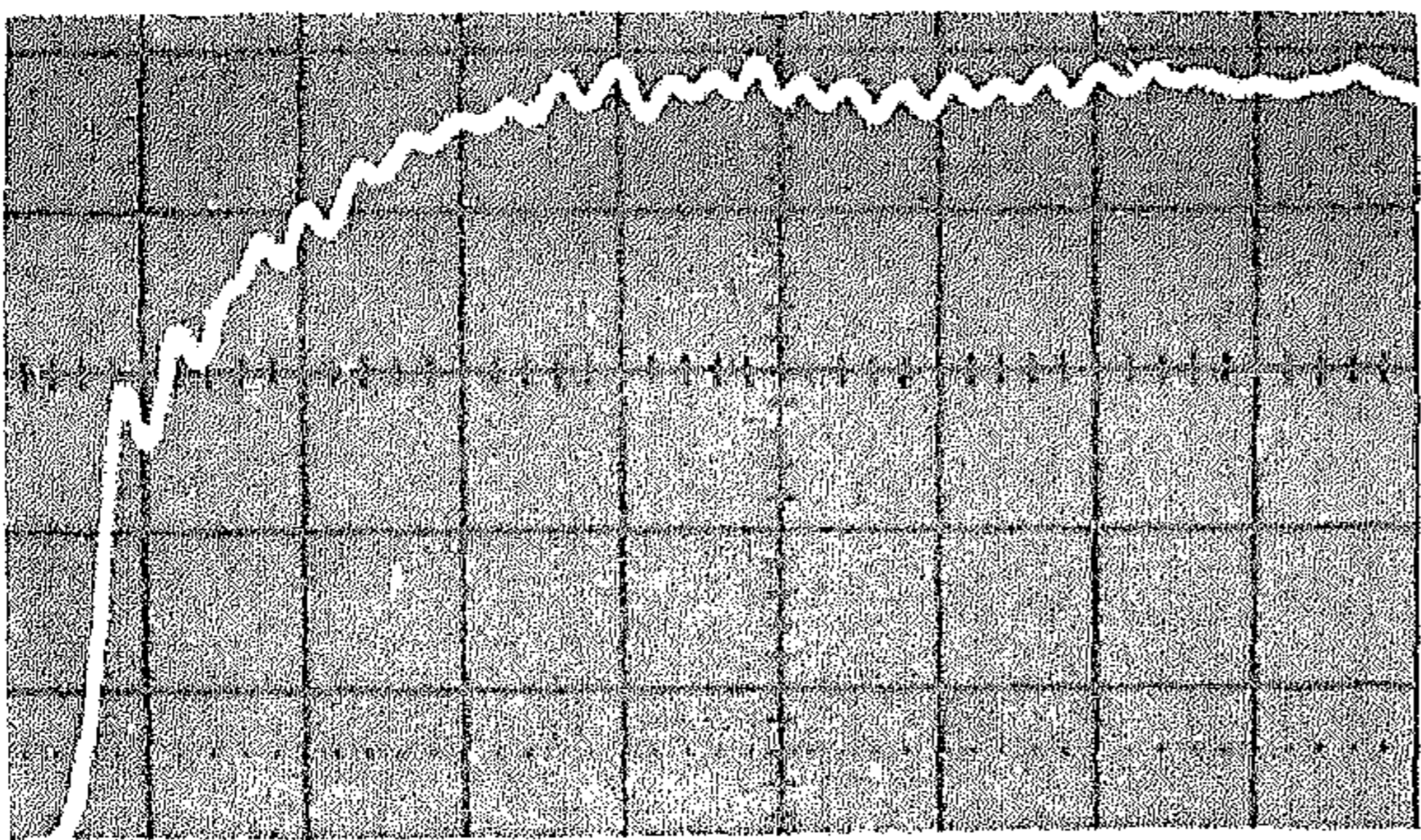
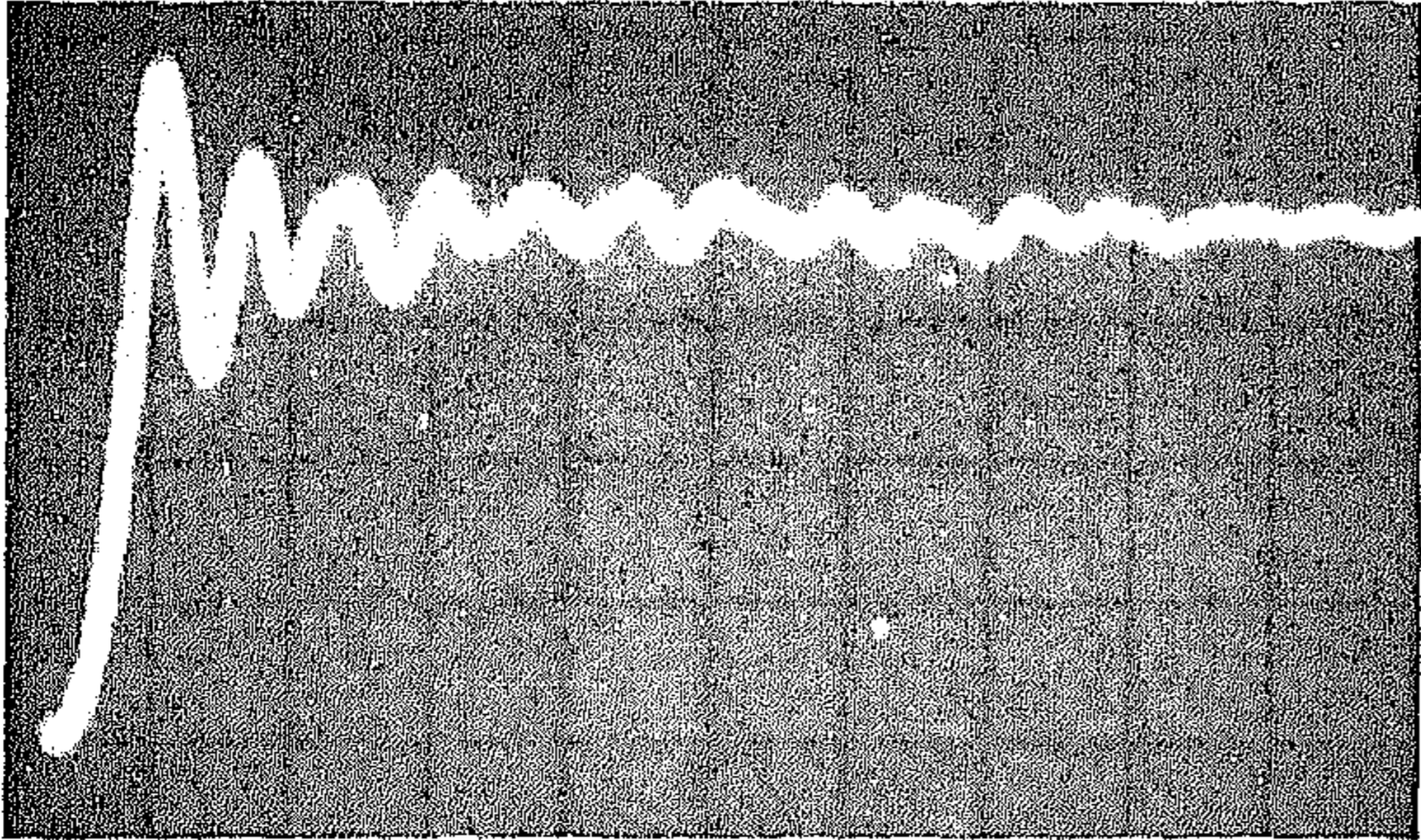


Figure 4:

