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Arun Kumar Chaudhary
Associate Professor, Department of Management Science, Nepal Commerce Campus, Tribhuvan University, Kathmandu, Nepal

Laxmi Prasad Sapkota
Ph.D. Scholar, Department of Mathematics and Statistics DDU Gorakhpur University, Gorakhpur, Uttar Pradesh, India

Vijay Kumar
Professor, Department of Mathematics and Statistics, DDU Gorakhpur University, Gorakhpur, Uttar Pradesh, India

Corresponding Author:
Arun Kumar Chaudhary
Associate Professor, Department of Management Science, Nepal Commerce Campus, Tribhuvan University, Kathmandu, Nepal

Some properties and applications of half Cauchy extended exponential distribution

Arun Kumar Chaudhary, Laxmi Prasad Sapkota and Vijay Kumar

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Abstract
In this article, we have introduced a new probability distribution having three parameters using half Cauchy family of distribution named half Cauchy extended exponential distribution. The statistical properties and characteristics of the proposed distribution like the hazard rate function (HRF), cumulative hazard function, and the probability density function (PDF), and the cumulative distribution function (CDF), quantile function and the skewness, kurtosis are provided. The parameters of the proposed distribution are estimated using the Cramer-Von-Mises (CVM) least-square estimation (LSE), and maximum likelihood estimators (MLE) methods. A real data set is analyzed to test the goodness-of-fit of the proposed distribution. It is found that the half Cauchy extended exponential distribution performed well as compared to some competing distributions.

Keywords: Hazard function, extended exponential distribution, half Cauchy distribution, estimation method

Introduction

The exponential distribution (ED) plays a significant role in the modeling of survival and reliability data in applied statistics and probability theory. It has the memoryless property and is a particular case of the geometric and gamma distributions. In addition it can be applied for the study of the Poisson point processes. The ED has been widely utilized as a basis distribution during the past few decades to construct a more adaptable family of distributions. Researchers from several fields presented the ED's modifications and extensions , such as, Nadarajah and Kotz (2006) [26] have defined beta exponential, generalized exponential by (Gupta & Kundu, 2007) [13], Kumar(2010) [17] has presented Exponential extension(EE) distribution, the reliability estimation of the generalized inverted ED by (Abouammoh & Alshingiti, 2009) [2], Kumaraswamy exponential (Cordeiro & de Castro, 2011) [11], beta generalized exponential (Barreto-Souza *et al*., 2010) [5], an extension of the ED by (Nadarajah & Haghighi, 2011) [25]. Transmuted EE distribution has presented by (Merovci, 2013) [22], Gamma EE presented by (Ristic and Balakrishnan, 2012) [28], a novel exponential-type model with a bathtub-shaped failure rate function has been described by (Lemonte, 2013) [19]. It contains four functions: declining, rising, constant, and upside-down. Gomez *et al.* (2014) [12] and Louzada *et al*. (2014) [20] presented a novel extension of the ED known as the exponentiated exponential geometric. Kumaraswamy transmuted ED (Afify *et al*., 2016) [3]. Mahdavi and Kundu (2017) [21] have developed a new method for extension of the distribution by applying the ED. In the present, the Alpha power transformed extended exponential distribution has introduced by (Almarashi *et al*., 2019) [4] and a novel extension of the exponential distribution with various statistical properties has been introduced by (Hassan et al., 2018) [15]. The Type II half-logistic exponentiated exponential distribution was introduced by (Abdulkabir & Ipinyomi, 2020) [1]. Chaudhary and Kumar (2020) [7] has defined the extension of ED called the half logistic exponential extension distribution. Another extension of ED was presented by (Chaudhary *et al.* 2020) [8] named the truncated Cauchy power–exponential distribution.

The half-Cauchy distribution, a specific case of the Cauchy distribution, was used in this article by breaking down the curve at the origin to only take into account non-negative values. As an alternative to modeling spreading distances, Shaw (1995) [29] employed the half-Cauchy distribution with a strong tail because it can predict more frequent long-distance spreading occurrences. In addition, the half-Cauchy distribution is also used by (Paradis *et al*. 2002) [27] to model ringing data on tits having two species in Ireland and Britain. Chaudhary and Kumar (2022) [9] also introduced half Cauchy modified exponential distribution using half Cauchy family of distribution.

