# SOME PROBLEMS OF DIOPHANTINE APPROXIMATION: AN ADDITIONAL NOTE ON THE TRIGONOMETRICAL SERIES ASSOCIATED WITH THE ELLIPTIC THETA-FUNCTIONS.

By

G. H. HARDY and J. E. LITTLEWOOD

New College, Oxford. Trinity College, Cambridge.

1. In this note we give an alternative and more instructive proof of the fundamental theorem on which our earlier researches in this field[1] were based. The theorem may be stated as follows.

**Theorem A.** *Suppose that*

$$0<x<1, \quad 0\leqq\theta\leqq 1, \quad \omega>1 \tag{1.1}$$

*and*

$$s(\omega)=s(\omega, x, \theta)=\sum_{0\leqq n^2\leqq\omega} e^{-n^2\pi i x}\cos 2n\pi\theta. \tag{1.2}$$

*Then*

$$s(\omega, x, \theta)-\frac{e^{-\frac{1}{4}\pi i}}{\sqrt{x}}e^{\frac{\pi i\theta^2}{x}}s\left(x^2\omega, -\frac{1}{x}, \frac{\theta}{x}\right)=O\left(\frac{1}{\sqrt{x}}\right) \tag{1.3}$$

*uniformly in $\omega$ and $\theta$.*[2]

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[1] G. H. Hardy and J. E. Littlewood, 'Some problems of Diophantine Approximation', *Acta mathematica,* 37 (1914), 193—238, and *Proc. Cambridge Phil. Soc.,* 21 (1923), 1—5.

[2] That is to say, the absolute value of the left hand side is less than $Ax^{-\frac{1}{2}}$, where $A$ is an absolute constant.

Our earlier proof, which followed the classical lines of the calculus of residues, as exposed in Lindelöf's book[1], was fairly straightforward, but very long. Two other proofs have been given recently by van der Corput.[2] The proof which we give here proceeds on lines different from any of these, and seems to us in some ways the most natural. It has also the advantage of being applicable, in principle at any rate, to the sums associated with any power of a a theta-function, such as the sum

$$\sum_{0\leqq n\leqq \omega} r(n)e^{-n\pi i x},$$

where $r(n)$ is the number of representations of $n$ as a sum of two squares.

The proof which we give here owes very much of its comparative simplicity to the criticism of Mr. A. E. Ingham, to whom we submitted our original version. In particular Mr. Ingham pointed out to us the usefulness of the elementary identity (5. 2), and we have rewritten the whole of §§ 5—7 in accordance with his suggestions.

2. We begin by showing that we may assume certain supplementary hypotheses without prejudice to the generality of the theorem.

In the first place, since we are aiming at a result which holds uniformly in $\theta$, we may suppose that $0<\theta<1$. The result for $\theta=0$ or $\theta=1$ will then follow by continuity.

Next, we may suppose that

$$\lambda=\sqrt{\omega} \tag{2. 1}$$

is excluded from each of the sets of intervals

$$(i_m) \quad \frac{m-\theta}{x}-\frac{\delta}{\sqrt{x}}\leqq\lambda\leqq\frac{m-\theta}{x}+\frac{\delta}{\sqrt{x}}, \tag{2. 21}$$

$$(j_m) \quad \frac{m+\theta}{x}-\frac{\delta}{\sqrt{x}}\leqq\lambda\leqq\frac{m+\theta}{x}+\frac{\delta}{\sqrt{x}}, \tag{2. 22}$$

[1] E. Lindelöf, *Le calcul des résidus*, 1905.

[2] J. G. van der Corput, 'Über Summen, die mit den elliptischen $\vartheta$-Funktionen zusammenhängen', *Math. Annalen*, 87 (1922), 66—77, and 90 (1923), 1—18. van der Corput proves a good deal more than is asserted by the theorem, and his proofs appear for this reason to be more elaborate than they are. The first proof is based on the theory of Fourier series, while the second follows lines more like those of our original proof, on which it is (when reduced to its simplest terms) a considerable improvement.

where $\delta$ is an appropriate positive constant. Here $m = 0, 1, 2, \ldots$, and any negative part of any interval is to be discarded as irrelevant.