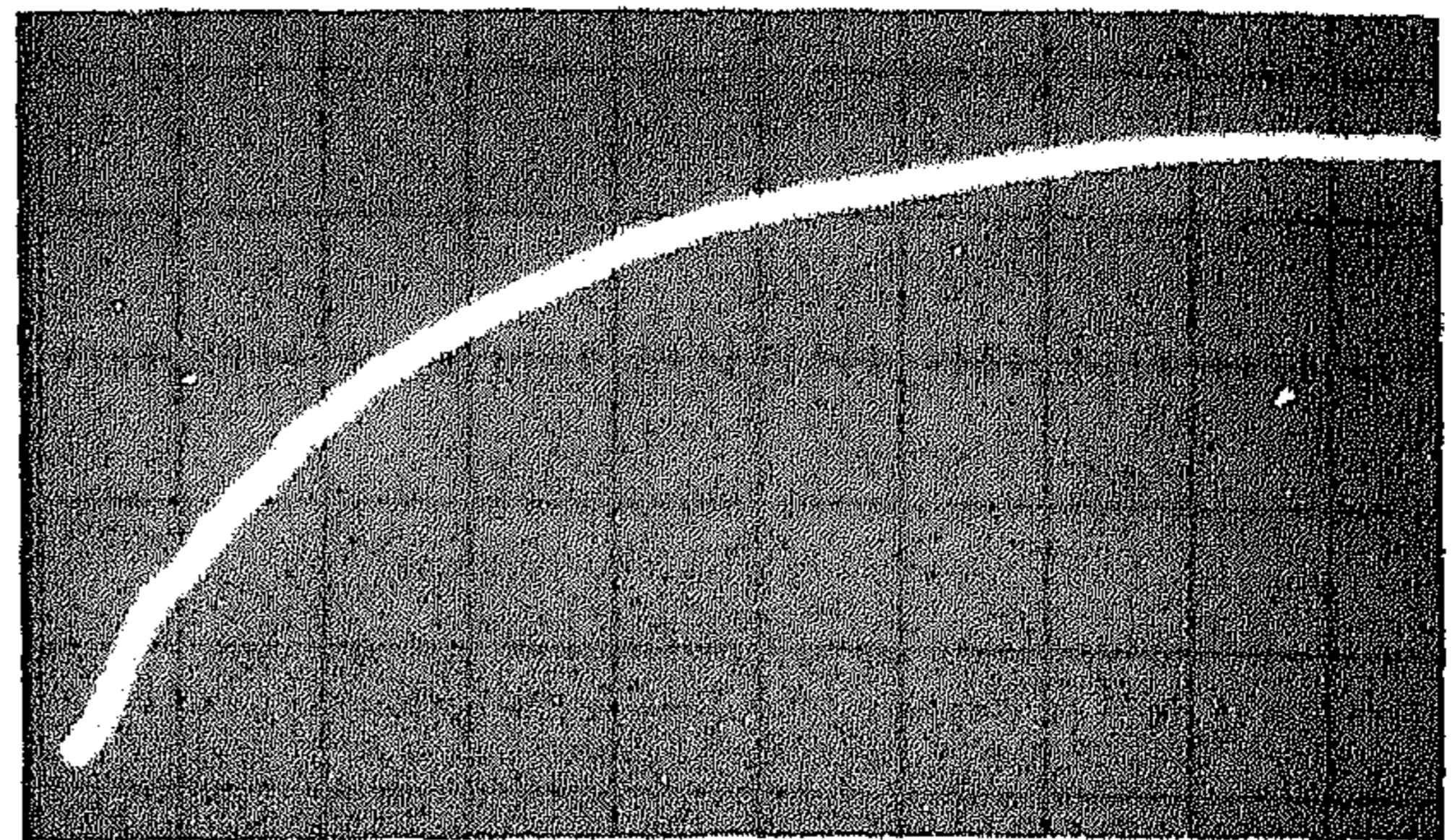
Three types of shocks in a bubbly flow, from [11].

In type A there is no relaxation.

B shows a partly dispersed shock.

C shows a fully dispersed shock.

C.



The oscillations present in type A and B but especially in A are due to bubble oscillations. These, rather than dissipation, determine the overall thickness of the type A shock. Both the analytical and the experimental results discussed in this section apply to disturbances in a two phase mixture which is at rest, so that there are no large relative velocities. For bubbly flow we may conclude that analytical predictions on acoustic waves and weak shock waves, are well supported by experimental observations.

Stratified flow.

We consider a situation as in Figure 5 and write down separate equations for each of the two phases.

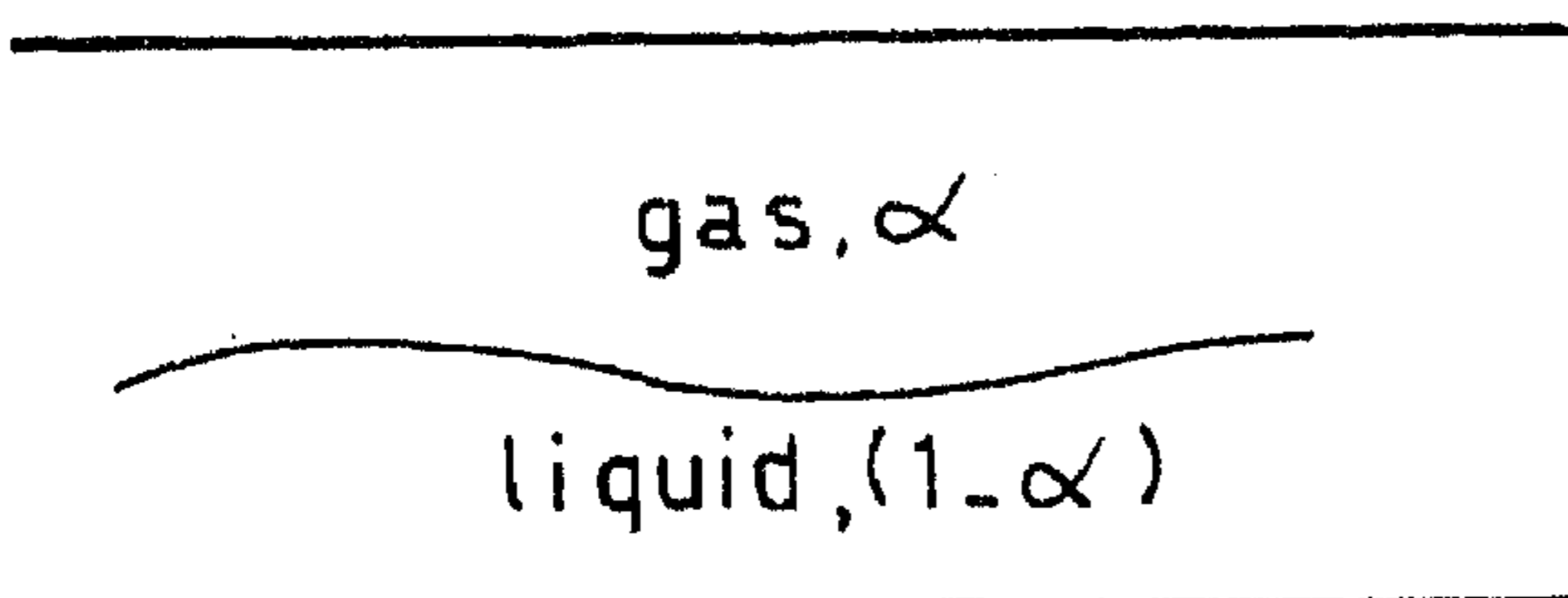


Figure 5: Two phase flow with separated phases with which in mind most model equations are formulated.