Let X be a non-negative random variable that follows the half-Cauchy distribution and its cumulative distribution function (CDF) can be expressed as

$$R(x;\theta) = \frac{2}{\pi} tan^{-1}\left(\frac{x}{\theta}\right), x > 0, \theta > 0. \tag{1}$$

and the probability density function (PDF) corresponding to (1) is,

$$r(x;\theta) = \frac{2}{\pi}\left(\frac{\theta}{\theta^2+x^2}\right), x > 0, \theta > 0. \tag{2}$$

Therefore we are interested to generate new distribution using half-Cauchy family of distribution. The generating family of distribution developed by (Zografas & Balakrishnan, 2009) [32] and CDF of family of distribution can be obtained as

$$F(x) = \int_0^{-ln[1-G(x)]} r(t)\ dt, \tag{3}$$

here $G(x)$ is the CDF of any baseline distribution and $r(t)$ is the PDF of any distribution. The family of half-Cauchy distribution whose CDF can be defined by using $r(t)$ as PDF of half-Cauchy distribution defined in (2) as

$$F(x) = \int_0^{-ln[1-G(x)]} \frac{2}{\pi}\frac{\theta}{\theta^2+t^2}\ dt$$

$$= \frac{2}{\pi} arctan\left(-\frac{1}{\theta} ln[1-G(x)]\right);\ x > 0, \theta > 0 \tag{4}$$

The PDF corresponding to (4) can be expressed as

$$f(x) = \frac{2}{\pi\theta}\frac{g(x)}{1-G(x)}\left[1+\left\{-\frac{1}{\theta}log[1-G(x)]\right\}^2\right]^{-1} ; x > 0, \theta > 0 \tag{5}$$

The rest part of this article is organized as, In Section 2, the half Cauchy extended exponential distribution is defined and also we present the statistical properties of the proposed distribution such as survival function, probability density function, hazard function, cumulative distribution function, cumulative hazard function, quantiles, the measures of skewness based on quartiles and kurtosis based on octiles. In Section 3 the estimation of the parameters of the proposed distribution is carried out using the three widely used estimation technique namely maximum likelihood estimators (MLE), Cramer-Von-Mises (CVM) and least-square (LSE) methods. The application of the proposed model is presented in Section 4. Finally some concluding explanations are entered in Section 5.

The Half Cauchy Extended Exponential (HCEE) distribution

The extension of the exponential distribution has defined by (Joshi, 2015) [15] named it as extended exponential distribution. The CDF of extended exponential distribution is

$$G(x;\beta,\lambda) = 1 - exp\left(-\beta x e^{-\frac{\lambda}{x}}\right); x > 0, (\beta,\lambda) > 0 \tag{6}$$

The PDF corresponding to (6) can be written as

$$g(x;\beta,\lambda) = \beta\left(1+\frac{\lambda}{x}\right) e^{-\frac{\lambda}{x}} exp\left(-\beta x e^{-\frac{\lambda}{x}}\right); x > 0, (\beta,\lambda) > 0 \tag{7}$$

Substituting (6) and (7) in (4) and (5) we get the CDF of HCEE distribution, which is defined as

$$F(x) = \frac{2}{\pi} arctan\left\{-\frac{1}{\theta}\beta x e^{-\lambda/x}\right\}; x > 0, \beta, \lambda, \theta > 0. \tag{8}$$

And the PDF of half-Cauchy exponential extension can be expressed as

$$f(x) = \frac{2}{\pi}\frac{\beta}{\theta}\left(1+\frac{\lambda}{x}\right) e^{-\lambda/x}\left\{1+\left(\frac{1}{\theta}\beta x e^{-\lambda/x}\right)^2\right\}^{-1} \tag{9}$$

Reliability function
The reliability function of HCEE is

$$r(x) = 1 - \frac{2}{\pi} arctan\left\{-\frac{1}{\theta}\beta x e^{-\lambda/x}\right\}, x > 0, \beta, \lambda, \theta > 0. \quad (10)$$

Hazard rate function
Hazard rate function of HCEE distribution with parameters (β, λ, θ) is
$h(t) = \frac{f(t)}{1-F(t)};\ 0 < t < \infty$

$$= \frac{2}{\pi}\frac{\beta}{\theta}\left(1 + \frac{\lambda}{x}\right) e^{-\lambda/x}\left\{1 + \left(\frac{1}{\theta}\beta x e^{-\lambda/x}\right)^2\right\}^{-1}\left[1 - \frac{2}{\pi} arctan\left\{-\frac{1}{\theta}\beta x e^{-\lambda/x}\right\}\right]^{-1} \quad (11)$$

Reverse hazard function of HCEE
The reverse hazard function of HCEE can be defined as

$$\boldsymbol{h_{rev}(x)} = \frac{\boldsymbol{f(x)}}{\boldsymbol{1-r(x)}}$$

$$= \frac{2}{\pi}\frac{\beta}{\theta}\left(1 + \frac{\lambda}{x}\right) e^{-\lambda/x}\left\{1 + \left(\frac{1}{\theta}\beta x e^{-\lambda/x}\right)^2\right\}^{-1}\left[\frac{2}{\pi} arctan\left\{-\frac{1}{\theta}\beta x e^{-\lambda/x}\right\}\right]^{-1} \quad (12)$$

The various shapes of PDF and hazard rate function of HCEE$(\beta, \lambda, \theta)$with different values of parameters are shown in Figure 1.