To prove this, consider the interval $I$ defined by

$$0 \leqq M < \lambda < M + \left[\frac{1}{\sqrt{x}}\right] = M + \mu.$$

*If $\delta$ is small enough, this interval will necessarily include a $\lambda$ external to the $i$'s and $j$'s.* For each $i$ is of length $\frac{2\delta}{\sqrt{x}}$ and is followed by a complementary interval $k$ of length

$$\frac{1}{x} - \frac{2\delta}{\sqrt{x}} > \frac{1-2\delta}{\sqrt{x}}.$$

There may be no complete $k$ inside $I$. In this case there is at most one (complete or incomplete) $i$, and at most

$$\frac{2\delta}{\sqrt{x}} < 4\delta\mu$$

of $I$ is inside $i$'s. Otherwise $I$ contains at least one complete $k$, and the number of complete or incomplete $i$'s does not exceed twice the number of complete $k$'s. The ratio of the total length of the $i$'s to that of the $k$'s is accordingly less than

$$\frac{4\delta}{1-2\delta}$$

which is less than $8\delta$ if $\delta < \frac{1}{4}$; and then the length of the $i$'s is not greater than $8\delta\mu$. A similar argument applies to the $j$'s, and the part of $I$, inside one interval or another of the two systems, can in no case exceed $16\delta\mu$. Our conclusion follows if $\delta < \frac{1}{16}$.[1]

Suppose now that Theorem A has been established for $\lambda$'s excluded from the $i$'s and $j$'s. Any given value of $\lambda$ lies in an $I$, and there is therefore a $\lambda'$ subject to our restrictive conditions and differing from $\lambda$ by less than $\mu$. It is plain that, if we change $\lambda$ into $\lambda'$, the alteration of the left hand side of (1.3) is

[1] It would naturally be easy to improve on this number if it were necessary.

$$O\left(\frac{1}{\sqrt{x}}\right) + O\left(\frac{1}{\sqrt{x}}\right) O(1) = O\left(\frac{1}{\sqrt{x}}\right),$$

so that the theorem, if true for $\lambda'$, is true for $\lambda$.

3. We write

(3. 1) $$f(s) = f(s, \theta) = 1 + 2\sum_{1}^{\infty} e^{-n^2 \pi s} \cos 2n\pi\theta,$$

where $s = \sigma + it$, $\sigma > 0$. It is well-known[1] that

(3. 2) $$f(s) = \frac{1}{\sqrt{s}} \sum_{-\infty}^{\infty} e^{-\frac{\pi}{s}(n-\theta)^2}.$$

We write also

$$\varepsilon_n = 1 \quad (n = 0), \qquad \varepsilon_n = 2 \quad (n > 0).$$

Then, if $c$ is positive, we have[2]

$$\pi \sum_{0 \leqq n \leqq \lambda} \varepsilon_n (\omega - n^2) e^{-n^2 \pi i x} \cos 2n\pi\theta = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{\omega\pi s}}{s^2} f(s + ix) ds$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{\omega\pi(s-ix)}}{(s-ix)^2 \sqrt{s}} \sum_{-\infty}^{\infty} e^{-\frac{\pi(n-\theta)^2}{s}} ds,$$

on writing $s - ix$ for $s$ and using (3. 2).

We may invert the order of integration and summation; for, when $t$ is large,

$$\left| \frac{1}{(s-ix)^2 \sqrt{s}} \right| = O\left(|t|^{-\frac{5}{2}}\right),$$

$$\sum \left| e^{-\frac{\pi(n-\theta)^2}{s}} \right| = \sum exp\left(-\frac{\pi\sigma(n-\theta)^2}{\sigma^2 + t^2}\right)$$

$$= O(\sqrt{\sigma^2 + t^2}) = O(|t|),$$

and

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[1] See for example E. LANDAU, *Handbuch der Lehre von der Verteilung der Primzahlen*, 277.