We first ignore viscous forces between liquid and gas, assume again that thermodynamic changes in the gas are isothermal and that the liquid is incompressible. When the equations for the behaviour of the disturbances of the interface have to be formulated, a two dimensional theory is in order. One has to prescribe the condition of equal pressure at the interface together with the kinematical condition. The equations for gas/liquid flows are usually restricted to one dimensional flow. The kinematical condition is equivalent to the continuity equation. Equal pressures are prescribed in both phases. One should expect that this is a reasonable approximation to the true two dimensional situation for wave lengths which are large with respect to the width of the duct. In the absence of frictional forces between the phases one cannot prescribe equal velocity disturbances in the two fluids, because fluids of different densities assume different velocities when subjected to the same acceleration. Denoting liquid and gas velocities with u and v respectively, we have,

$$\frac{\partial}{\partial t} \rho_1(1-\alpha) + \rho_1(1-\alpha) \frac{\partial u}{\partial x} = 0, \quad (2.5)$$

$$p \frac{\partial \alpha}{\partial t} + \alpha \frac{\partial p}{\partial t} + \alpha p \frac{\partial v}{\partial x} = 0, \quad (2.6)$$

$$\rho_1(1-\alpha)\frac{\partial u}{\partial t} + (1-\alpha)\frac{\partial p}{\partial x} = 0, \quad (2.7)$$

$$\rho_g\alpha\frac{\partial v}{\partial t} + \alpha\frac{\partial p}{\partial x} = 0. \quad (2.8)$$

There is no equation of state $p=p(\rho)$, so that we cannot define a sound velocity as $dp/d\rho$. We can however compute the characteristics of (2.5)-(2.8) since these give us the velocity with which discontinuities propagate. They are $dx/dt = 0$ and

$$\frac{dx}{dt} = \pm \left[\frac{p\{\alpha\rho_1 + (1-\alpha)\rho_g\}}{\alpha\rho_1\rho_g} \right]^{1/2}. \quad (2.9)$$

The exact two dimensional problem has been analysed by Morioka and Matsui [13]. They obtained the dispersion relation for the velocity of propagation of pressure waves of small amplitude. In the limit of long wave lengths their dispersion relation reduces, as was anticipated above, to (2.9). For $\rho_g/\rho_1 \ll 1$ the right hand side of (2.9) reduces to $\pm (p/\rho_g)^{1/2}$, which is the sound velocity c_g of the gas. I have not been able to find experiments pertaining to this situation, which could verify this result. It seems physically plausible that when the gas is not interrupted by liquid nor inhibited in its motion by that liquid, pressure disturbances travel approximately with the speed of sound in the gas. While (2.9) in stratified flow is the counterpart of (2.4) for bubbly flow, we may also for stratified flow look at the situation when viscous forces are so large at the interface that $u \approx v$. Then (2.5) and (2.6) give directly $(p\alpha)/(1-\alpha) = \text{constant}$, which is identical with (2.3) for bubbly flow. The sound speed is accordingly given by (2.1). We conclude therefore that in the limit of infinite viscous forces at the interface the stratified flow and bubbly flow models predict the same velocity of sound c . However the corresponding speeds for negligibly small viscous interaction are widely different.

3. Stratified flow model with relative velocities.

In many practical situations, for example the flow of a gas/liquid mixture through a Laval nozzle there is a significant difference between the velocities of gas and liquid. It is natural therefore that one has tried to extend the above considerations to flows in motion with different velocities. The most simplified version of the equations of motion, without transfer of mass and momentum between the phases, then is in a two fluid model (with stratified flow in mind)

$$\frac{\partial}{\partial t} \rho_1(1-\alpha) + \frac{\partial}{\partial x} \rho_1(1-\alpha)u = 0, \quad (3.1)$$

$$\frac{\partial}{\partial t} \rho_g\alpha + \frac{\partial}{\partial x} \rho_g\alpha v = 0, \quad (3.2)$$

$$\frac{\partial}{\partial t} \rho_1(1-\alpha)u + \frac{\partial}{\partial x} \rho_1(1-\alpha)u^2 + (1-\alpha)\frac{\partial p}{\partial x} = 0, \quad (3.3)$$

$$\frac{\partial}{\partial t} \rho_g\alpha v + \frac{\partial}{\partial x} \rho_g\alpha v^2 + \alpha\frac{\partial p}{\partial x} = 0, \quad (3.4)$$

$$\left(\frac{\partial p}{\partial \rho_g}\right) = c_g^2. \quad (3.5)$$

Equation (3.5) specifies ρ_g when p is known, ρ_1 may be considered as constant, so (3.1) - (3.4) form a set of 4 equations for the 4 unknown quantities p , α , u , and v . They are partial differential equations of first order and belong to the quasilinear type. Hence when at $t=0$ the values of p , α , u and v are given the subsequent development in space and time can be conveniently computed with the method of characteristics. With standard methods we find the characteristic directions λ of (3.1) - (3.4) to be given by

$$\{(\lambda-u)^2 - \frac{p}{\rho_1} \frac{1-\alpha}{\alpha}\} \{(\lambda-v)^2 - c_g^2\} = \frac{pc_g^2}{\rho_1} \frac{1-\alpha}{\alpha}. \quad (3.6)$$

This equation has two real and two complex roots, which means that the system (3.1)-(3.4) is not hyperbolic. Mathematically this means that the Cauchy problem for these equations is ill-posed. What this means can perhaps be illustrated in the best way with the example originally given by Hadamard (see Garabedian [14]). Consider the equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0,$$

which has imaginary characteristics. The solution satisfying the initial conditions

$$u(0,x) = 0, \\ u_t(0,x) = \frac{\partial u}{\partial t}(0,x) = n^{-1} \sin nx$$

is

$$u = n^{-2} \sinh nt \sin nx.$$

When $n \rightarrow \infty$, the value of $u(0,x)$ tends uniformly to zero whereas $u(t,x)$ oscillates between unbounded limits. The solution therefore does not depend continuously on the initial data. This is meant by "ill-posed". Many workers in two phase flow have found that the equations for the separated flow model for gas and liquids have complex characteristics. Also for more sophisticated versions of the equations than those given above, this is in general the case. It is mentioned in Wallis's book [7].

x

No such problems arise in the flow of dusty gases [15], [16].

More recently it was the subject of a lively discussion [4] during the 5th International Heat Transfer Conference at Tokyo in September 1974. In a detailed survey by Lyczowski et al [6] it was concluded that in general the characteristics for the equations of motion in the separated flow model are complex. Several proposals have been put forward to get around this situation, which of course makes numerical computation a hazardous enterprise. We mention here

i) An inspection of the effect of the various terms in (3.1) - (3.4) shows that in some cases [6], the characteristics become real when one writes in the pressure terms in (3.3) and (3.4) $\partial/\partial x(1-\alpha)p$ and $\partial/\partial x(\alpha p)$ instead of $(1-\alpha)\partial p/\partial x$ and $\alpha\partial p/\partial x$. There is however no physical justification for this modified terms. It is obvious that in the situation of Figure 5 the force on a slice dx of the liquid is $\alpha\partial p/\partial x$ and not $\partial/\partial x(\alpha p)$. Situations in which the latter term is the appropriate one are hard to imagine.

ii) One may extend the equations (3.1)-(3.4) with additional terms in the right hand sides to represent transfer of mass and momentum. If these additional terms do only include the dependent variables, the characteristics are unchanged. When they include first order derivatives the characteristics are modified and might become real. The view that such terms should be included in a proper set of equations is taken by the Belgian - French School as represented by Giot and Fritte [17] and Bouré et al. [18]. The equation would surely model the real flow better when transfer processes were included. A difficulty, however, is that till thus far one has only been able to do this in some empirical way, based on experience, insight, but invariably with unknown constants and more or less ad hoc introduced functional relationships. It is therefore open to discussion what the value is of equations which are necessarily very complicated and which contain so many more or less guessed terms. But even if transport processes could be represented more accurately, and research in this direction is badly needed, it seems unlikely that real characteristics are obtained in general. Therefore, in spite of the incompleteness of equations like (3.1) - (3.4), we may worry about its complex characteristics.

iii) Although they were originally not developed in connection with the present problem, one might mention here some models in which no complex characteristics occur. These are the so called "drift models", in which a relation is assumed between the ratio of the velocities of the two phases and one or more of the other dependent variables. Examples are the models proposed by Fauske [19] and Moody [20], especially in relation with critical flow in the throat of a converging - diverging nozzle (Laval nozzle). The first takes at critical flow (u/v) proportional to $(\rho_g/\rho_l)^{1/2}$, the second to $(\rho_g/\rho_l)^{1/3}$.