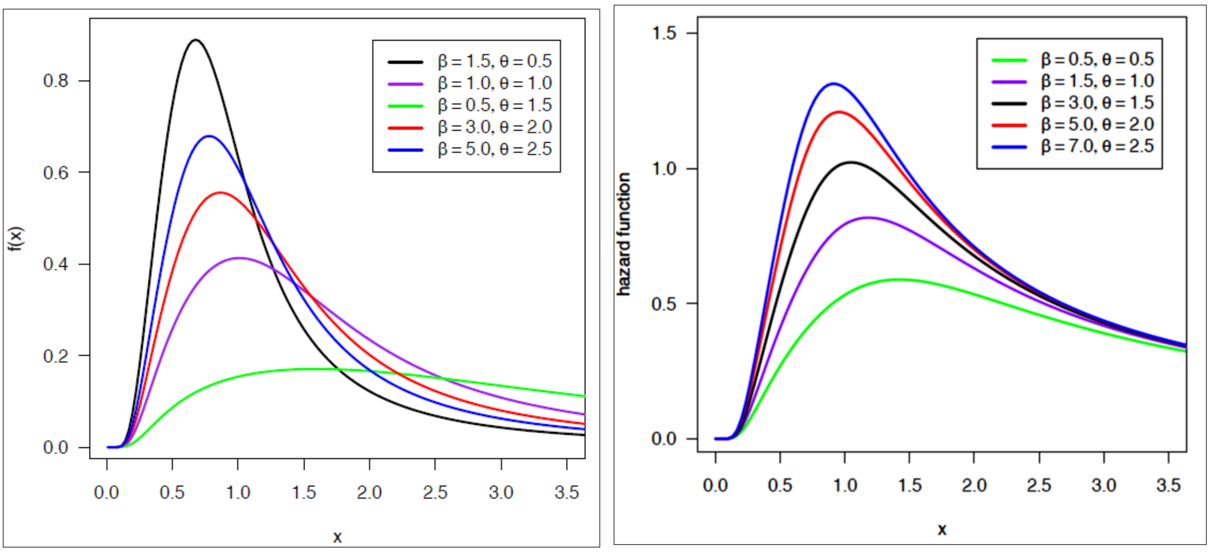

Fig 1: PDF (left panel) and hazard function (right panel) for fixed λ, and different values of β and θ.

Cumulative hazard function (chf)
The chf of the HCEE$(\beta, \lambda, \theta)$is defined as

$$H(x) = \int_{-\infty}^{x} h(y)dy$$

$$= -log[1 - F(x)]$$

$$= -log\left[1 - \frac{2}{\pi} arctan\left\{-\frac{1}{\theta}\beta x e^{-\lambda/x}\right\}\right] \quad (13)$$

Quantile function
Let X be a positive random variable with CDF $F(x)$ then quantile function can be defined as

$$Q(u) = F^{-1}(u)$$

$$-\beta x e^{-\frac{\lambda}{x}} + \theta\, tan\left(\frac{\pi u}{2}\right) = 0\ ;\ 0 < u < 1 \quad (14)$$

The random deviate generation for the HCEE$(\beta, \lambda, \theta)$is,

$$-\beta x e^{-\frac{\lambda}{x}} + \theta\, tan\left(\frac{\pi v}{2}\right) = 0\ ;\ 0 < v < 1 \quad (15)$$

Skewness and Kurtosis
The Bowley's coefficient of skewness based on quartiles is,

$$S_k(B) = \frac{Q(3/4)+Q(1/4)-2Q(1/2)}{Q(3/4)-Q(1/4)}, \text{ and}$$

Coefficient of kurtosis based on octiles defined by (Moors, 1988) [22] is

$$K - Moors = \frac{Q(0.875)-Q(0.625)-Q(0.125)+Q(0.375)}{Q(3/4)-Q(1/4)}$$

Parameter estimation
Maximum Likelihood Estimation (MLE)
Here, we have presented the ML estimators (MLE's) of the HCEE distribution are estimated by using MLE method. Let $\underset{\sim}{x} = (x_1, \dots, x_n)$ be a random sample of size 'n' from HCEE(β, λ, θ) then the log likelihood function can be written as,

$$\ell(\lambda,\beta,\theta|\underline{x}) = n\,ln\left(\frac{2}{\pi}\right) + n\,ln\,\beta + n\,ln\,\theta + \sum_{i=1}^{n} ln\left(1+\frac{\lambda}{x_i}\right) - \sum_{i=1}^{n}\frac{\lambda}{x_i} - \sum_{i=1}^{n} ln\left\{\theta^2 + \left(\beta x_i e^{-\lambda/x_i}\right)^2\right\} \quad (16)$$

Differentiating (16) with respect to β, λ and θ, we get

$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} - 2\beta\left[\sum_{i=1}^{n}\left(x_i e^{-\lambda/x_i}\right)^2\left\{\theta^2 + \left(\beta x_i e^{-\lambda/x_i}\right)^2\right\}^{-1}\right]$$

$$\frac{\partial\ell}{\partial\lambda} = \sum_{i=1}^{n}\left[\left(1+\frac{\lambda}{x_i}\right)^{-1} + 2\left\{\theta^2 + \left(\beta x_i e^{-\lambda/x_i}\right)^2\right\}^{-1}\left(\beta x_i e^{-\lambda/x_i}\right)^2 - 1\right]\left(\frac{1}{x_i}\right)$$

$$\frac{\partial\ell}{\partial\theta} = \frac{n}{\theta} - 2\theta\sum_{i=1}^{n}\left\{\theta^2 + \left(\beta x_i e^{-\lambda/x_i}\right)^2\right\}^{-1}$$