[2] See G. H. HARDY and M. RIESZ, *The general theory of Dirichlet's series*, 50 (Theorem 39). We require the result of the theorem for absolutely convergent series only.

$$\int^{\infty} t^{-\frac{5}{2}}.t.dt$$

is convergent. We have thus

(3. 3)
$$\pi \sum_{0\leq n \leq \lambda} \varepsilon_n(\omega - n^2) e^{-n^2\pi i x} \cos 2 n \pi \theta = \sum_{-\infty}^{\infty} I_n,$$

where

(3. 4)
$$I_n = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\Phi(s)} \frac{ds}{(s-ix)^2 \sqrt{s}},$$

(3. 41)
$$\Phi(s) = \omega\pi(s-ix) - \frac{\pi(n-\theta)^2}{s}.$$

If we suppose now that $\lambda = \sqrt{\omega}$ is non-integral, as plainly we may do without loss of generality, and differentiate (3. 3) formally with respect to $\omega$, we obtain

(3. 5)
$$-1 + 2s(\omega) = \sum_{-\infty}^{\infty} J_n,$$

where

(3. 51)
$$J_n = \frac{dI_n}{d\omega} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\Phi(s)} \frac{ds}{(s-ix)\sqrt{s}}.$$

This process is certainly legitimate if $\Sigma J_n$ is uniformly convergent in a neighbourhood of the particular value of $\omega$ considered. That this is so will appear incidentally in the sequel.

4. Our main idea is to approximate to $J_n$, in (3. 5), by the saddle-point method or 'method of steepest descents'.

The saddle-points of $e^{\Phi(s)}$ are given by

$$\Phi'(s) = 0$$

or

$$s = \pm i \frac{|n-\theta|}{\lambda} = \pm i N,$$

[1] See G. N. WATSON, *Theory of Bessel Functions*, 235, for a general account of the method.

say. The curve of 'zero level', given by $\Re\,\Phi(s)=0$, has the equation

$$\sigma(\sigma^2+t^2-N^2)=0,$$

and consists of the imaginary axis and the circle described on the line joining the saddle-points as diameter. It divides the plane into four regions, the 'low' regions, for which $\Re\,\Phi(s)<0$, being the right hand inside and the left hand outside regions.

We define the path $C=C_1+C_2$ by the lines $C_1$ and $C_2$ from the point $s=\frac{1}{2}N$ to infinity through the upper and lower saddle-points respectively. The whole path lies in low ground, except at the saddle-points, and it cannot pass through $s=ix$, since (owing to the restrictions of § 2)

$$x\neq\frac{|n-\theta|}{\lambda}$$

for any value of $n$. We can deform the path of integration in (3.51) into $C$, if we introduce the appropriate correction when this deformation involves crossing a pole. This is so if and only if $x>N$, *i. e.* if $|n-\theta|<\lambda x$, and the correction required is accordingly

$$\frac{1}{\sqrt{ix}}\sum_{|n-\theta|<\lambda x}e^{\frac{\pi i(n-\theta)^2}{x}},$$

which differs from

$$2\frac{e^{-\frac{1}{4}\pi i}}{\sqrt{x}}e^{\frac{\pi i\theta^2}{x}}s\left(x^2\omega,\,-\frac{1}{x},\,\frac{\theta}{x}\right)$$

by $O\left(\frac{1}{\sqrt{x}}\right)$. We thus obtain

$$s(\omega,x,\theta)-\frac{e^{-\frac{1}{4}\pi i}}{\sqrt{x}}e^{\frac{\pi i\theta^2}{x}}s\left(x^2\omega,\,-\frac{1}{x},\,\frac{\theta}{x}\right)=\sum_{-\infty}^{\infty}\bar{J}_n+O\left(\frac{1}{\sqrt{x}}\right),$$

where $\bar{J}_n$ differs from $J_n$ in being taken along $C$; and the proof of Theorem A is reduced to a proof that