The physical basis for these assumptions is rather shaky and they have, as Prof. Wallis told the participants in a Workshop on Two Phase Flows recently, been repudiated even by their authors. All the available evidence, indicates that the characteristics of the separated flow model are in general complex. The question arises: Why is this so? It has been suggested during the 1974 Round Table Discussion at Tokyo [4] that it has to do with Helmholtz instability. This seems to be probable indeed. The classical problem studied by Helmholtz [21] is the flow of two immiscible, inviscid, incompressible liquids with different velocities at the two sides of the interface. It turns out that in the absence of gravity and other forces, the interface is unstable with respect to disturbances of any wavelength. For long wave lengths the problem considered by Helmholtz becomes, as pointed out earlier approximately the one dimensional problem formulated in the separated flow model. For short wavelengths the behaviour in the two situations need not be the same, because the dispersion which occurs in the true two or threedimensional situation is absent in the one dimensional approximation. It is plausible therefore that the complex characteristics reflect the instability of the interface against disturbances of long wave lengths. Gravity, viscosity, surface tension and other physical mechanisms certainly affect the instability but it is not likely that they may prevent instability under all circumstances.

4. Bubbly flow with relative velocity.

Also in the case of bubbly flow with relative velocities between the phases, we have a "two fluid" model. The distinction with respect to the model of the previous section is that it is possible, at least in the simple case of spherical bubbles, to represent more specifically some of the physics of the dynamic interaction between the gas and liquid. With a view to the different velocities v of the gas and u of the liquid, we have to distinguish between the material derivative pertaining to the gas and that pertaining to the liquid. We define therefore

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x}, \quad (4.1)$$

and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v \frac{\partial}{\partial x}. \quad (4.2)$$

Neglecting the contribution of the gas to the mass and momentum flow we have

$$\frac{d}{dt} (1-\alpha) + (1-\alpha) \frac{\partial u}{\partial x} = 0, \quad (4.3)$$

$$\frac{du}{dt} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0. \quad (4.4)$$

When the bubbles do not break up, coalesce or dissolve, we have for their number density n

$$\frac{Dn}{Dt} + n \frac{\partial v}{\partial x} = 0, \quad (4.5)$$

which equation, together with the isothermal assumption $p\tau = \text{const.}$, yields

$$\frac{D}{Dt} p\alpha + p\alpha \frac{\partial v}{\partial x} = 0. \quad (4.6)$$

Mass and momentum conservation for the mean liquid velocity is given by (4.3) and (4.4), whereas (4.6) may be interpreted as the mass conservation equation for the gas. To complete the set of equations for p , α , v and u , we try to formulate the dynamic interaction between bubbles and liquid. First we ignore frictional forces altogether and consider the motion of a sphere in an incompressible inviscid liquid. At first sight one might think that this is largely covered in the textbooks of classical hydrodynamics. Landau and Lifshitz [22], for example, write as equation for the dynamic interaction

$$\frac{D}{Dt} m(v-u) = \rho_1 \tau \frac{Du}{Dt}, \quad (4.7)$$

where m is the virtual mass, and where the mass of the sphere is neglected. We have written $\frac{D}{Dt}$ in order not to attach the meaning either of (4.1) or (4.2) to this time derivative. As long as u is uniform in space and depends only on time there is no ambiguity in the interpretation of (4.7), D/Dt is the same as either d/dt or D/Dt . As soon as u is a function of both t and x , the textbooks leave you alone and you have to find out for yourself. I have heard convincing arguments for each of the forms which you obtain by interpreting Du/Dt in (4.7) either as du/dt or Du/Dt . The problem is far from trivial however and time does not permit to reproduce all the aspects here. Without pretending that this is the last word about this fascinating problem, I will stick here to the form adopted in Prosperetti and Van Wijngaarden [23], in which D/Dt is D/Dt at both sides of (4.7). The argument for this boils down roughly speaking to this: The liquid motion is incompressible and governed by Laplace's equation for the potential. This does not contain the time which enters through the momentary value of u and v alone. There is no time scale in the relative motion and this is the same therefore as when the particular relative motion was instantaneously started from rest. When a sphere is placed with centre in x_0 and the liquid is accelerated to motion with potential ϕ in the absence of the sphere, the impulsive force exerted on the sphere is $\rho_1 (\nabla \phi)_{x=x_0} \tau$ (Van Wijngaarden [24]), or $\rho_1 u \tau$,

u being the velocity which the undisturbed velocity would have in x_0 . Since the sphere is massless it cannot support any force and starts to move instantaneously with such a velocity that the impulsive force generated by the relative motion $-m(v-u)$, cancels the impulsive force $\rho_1 u \tau$.

This must be the case at any time whence at all times the rate of change

$$\frac{D}{Dt} m(v-u)$$

of the impulse of the relative motion must be equal to the rate of change $\frac{D}{Dt} \rho_1 u \tau$.

Apart from this interaction there is a viscous force. If the Reynolds number of the relative flow is large enough and if there are no surface active agents in the liquid the flow remains approximately described by the potential flow around the bubble. The drag, in steady flow is given by Levich [22], [25] as $-12\pi\mu a(v-u)$, obtained from calculation of the dissipation in the (potential) flow. Since this is instantaneously generated it might seem that this expression for the drag holds good in unsteady flow as well. This is not so however. The potential flow has a nonvanishing shear stress at the interface with the bubble, whereas the stress at this interface should be zero. There is accordingly a boundary layer for the stress and the diffusion of the initial discontinuity in the stress must give rise to a unsteady resistance term. This term has been recently calculated by Chen [26] for impulsively started flow. The timevarying part of the drag is as shown in Figure 5^b relatively small at a Reynolds number of order 100 and we leave it out therefore in the present discussion. The equation for the relative motion becomes in this way

$$\frac{D}{Dt} m(v-u) = \rho_1 \tau \frac{Du}{Dt} - 12\pi\mu a(v-u). \quad (4.8)$$

In an unbounded liquid the virtual mass of a sphere is $\frac{1}{2}\rho_1 \tau$, which we shall adopt here, postponing a discussion of the effect of neighbouring bubbles on m to section 5. With this value for m it follows from (4.8) that for a rigid massless sphere $v=3u$ in the absence of viscosity, a result we used earlier in discussing the meaning of c_f (cf. paragraph following eqn. 2.4).

When viscosity is included there appears to be a characteristic time T , given by

$$T = \frac{18a^2}{v}, \quad (4.9)$$

which it takes the liquid to slow down a faster moving bubble. Together with (4.4) and (4.5), the above relation (4.8) gives

$$\frac{D}{Dt} \alpha(v-u) + \alpha(v-u) \frac{\partial v}{\partial x} = -2\alpha(v-u) \frac{\partial u}{\partial x} = -\frac{2\alpha}{\rho_1(1-\alpha)} \frac{\partial p}{\partial x} - \frac{\alpha(v-u)}{T}. \quad (4.10)$$

The equations for our two fluid model now are (4.3), (4.4), (4.6) and (4.10) for the dependent variables α , p , v and u . It is useful to look once more at the analogy with gasdynamic flow with a chemical reaction. A lucid analysis of that type of flow has been given by Broer in [10].