Solving $\frac{\partial\ell}{\partial\beta} = \frac{\partial\ell}{\partial\lambda} = \frac{\partial\ell}{\partial\theta} = 0$ for the β, λ and θ we get the ML estimators of the HCEE(β,λ,θ) distribution. But normally, it is not possible to solve non-linear equations above so with the aid of suitable computer software one can solve them easily. Let $\underline{\Theta} = (\beta,\lambda,\theta)$ denote the parameter vector of HCEE(β,λ,θ) and the corresponding MLE of $\underline{\Theta}$ as $\hat{\underline{\Theta}} = (\hat{\beta},\hat{\lambda},\hat{\theta})$ then the asymptotic normality results in, $(\hat{\underline{\Theta}} - \underline{\Theta}) \rightarrow N_3\left[0, \left(I(\underline{\Theta})\right)^{-1}\right]$ where $I(\underline{\Theta})$ is the Fisher's information matrix given by,

$$I(\underline{\Theta}) = -\begin{pmatrix} E\left(\frac{\partial^2 l}{\partial\beta^2}\right) & E\left(\frac{\partial^2 l}{\partial\beta\partial\lambda}\right) & E\left(\frac{\partial^2 l}{\partial\beta\partial\theta}\right) \\ E\left(\frac{\partial^2 l}{\partial\beta\partial\lambda}\right) & E\left(\frac{\partial^2 l}{\partial\lambda^2}\right) & E\left(\frac{\partial^2 l}{\partial\lambda\partial\theta}\right) \\ E\left(\frac{\partial^2 l}{\partial\beta\partial\theta}\right) & E\left(\frac{\partial^2 l}{\partial\lambda\partial\theta}\right) & E\left(\frac{\partial^2 l}{\partial\theta^2}\right) \end{pmatrix}$$

In practice, we don't know $\underline{\Theta}$ hence it is useless that the MLE has an asymptotic variance$\left(I(\underline{\Theta})\right)^{-1}$. Hence we approximate the asymptotic variance by plugging in the estimated value of the parameters. The observed fisher information matrix $O(\widehat{\underline{\Theta}})$ is used as an estimate of the information matrix $I(\underline{\Theta})$ given by

$$O(\widehat{\underline{\Theta}}) = -\begin{pmatrix} \frac{\partial^2 l}{\partial\hat{\beta}^2} & \frac{\partial^2 l}{\partial\hat{\beta}\partial\hat{\lambda}} & \frac{\partial^2 l}{\partial\hat{\beta}\partial\hat{\theta}} \\ \frac{\partial^2 l}{\partial\hat{\beta}\partial\hat{\lambda}} & \frac{\partial^2 l}{\partial\hat{\lambda}^2} & \frac{\partial^2 l}{\partial\hat{\theta}\partial\hat{\lambda}} \\ \frac{\partial^2 l}{\partial\hat{\beta}\partial\hat{\theta}} & \frac{\partial^2 l}{\partial\hat{\theta}\partial\hat{\lambda}} & \frac{\partial^2 l}{\partial\hat{\theta}^2} \end{pmatrix}_{|(\hat{\beta},\hat{\lambda},\hat{\theta})} = -H(\underline{\Theta})_{|(\underline{\Theta}=\hat{\underline{\Theta}})}$$

where H is the Hessian matrix.
The Newton-Raphson algorithm to maximize the likelihood produces the observed information matrix. Therefore, the variance-covariance matrix is given by,

$$\left[-H(\underline{\Theta})_{|(\underline{\Theta}=\hat{\underline{\Theta}})}\right]^{-1} = \begin{pmatrix} var(\hat{\beta}) & cov(\hat{\beta},\hat{\lambda}) & cov(\hat{\beta},\hat{\theta}) \\ cov(\hat{\beta},\hat{\lambda}) & var(\hat{\lambda}) & cov(\hat{\lambda},\hat{\theta}) \\ cov(\hat{\beta},\hat{\theta}) & cov(\hat{\lambda},\hat{\theta}) & var(\hat{\theta}) \end{pmatrix} \quad (17)$$

Hence from the asymptotic normality of MLEs, approximate *100(1-b) %* confidence intervals for β, λ and θ of HCEE(β,λ,θ) can be constructed as,

$\hat{\beta} \pm Z_{b/2}\sqrt{var(\hat{\alpha})}$, $\hat{\lambda} \pm Z_{b/2}\sqrt{var(\hat{\lambda})}$ and $\hat{\theta} \pm Z_{b/2}\sqrt{var(\hat{\theta})}$.

where $Z_{b/2}$is the upper percentile of standard normal variate.