(4.1) $$\Sigma\bar{J}_n=O\left(\frac{1}{\sqrt{x}}\right).$$

5. It is convenient to write

$$\xi = |n-\theta| = N\lambda > 0,$$

$$X = \frac{x}{N} = \frac{\lambda x}{\xi} > 0, \quad Y = \lambda\xi = \omega N > 0,$$

and to transform $\bar{J}_n$ by writing $Ns$ for $s$. We thus obtain

(5. 1) $$\bar{J}_n = e^{-\omega\pi i x}\sqrt{\frac{\lambda}{\xi}}\frac{1}{2\pi i}\int e^{\pi Y\left(s-\frac{1}{s}\right)}\frac{ds}{(s-iX)\sqrt{s}},$$

where the path of integration $\Gamma = \Gamma_1 + \Gamma_2$ is $C = C_1 + C_2$ reduced in the ratio $1 : N$, so that it passes through the points $-i$, $\frac{1}{2}$, $i$.

Since

(5. 2) $$\frac{1}{s-iX} = \frac{i(s-i)}{2s(1+X)} - \frac{i(s+i)}{2s(1-X)} + \frac{i(s^2+1)X}{s(s-iX)(1-X^2)},$$

we may write (5. 1) in the form

(5. 3) $$\bar{J}_n = e^{-\omega\pi i x}\sqrt{\frac{\lambda}{\xi}}(K_{n,1} - K_{n,2} + L_n),$$

where

(5. 31) $$K_{n,1} = \frac{1}{4\pi(1+X)}\int e^{\pi Y\left(s-\frac{1}{s}\right)}\frac{s-i}{s^{\frac{3}{2}}}\,ds,$$

(5. 32) $$K_{n,2} = \frac{1}{4\pi(1-X)}\int e^{\pi Y\left(s-\frac{1}{s}\right)}\frac{s+i}{s^{\frac{3}{2}}}\,ds,$$

(5. 33) $$L_n = \frac{X}{2\pi(1-X^2)}\int e^{\pi Y\left(s-\frac{1}{s}\right)}\frac{s^2+1}{(s-iX)s^{\frac{3}{2}}}\,ds.$$

6. The integrals $K_{n,1}$ and $K_{n,2}$ are linear combinations of Bessel functions of orders $\frac{1}{2}$ and $-\frac{1}{2}$, and may accordingly be evaluated as elementary functions.[1] We have in fact

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[1] See WATSON, *loc. cit.*, 175 *et seq.*

$$K_{n,1} = \frac{i}{2\pi(1+X)\sqrt{Y}} e^{-2\pi i Y} = \frac{i}{2\pi}\sqrt{\frac{\xi}{\lambda}}\frac{e^{-2\pi i\lambda\xi}}{\lambda x+\xi},$$

$$K_{n,2} = \frac{i}{2\pi(1-X)\sqrt{Y}} e^{2\pi i Y} = -\frac{i}{2\pi}\sqrt{\frac{\xi}{\lambda}}\frac{e^{2\pi i\lambda\xi}}{\lambda x-\xi}.$$

Thus the contribution to $\Sigma \bar{J}_n$ of these integrals is

$$\frac{i}{2\pi}e^{-\omega\pi i x}\left(\sum_{-\infty}^{\infty}\frac{e^{-2\pi i\lambda|n-\theta|}}{\lambda x+|n-\theta|}+\sum_{-\infty}^{\infty}\frac{e^{2\pi i\lambda|n-\theta|}}{\lambda x-|n-\theta|}\right)$$

$$=\frac{i}{2\pi}e^{-\omega\pi i x}\left(\sum_{-\infty}^{\infty}\frac{e^{-2\pi i\lambda(n-\theta)}}{\lambda x+n-\theta}+\sum_{-\infty}^{\infty}\frac{e^{2\pi i\lambda(n-\theta)}}{\lambda x-n+\theta}\right),$$

when we combine the positive half of each series with the negative half of the other.

Now[1]

$$\sum_{-\infty}^{\infty}\frac{e^{-2\pi i\lambda(n-\theta)}}{\lambda x+n-\theta}=\pi e^{2\pi i\lambda\theta}\frac{e^{2\pi i\Lambda(\lambda x-\theta)}}{\sin(\lambda x-\theta)\pi},$$

where

$$\Lambda=\lambda-[\lambda]-\frac{1}{2},$$

and this is

$$O\left(\frac{1}{|\sin(\lambda x-\theta)\pi|}\right)=O\left(\frac{1}{\sqrt{x}}\right),$$

in virtue of the restrictions of § 2. The second series may be treated in the same way, so that the total contribution of $K_{n,1}$ and $K_{n,2}$ is $O\left(\frac{1}{\sqrt{x}}\right)$.