The flow is governed by the usual equations for conservation of mass momentum and energy. In addition there is an equation which describes the behaviour of q , a parameter that determines the chemical composition. Broer [10] uses as this rate equation

$$\frac{dq}{dt} = -\frac{q-q_0}{T} \quad (4.11)$$

Here q_0 is the equilibrium value of q for given pressure and density. Substituting solutions of the form $\exp(ikx - \omega t)$ in the linearized equations, two sound speeds are found, the equilibrium sound speed and the frozen sound speed. We mentioned earlier the analogy between c_0 and c_f in (2.1) and (2.4) on one hand and these two speeds at the other. We may, pursuing the analogy further, compare (4.10) with the rate equation (4.11). The reaction parameter is the gas velocity v , the equilibrium value of v is the liquid velocity u . In the gasdynamic problem the characteristics are for finite T , $u + c_f$, u being the gas velocity. With a view to the analogy we may expect the "frozen" sound speed c_f (in 2.4) to appear in the characteristics directions of the system (4.3), (4.4), (4.6) and (4.10). As Prosperetti and Van Wijngaarden [23] found, this is the case indeed. Moreover they found all four characteristics to be real, in contrast to the situation in stratified flow, which was discussed in the previous section. For small values of $(v-u)/c_f$, the characteristics can be obtained analytically and are

$$\frac{dx}{dt} = v, \quad \frac{dx}{dt} = v - \frac{\alpha\beta}{\delta} (v-u) \quad (4.12)$$

$$\frac{dx}{dt} = \frac{1}{2} (v+u) + \frac{1}{2} \frac{\alpha\beta}{\delta} (v-u) \pm c_f.$$

In these relations β and δ are of unit order and depend weakly on α . It is interesting to make a further simplification and consider the case where α is small. Then the characteristics are v , and $\frac{1}{2} (v+u) + c_f$. In nonequilibrium gasdynamics the characteristics are the particle velocity v and the directions $v + c_f$. We could expect, as outlined earlier, the occurrence of c_f in the characteristics for bubbly flow but it comes as a bit of surprise that the convective part is for small α , just the arithmetic mean of u and v . Calculations carried out at CalTech by Lees and Kubota (private communication) have shown that in case Du/Dt in (4.7) is interpreted as du/dt the characteristics corresponding with the ones in the first line in (4.12) become complex. There is therefore good reason to try and find out what the correct form of (4.7) is, in other words what the correct form of the equation of motion of a body in a nonuniform accelerated flow is.

Comparison with experiment.

A good experiment to verify some of the predictions of a two fluid theory for bubbly mixtures, is the flow through a de Laval nozzle. A few experiments of this type have been reported [27], [28]. The work by Muir and Eichhorn [27] gives the most detailed information and has been used therefore more than once by others as a yardstick for theoretical results. These are concerned mostly with one aspect of the dynamics with the neglect of others, and therefore comparison with experiment should be done with caution.

Muir and Eichhorn measured void fraction, pressure, bubble velocity and liquid velocity along their converging diverging nozzle.

Figure 6 shows the observed pressure in the throat as a function of the void fraction in the reservoir, together with the result following from homogeneous fluid theory ($v=u$). Apparently the latter is lower. There are two effects in the real flow which could cause this difference, the relative radial velocity of the bubbles and the relative translational velocity.

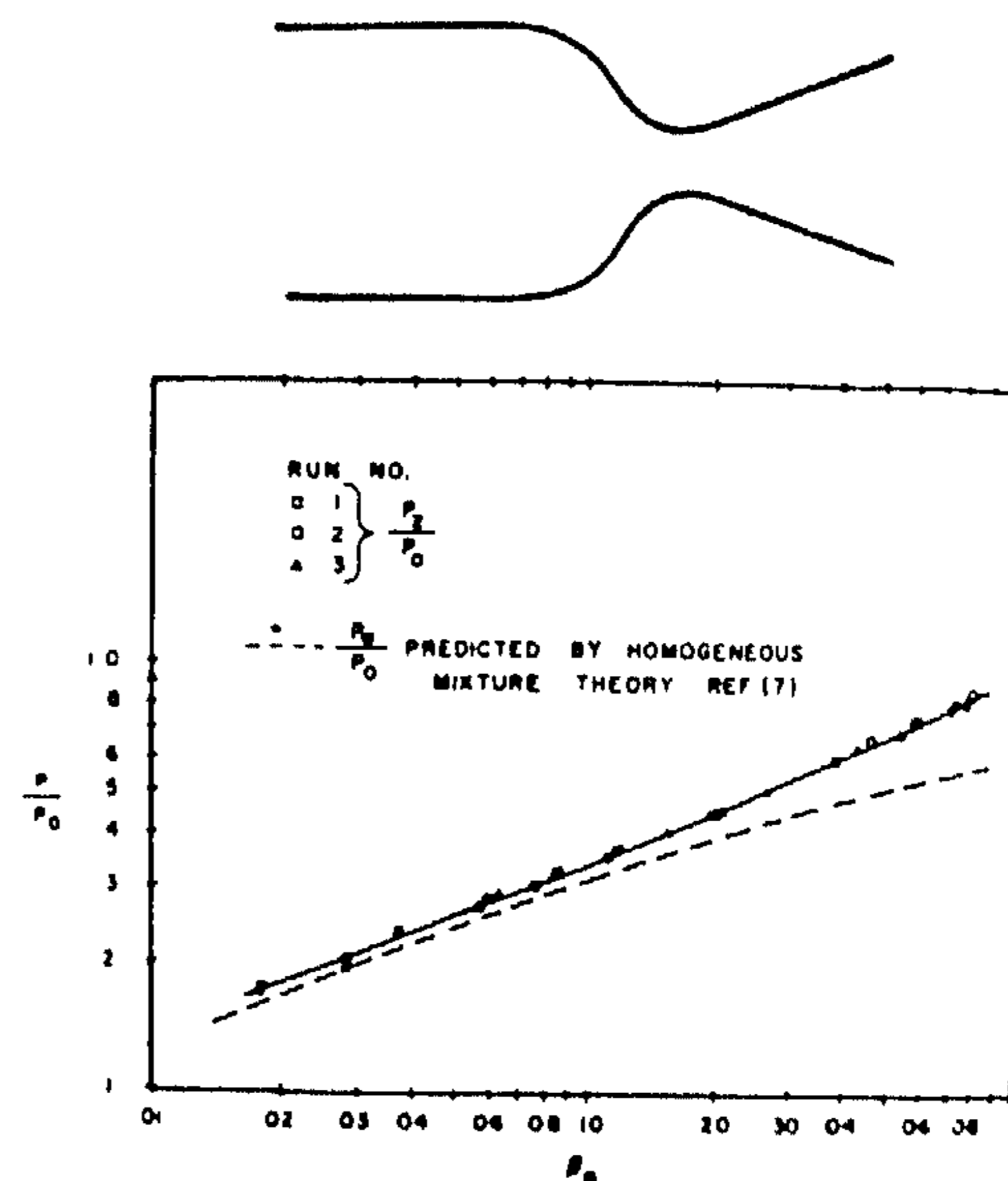


Figure 6. Bubbly flow through a Laval nozzle: Pressure in throat as measured by Muir and Eichhorn and as following from homogeneous theory. β_0 is void fraction in reservoir.

Baum [28] attributed the difference to relative radial velocity (dispersion). His point is that since the time necessary for a bubble to pass the converging - diverging part of the nozzle is comparable to the natural period of oscillation of a bubble, the pressure in the bubble cannot follow immediately the pressure of the liquid.

There may also be, however, an effect of the relative translational motion. This has not been investigated. There is certainly a need for more experiments of this type. Figure 7 shows another result from the paper by Muir and Eichhorn [27].