Method of Least-Square Estimation (LSE)
The another method of estimation we have used is least-square estimation to estimate the unknown parameters β, λ and θ of HCEE distribution and can be calculated by minimizing

$$T(X;\beta,\lambda,\theta) = \sum_{i=1}^{n}\left[F(X_i) - \frac{i}{n+1}\right]^2 \quad (18)$$

with respect to unknown parameters β, λ and θ.
Suppose $F(X_i)$ denotes the CDF of the ordered random variables $X_{(1)} < X_{(2)} < \dots < X_{(n)}$ where $\{X_1, X_2, \dots, X_n\}$ is a random sample of size n from a distribution function F(.). The least-square estimators of β, λ and θ say $\hat{\beta}, \hat{\lambda}$ *and* $\hat{\theta}$ respectively, can be obtained by minimizing

$$T(X;\beta,\lambda,\theta) = \sum_{i=1}^{n}\left[\frac{2}{\pi}arctan\left\{-\frac{1}{\theta}\beta x_i e^{-\lambda/x_i}\right\} - \frac{i}{n+1}\right]^2; x > 0, \beta, \lambda, \theta > 0. \quad (19)$$

with respect to β, λ and θ. Differentiating (19) with respect to β, λ and θ we get,

$$\frac{\partial T}{\partial \beta} = \frac{-4}{\pi\theta}\sum_{i=1}^{n} x_i e^{-\lambda/x_i}\left[\frac{2}{\pi}arctan\{M(x_i)\} - \frac{i}{n+1}\right][1 + \{M(x_i)\}^2]^{-1}$$

$$\frac{\partial T}{\partial \lambda} = \frac{4\beta}{\pi\theta}\sum_{i=1}^{n} e^{-\frac{\lambda}{x_i}}\left[\frac{2}{\pi}arctan\{M(x_i)\} - \frac{i}{n+1}\right][1 + \{M(x_i)\}^2]^{-1}$$

$$\frac{\partial T}{\partial \theta} = \frac{4\beta}{\pi\theta^2}\sum_{i=1}^{n} x_i e^{-\lambda/x_i}\left[\frac{2}{\pi}arctan\{M(x_i)\} - \frac{i}{n+1}\right][1 + \{M(x_i)\}^2]^{-1}$$

where $M(x_i) = -\frac{1}{\theta}\beta x_i e^{-\lambda/x_i}$.

Similarly the weighted least square estimators is computed by minimizing

$$B(X;\beta,\lambda,\theta) = \sum_{i=1}^{n} w_i\left[F(X_{(i)}) - \frac{i}{n+1}\right]$$

with respect to β, λ and θ. The weights w_i are $w_i = \frac{1}{Var(X_{(i)})} = \frac{(n+1)^2(n+2)}{i(n-i+1)}$. Hence, the weighted least square estimators of β, λ and θ respectively can be obtained by minimizing,

$$B(X;\beta,\lambda,\theta) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\frac{2}{\pi}arctan\left\{-\frac{1}{\theta}\beta x_i e^{-\lambda/x_i}\right\} - \frac{i}{n+1}\right]^2 \quad (20)$$

with respect to β, λ and θ.

Method of Cramer-Von-Mises estimation (CVME)
The Cramer-Von-Mises estimators of β, λ and θ are obtained by minimizing the function

$$K(X;\beta,\lambda,\theta) = \frac{1}{12n} + \sum_{i=1}^{n}\left[F(x_{i:n}|\beta,\lambda,\theta) - \frac{2i-1}{2n}\right]^2$$

$$= \frac{1}{12n} + \sum_{i=1}^{n}\left[\frac{2}{\pi}arctan\left\{-\frac{1}{\theta}\beta x_i e^{-\lambda/x_i}\right\} - \frac{2i-1}{2n}\right]^2 \quad (21)$$

Differentiating (21) with respect to β, λ and θ we get,

$$\frac{\partial K}{\partial \beta} = \frac{-4}{\pi\theta}\sum_{i=1}^{n} x_i e^{-\lambda/x_i}\left[\frac{2}{\pi}arctan\{M(x_i)\} - \frac{2i-1}{2n}\right][1 + \{M(x_i)\}^2]^{-1}$$

$$\frac{\partial K}{\partial \lambda} = \frac{4\beta}{\pi\theta}\sum_{i=1}^{n} e^{-\lambda/x_i}\left[\frac{2}{\pi}arctan\{M(x_i)\} - \frac{2i-1}{2n}\right][1 + \{M(x_i)\}^2]^{-1}$$

$$\frac{\partial K}{\partial \theta} = \frac{4\beta}{\pi\theta^2}\sum_{i=1}^{n} x_i e^{-\lambda/x_i}\left[\frac{2}{\pi}arctan\{M(x_i)\} - \frac{i}{n+1}\right][1 + \{M(x_i)\}^2]^{-1}$$

Where $M(x_i) = -\frac{1}{\theta}\beta x_i e^{-\lambda/x_i}$. After solving non-linear equations $\frac{\partial K}{\partial \beta} = 0, \frac{\partial K}{\partial \lambda} = 0$ *and* $\frac{\partial K}{\partial \theta} = 0$ simultaneously we will get the CVM estimators.