7. It remains only to discuss the contribution of $L_n$. It appears at once from a figure that, on either $\Gamma_1$ or $\Gamma_2$,

(7. 1) $$|s-iX|>A|1-X|,\ \left|s^{\frac{3}{2}}\right|>A,\ |s^2+1|<A|\sigma|$$

if $-\frac{1}{2}\leqq\sigma\leqq\frac{1}{2}$, and

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[1] See, for example, T. J. I'A. BROMWICH, *Infinite series.*

$$(7.2) \qquad |s-iX|>A|1-X|, \quad \left|s^{\frac{3}{2}}\right|>A|\sigma|^{\frac{3}{2}}, \quad |s^2+1|<A\sigma^2$$

if $\sigma<-\frac{1}{2}$, the $A$'s being absolute constants. Also

$$\Re\left(s-\frac{1}{s}\right)=\sigma\left(1-\frac{1}{\sigma^2+t^2}\right)=\sigma\left(1-\frac{1}{\sigma^2+(1-2\sigma)^2}\right)$$

is negative except when $\sigma=0$, and

$$(7.3) \qquad \Re\left(s-\frac{1}{s}\right)<-A\sigma^2 \quad \left(-\frac{1}{2}\leqq\sigma\leqq\frac{1}{2}\right), \qquad \Re\left(s-\frac{1}{s}\right)<A\sigma \quad \left(\sigma<-\frac{1}{2}\right).$$

It follows from (7.1) — (7.3) that

$$|L_n|<\frac{AX}{(1-X)^2(1+X)}\int_{-\frac{1}{2}}^{\frac{1}{2}}|\sigma|e^{-AY\sigma^2}d\sigma+\frac{AX}{(1-X)^2(1+X)}\int_{-\infty}^{\frac{1}{2}}\sqrt{|\sigma|}\,e^{AY\sigma}d\sigma$$

$$<\frac{AX}{(1-X)^2(1+X)}\left(\frac{1}{Y}+\frac{1}{Y^{\frac{3}{2}}}\right)<\frac{A\sqrt{X}}{(1-X)^2\,Y}+\frac{AX}{|1-X|^3\,Y^{\frac{3}{2}}}$$

$$<A\sqrt{\frac{\xi}{\lambda}}\,\frac{\sqrt{x}}{(\lambda x-\xi)^2}+A\sqrt{\frac{\xi}{\lambda}}\,\frac{x}{|\lambda x-\xi|^3},$$

Thus the contribution of $L_n$ is

$$O\left(\sqrt{x}\sum_{-\infty}^{\infty}\frac{1}{(|n-\theta|-\lambda x)^2}\right)+O\left(x\sum_{-\infty}^{\infty}\frac{1}{||n-\theta|-\lambda x|^3}\right).$$

In these series there are at most four terms in which the denominator is less than unity, and the contribution of these terms is

$$O\left(\sqrt{x}\left(\frac{1}{\sqrt{x}}\right)^2\right)+O\left(x\left(\frac{1}{\sqrt{x}}\right)^3\right)=O\left(\frac{1}{\sqrt{x}}\right)$$

in virtue of the restrictions on $\lambda$. The contribution of the remaining terms is obviously $O(1)$. Thus the total contribution of $L_n$ is $O\left(\frac{1}{\sqrt{x}}\right)$.

This completes the proof of the theorem. It is only necessary to add one word in justification of the assumption made provisionally in § 3, that the series $\Sigma J_n$ is uniformly convergent in a neighbourhood of any particular value of $\omega$ under consideration. The series has in fact been decomposed into a number of parts, all of which have been proved uniformly convergent by direct estimation of their terms, except those which were summed in § 6. The uniform convergence of these last series is classical.