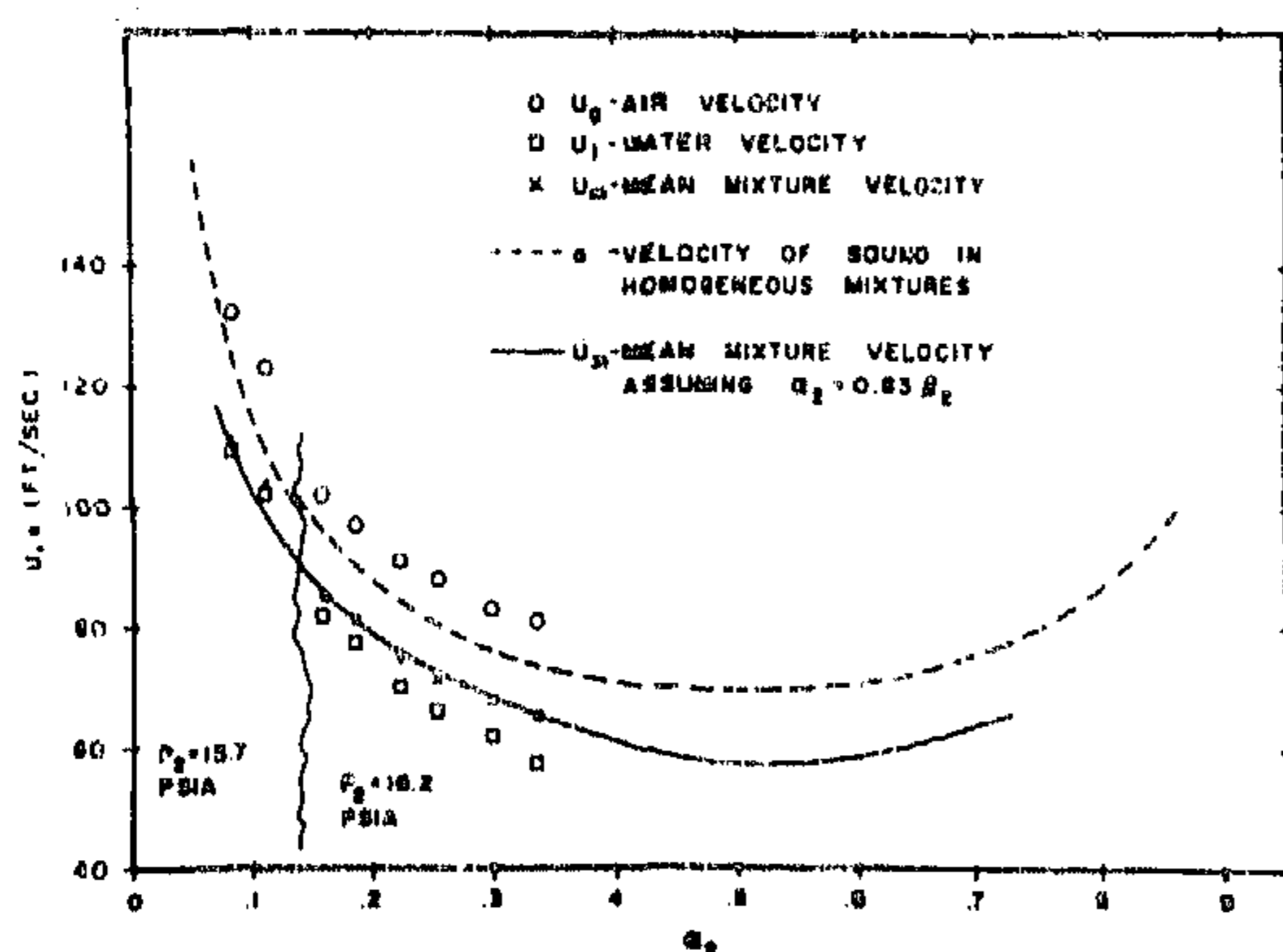


Figure 7. Liquid velocity, gas velocity and sound velocity measured by Muir and Eichhorn in throat of Laval nozzle. α_2 on the horizontal scale is the void fraction in the throat.

Shown are experimentally determined values of α , p , v , u and c_0 in the throat of the nozzle, under circumstances when supersonic flow exists in the diverging part. Under critical conditions the characteristic velocity in the throat is zero and it is therefore tempting to compare these results with what is predicted by the characteristics. They should give reasonably accurate results, when dispersion effects are weak. Prosperetti and I [23] could not resist this temptation. Under critical conditions ($dx/dt = 0$) the following relationship is given by the theory

$$c_f^2 = \delta c_0^2 = u^2 \left\{ 1 + (v-u) \frac{1 + \frac{\alpha\beta}{\delta}}{v - \frac{\alpha\beta}{\delta} (v-u)} \right\}$$

Upon inserting in this relation the measured value of α , v and u , a c_0 results. This calculated value was compared in [23] with the experimentally obtained one, that is to say the value which follows from (2.1) for the measured p and α , with $\gamma = 1$. The relative difference between measured and calculated c_0 is of the order of 10% which is quite satisfactory. In conclusion of this section we might say that the two fluid model of bubbly flow has been developed further than the corresponding one for stratified flow. Though for bubbly flow there are not many experimental results available, there are some at least. For stratified flow reliable experimental data are virtually absent.

5. Virtual mass and pressure in a two phase fluid.

In the final section of this lecture I want to discuss two more problems which have to do with the understanding of two phase flows with relative motion. The first is about the virtual mass. In equation (4.8) we used the virtual mass $\frac{1}{2}\rho_l \tau$ of a sphere in an unbounded liquid. It is to be expected that a particle when accelerated in the presence of many other particles, will possess a virtual mass which is different from the value in an infinite liquid. In the formulation of equations for liquid-gas or solid-liquid mixtures the virtual mass plays an important rôle and it is therefore of some interest to investigate the effect of boundaries in the flow, in particular those formed by neighbouring particles. Consider, as a simple example a sphere (radius a) which starts to move with velocity v in the centre of a larger sphere (radius b), filled with liquid, Figure 8.