Application to Real Dataset

The data given below represents the fatigue life of 6061-T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per seconds (cps) which consists of 101 observations with maximum stress per cycle 31,000 psi. This data set was originally analyzed by (Birnbaum & Saunders, 1969) [6].

70, 90, 96, 97, 99, 100, 103, 104, 104, 105, 107, 108, 108, 108, 109, 109, 112, 112, 113, 114, 114, 114, 116, 119, 120, 120, 120, 121, 121, 123, 124, 124, 124, 124, 124, 128, 128, 129, 129, 130, 130, 130, 131, 131, 131, 131, 131, 132, 132, 132, 133, 134, 134, 134, 134, 134, 136, 136, 137, 138, 138, 138, 139, 139, 141, 141, 142, 142, 142, 142, 142, 142, 144, 144, 145, 146, 148, 148, 149, 151, 151, 152, 155, 156, 157, 157, 157, 157, 158, 159, 162, 163, 163, 164, 166, 166, 168, 170, 174, 196, 212

By utilizing R software (R Core Team, 2020) [31] of the optim () function, we have calculated the MLEs of HCEE distribution by maximizing the likelihood function (16). We have obtained the value of Log-Likelihood is l = -458.5402, $\hat{\beta}$=29.6600, $\hat{\lambda}$=1018.6973 and $\hat{\theta}$=1.8073. We have depicted the graph of profile log-likelihood function for the parameters β, λ and θ in Figure 2 and found that the ML estimates can be calculated uniquely.

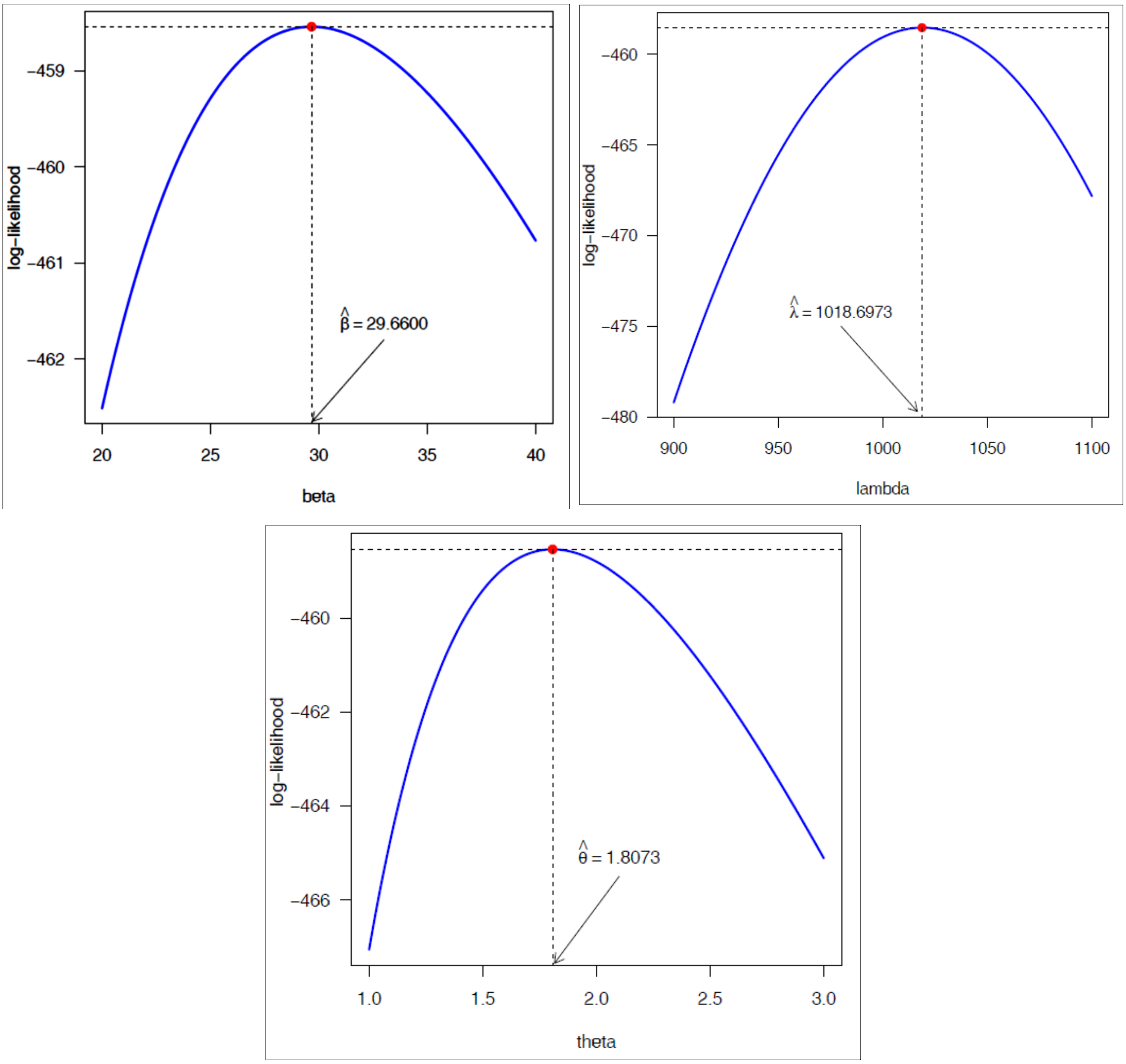

Fig 2: Profile log-likelihood function of the parameters β, λ and θ.