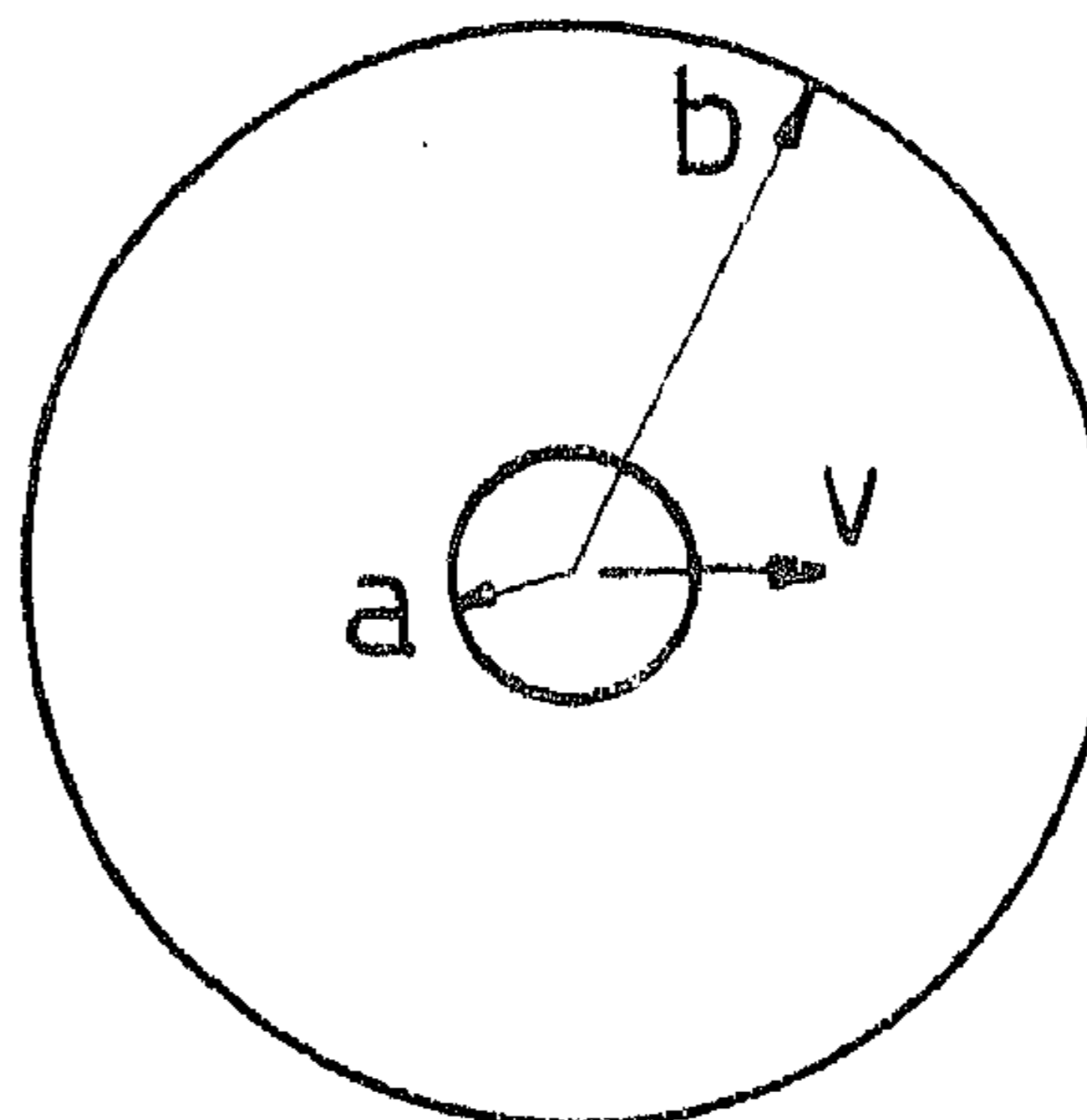


Figure 8: A sphere of radius a is accelerated to velocity v in the centre of a sphere, with radius b , filled with liquid.

In an unbounded liquid the kinetic energy of the liquid is, τ being the volume of the test sphere

$$T = \frac{1}{2} \rho_l \tau v^2$$

whence the virtual mass is $m_0 = \frac{1}{2}\rho_l \tau$. In the enclosure of the sphere with radius b , the kinetic energy associated with the motion of the small sphere is [29]

$$T = \frac{1}{2} \rho_l v^2 \tau \left(\frac{2a^3 + b^3}{b^3 - a^3} \right)$$

The apparent inertia of the small sphere is now $m_0 (2a^3 + b^3) / (b^3 - a^3)$. The "concentration" by volume α of the small sphere in the big one is a^3/b^3 , whence

$$m = m_0 \frac{1+2\alpha}{1-\alpha} \quad (5.1)$$

This value was used by Zuber [30] as an estimate for the virtual mass of spherical particles suspended in a liquid with density ρ . Since it is difficult to see how a large sphere approximates the effect of neighbouring spheres in a suspension, not much can be said at this point of the accuracy of (5.1) for a suspension of spherical particles. Let us consider the potential for the flow depicted in Figure 8. With $\alpha = a^3/b^3$ this

is, see 29

$$\phi = -\frac{va^3 \cos\theta}{2r^2} - \frac{\alpha v}{1-\alpha} \left(r \cos\theta + \frac{a^3 \cos\theta}{2r^2} \right). \quad (5.2)$$

The first term on the right hand side is the familiar potential for a sphere moving with velocity v in an unbounded liquid. The sphere displaces liquid and in order to maintain zero volume flow there is a back flow in the liquid with average velocity $-v\alpha/(1-\alpha)$. The potential of this uniform backflow is given by the second term on the right hand side of (5.2). The third term represents the additional dipole induced by the backflow in the centre of the small sphere. It happens that (5.2) satisfies also the boundary condition on the large sphere but this would not be so when the centre of the small sphere were not coinciding with that of the large sphere. The virtual mass m can be found either by calculation of the kinetic energy of the liquid or by calculation of the impulsive force which is necessary to generate the motion from rest and which is equal to $m v$. By making use of Bernoulli's Theorem we obtain

$$m v = \int \phi \, dA = v \tau \left(\frac{1}{2} + \frac{3}{2} \frac{\alpha}{1-\alpha} \right),$$

where dA is a surface element on the sphere. From the above relation (5.1) is obtained again. When used for a suspension of spheres in a liquid, (5.1) involves the neglect of the images of the test sphere in all other spheres. Otherwise said, only the boundary condition on the test sphere is satisfied, not the boundary condition on neighbouring spheres. To satisfy the latter, the interaction between spheres in a potential flow has to be calculated. Till thus far this is only possible for two spheres. This allows the calculation of the virtual mass of a sphere in a dilute mixture, correct in the first order of α . For interactions involving more than two particles contribute to order α^2 . When we have a well stirred suspension consisting of spherical bubbles of the same radius, the calculation of the interaction leads for the virtual mass to (Van Wijngaarden [24])

$$m = m_0 (1 + 2.78\alpha) + O(\alpha^2). \quad (5.3)$$

To the same accuracy (5.1) gives $m = m_0 (1 + 3\alpha)$.

This means that the estimate (5.1) is not bad, which is due to the fact that numerically the effect of the images of the test sphere in other spheres is apparently small with respect to the displacement effect involved in (5.1). The relation (5.3) was used in 23 for the calculation of the critical velocity in a converging - diverging nozzle. In a real mixture of liquid and gas bubbles, the bubbles are not perfectly spherical, nor are they all of the same size. It would be desirable therefore to extend theoretical work on interactions to nonspherical particles or bubbles and also to investigate for, e.g. spherical particles, the effect of a distribution of different radii. In the latter case a severe complication is the change in the probability distribution due to relative velocity between the phases.

The second problem of this section concerns the concept of pressure in a gas - liquid flow. In the equations of motion discussed in section 4, the pressure p_g in the gaseous phase is taken equal to the average pressure p in the liquid. In some loose way one feels that they are almost equal but, of course, that is not enough when one wants to construct reliable equations of motion. For a bubbly flow, the effect on the stress distribution caused by the bubbles is known (see Batchelor [31]) when there is no appreciable relative translational motion. The mixture may be considered as a homogeneous Newtonian fluid with a shear viscosity $\mu = \mu_l (1 + \alpha)$ and a bulk viscosity $4\mu_l/3\alpha$ ($\alpha \neq 0$). The pressure in the liquid has a unique meaning and can, if necessary, be related to the pressure inside the gas bubbles. When the phases have appreciable relative velocity, the situation is not clear at all and here is another problem which needs to be investigated. Apart from viscous effects, inertial effects are a cause for pressure differences between liquid and gas. To indicate of what kind these effects are, let us consider a well stirred bubbly flow, with average gas velocity \underline{v} , average liquid velocity \underline{U}_1 . The volume flow is supposed to be known

$$\underline{U}_0 = (1-\alpha)\underline{U}_1 + \alpha \underline{v}. \quad (5.4)$$

Again we assume that the relative motion can, approximately, be described by potential theory.

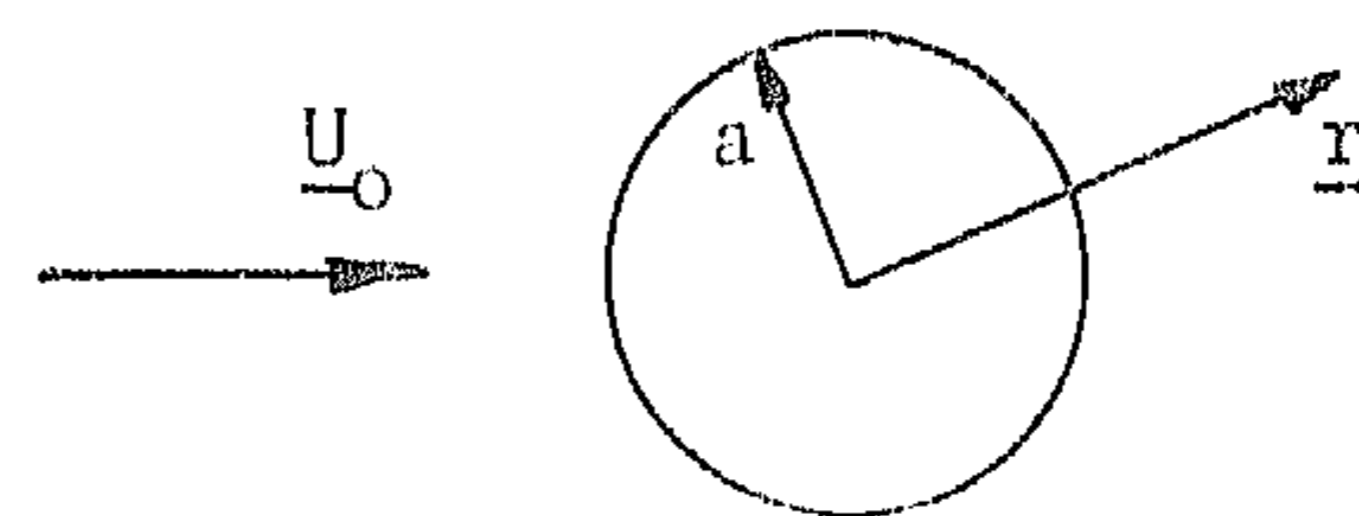


Figure 9.