We have presented the graph of P-P plot and Q-Q plot in Figure 3 and it is found that the HCEE distribution fits the data very well.

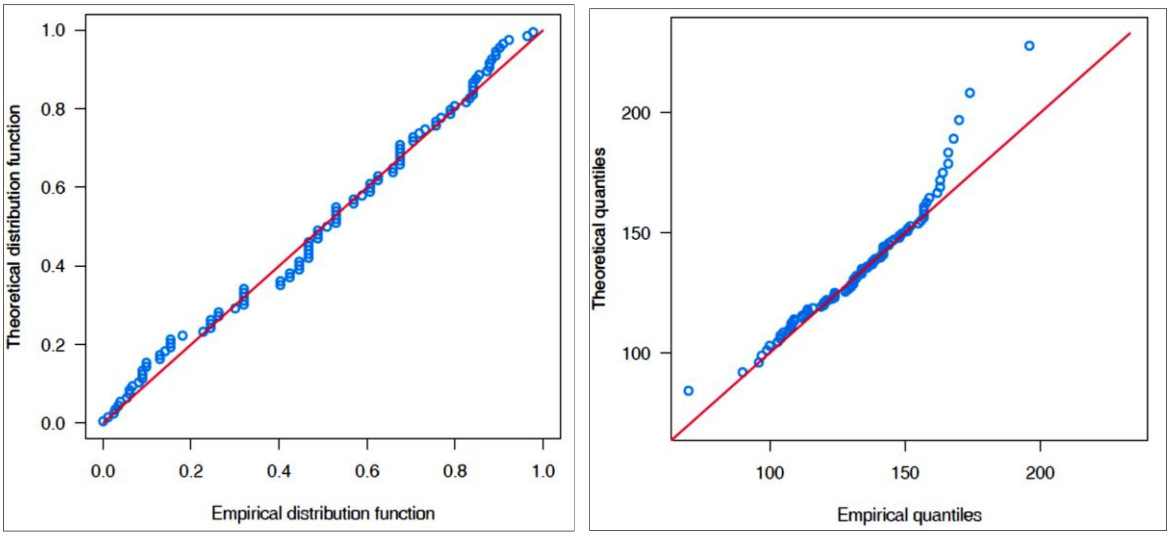

Fig 3: The P-P plot (left panel) and Q-Q plot (right panel) of the HCEE distribution.

Using MLE, LSE and CVE method we have displayed the estimated value of the parameters of HCEE distribution and their corresponding negative log-likelihood, and AIC criterion in Table 1.

Table 1: Estimated parameters, log-likelihood, and AIC

Method of Estimation	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	LL	AIC	HQIC
MLE	29.6600	1018.6973	1.8073	-458.5402	923.0805	926.2565
LSE	0.5322	1003.1930	0.0364	-458.5627	923.1254	926.3014
CVE	0.8713	1023.0540	0.0514	-458.5380	923.0760	926.2520

The KS, W and A^2 statistic with their corresponding p-value of MLE, LSE and CVE estimates we have presented in Table 2.

Table 2: The KS, W and A^2 statistic with a p-value

Method of Estimation	*KS(p-value)*	*W(p-value)*	*A^2(p-value)*
MLE	0.0642(0.7999)	0.0758(0.7177)	0.6866(0.5697)
LSE	0.0632(0.8141)	0.0764(0.7138)	0.6840(0.5719)
CVE	0.0650(0.7866)	0.0758(0.7179)	0.6888(0.5679)