In practice the pressure is usually measured with pressure transducers mounted in the wall of the duct in which the flow occurs. The quantity measured is the average pressure $\langle p \rangle$ in the liquid, which enters as p in the momentum equation.

The relation between $\langle p \rangle$ and p_g is of the form, apart from viscous effects,

$$\langle p \rangle = p_g - \frac{2\sigma}{a} + \rho_1 (\underline{v} - \underline{U}_0)^2 \{k_1 + k_2 \alpha\}, \quad (5.5)$$

where a is a representative bubble radius and k_1, k_2 are constants of unit order. This relation can be made plausible for spherical bubbles at small α . In a steady flow the average pressure $\langle p \rangle$ can be related to the potential ϕ , describing the flow relative to \underline{U}_0 , by

$$\langle p \rangle = c - \frac{1}{2} \rho_1 \langle (\nabla \phi - \underline{U}_0)^2 \rangle, \quad (5.6)$$

where c is the Bernoulli constant. At small α the averaging can be restricted to considering just one bubble in each realisation of the mixture. Then ϕ can be simplified to the potential associated with the motion of a sphere moving with relative velocity $\underline{v} - \underline{U}_0$,

$$\nabla \phi - \underline{U}_0 = \nabla \phi = \nabla \left\{ \frac{(\underline{v} - \underline{U}_0) \cdot \underline{a} r^3}{2r^2} \right\}$$

Carrying out the averaging over all space occupied by liquid, we obtain

$$\langle p \rangle = c - \frac{1}{2} \alpha \rho_1 (\underline{v} - \underline{U}_0)^2. \quad (5.7)$$

The Bernoulli constant can be, approximately, determined from the condition that at the interface of gas and liquid the pressure p must equal p_g , apart from surface tension. This means that averaged over the surface of a bubble

$$\langle p \rangle_{\text{surf}} = p_g - \frac{2\sigma}{a}. \quad (5.8)$$

The pressure p on the surface follows, again approximating $\nabla \phi - \underline{U}_0$ by $\nabla \phi$, from

$$p + \frac{1}{2} \rho_1 \{ \nabla \phi - (\underline{v} - \underline{U}_0) \}^2 = c + \frac{1}{2} \rho_1 (\underline{v} - \underline{U}_0)^2.$$

Averaging over the surface of a sphere gives with (5.8)

$$c = p_g - \frac{2\sigma}{a} + \frac{1}{4} \rho_1 (\underline{v} - \underline{U}_0)^2,$$

or with help of (5.4) and (5.7)

$$\langle p \rangle = p_g - \frac{2\sigma}{a} + \frac{1}{4} \rho_1 (\underline{v} - \underline{U}_0)^2 - \alpha \rho_1 (\underline{v} - \underline{U}_0)^2,$$

which is of the form (5.5). There are two effects: For a single bubble the pressure "at infinity" is larger than $(p_g - 2\sigma/a)$. This gives rise to the term with k_1 , which is $\frac{1}{4}$ for a sphere but much less for oblate shapes. The second effect is that the average pressure is less than the pressure "at infinity" for a single bubble.

This order of magnitude analysis, rough as it may be, shows that unless $\rho_1 (\underline{v} - \underline{U}_0)^2 / p_g$ is small, it is not permitted to identify the average pressure in the liquid with the gas pressure.

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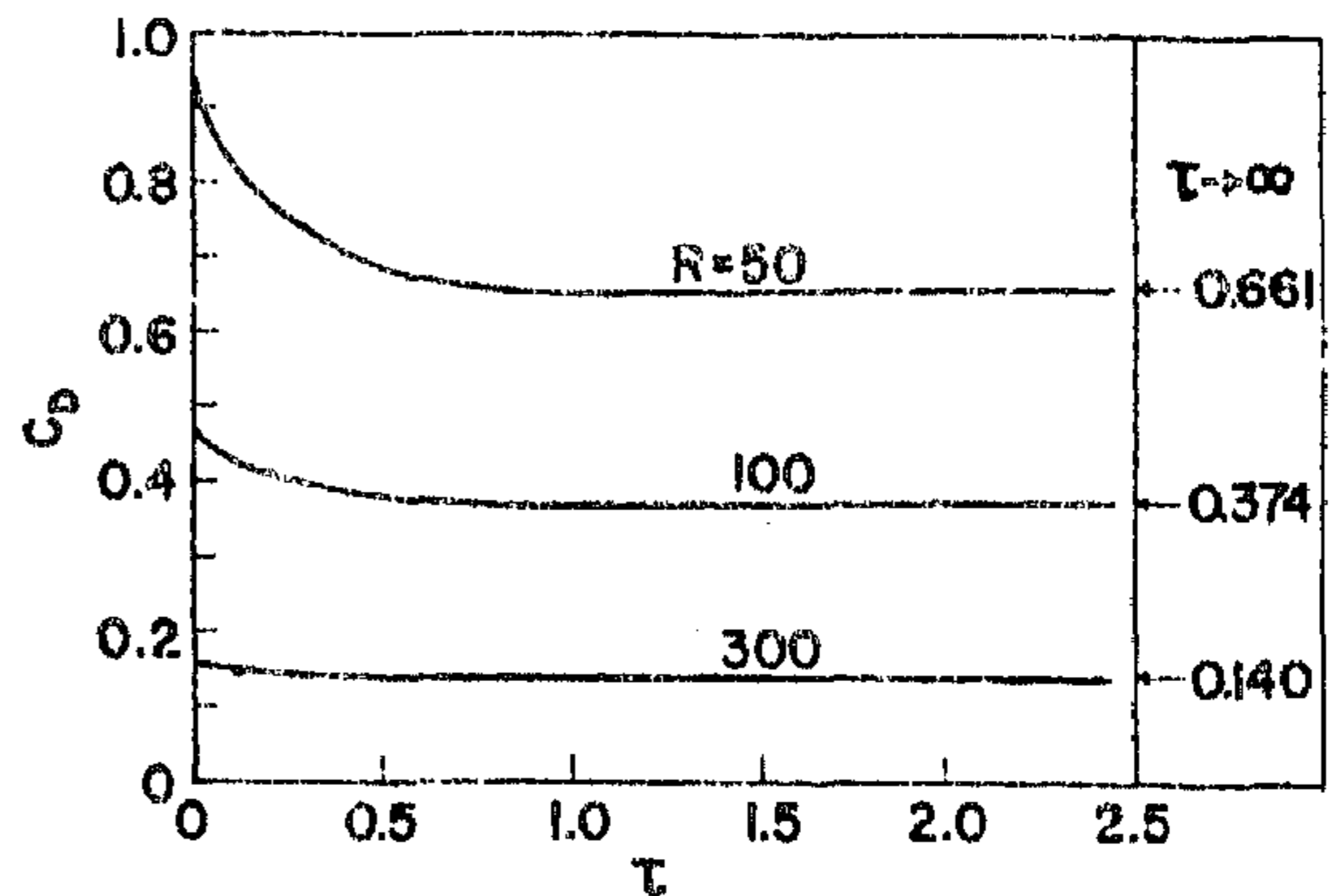


Figure 5^b: Variation of drag coefficient with dimensionless time for flow past spherical gas bubble starting impulsively from rest. From Chen [26].