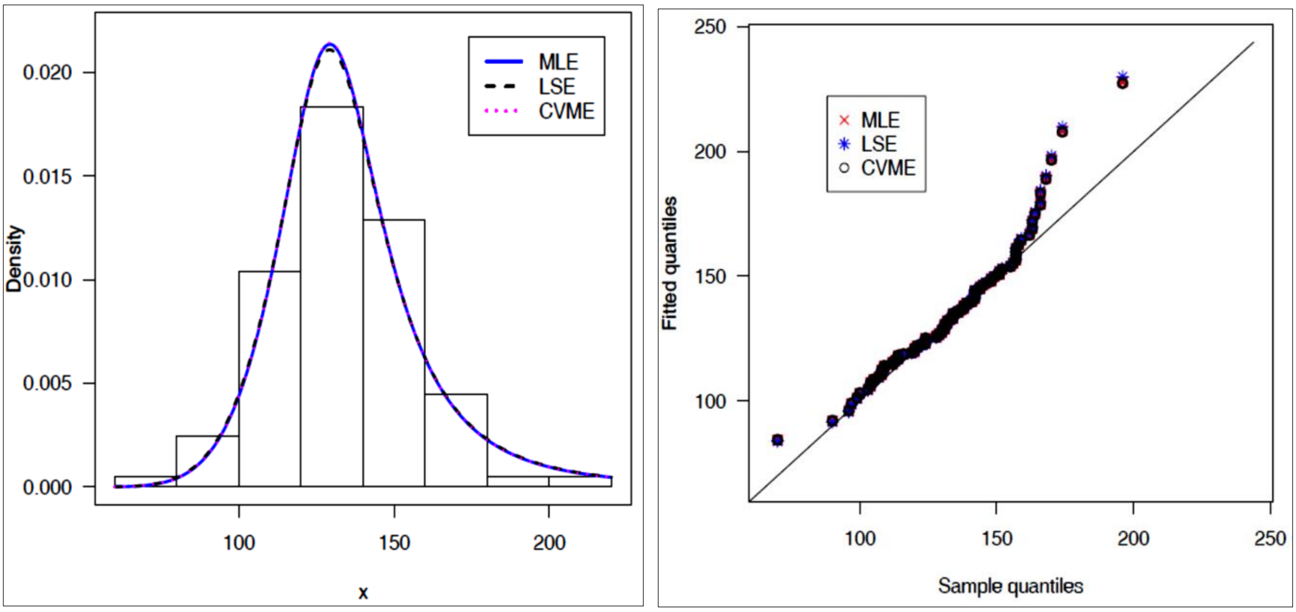

Fig 4: The Q-Q plot (right panel) and Histogram and the density function of fitted distributions (left panel) of estimation methods MLE, LSE and CVM of HCEE distribution.

In this section, we have illustrated the applicability of HCEE distribution using a real dataset used by earlier researchers. To compare the fit of the proposed model, we have taken the following six distributions.

i) Chen distribution

The probability density function of Chen distribution is presented by (Chen, 2000) [9] as

$$f_{CN}(x;\lambda,\ \beta) = \lambda\ \beta\ \ x^{\beta-1}\ e^{x^{\beta}}\ \boldsymbol{exp}\left\{\lambda\left(1-e^{x^{\beta}}\right)\right\}\quad ;\ (\lambda,\ \beta) > 0,\ x > 0.$$

ii) Exponential Extension (EE) distribution: NHE

The density of exponential extension (EE) distribution (Nadarajah & Haghighi, 2011) [24] with parameters α and λ is

$$f_{EE}(x) = \alpha\lambda\ (1+\lambda x)^{\alpha-1}\ \ exp\{1-(1+\lambda x)^{\alpha}\}\quad ; x \geq 0,\ \alpha > 0,\ \lambda\ > 0.$$

iii) Modified Weibull (MW)

The modified Weibull (MW) distribution was introduced by (Lai *et al*., 2003) [17] with probability density function (pdf)

$$f_{MW}(x) = \alpha(\lambda+\beta x)x^{\lambda-1}\, exp(\beta x - \alpha x^{\lambda} e^{\beta x}\, ; (\alpha\beta\lambda) > 0, x > 0$$

iv) Generalized Exponential (GE) distribution

The probability density function of generalized exponential distribution (Gupta & Kundu, 1999) [13].

$$f_{GE}(x;\alpha,\ \lambda) = \alpha\ \lambda\ e^{-\lambda\,x}\{1-e^{-\lambda\,x}\}^{\alpha-1}\ ;\ (\alpha,\ \lambda) > 0,\ x > 0.$$

v) Weibull Extension Model:

The probability density function of Weibull extension (WE) distribution (Tang *et al*., 2003) [32] with three parameters (α, β, λ) is

$$f_{WE}(x;\alpha,\ \beta,\lambda) = \ \ \lambda\beta\ \left(\frac{x}{\alpha}\right)^{\beta-1} exp\left(\frac{x}{\alpha}\right)^{\beta}\ \ exp\left\{-\lambda\alpha\ \left(exp\left(\frac{x}{\alpha}\right)^{\beta}-1\right)\right\}\quad ;\quad x > 0$$

$\alpha > 0, \beta > 0$ and $\lambda > 0$

vi) Exponentiated Weibull Distribution (EW)

The probability density function (PDF) of exponentiated Weibull distribution (EW) (Mudholkar & Srivastava, 1993) [23] is

$$f_{EW}(x) = \alpha\beta\lambda x^{\beta-1}\, exp\left(-\alpha x^{\beta}\right)\left\{1-exp\left(-\alpha x^{\beta}\right)\right\}^{\lambda-1}\ ; x > 0$$

We have illustrated the Akaike information criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike information criterion (CAIC), and Hannan-Quinn information criterion (HQIC) for the evaluation of the applicability of the HCEE distribution, which are displayed in Table 3.

Table 3: Log-likelihood (LL), AIC, BIC, CAIC and HQIC

Model	LL	AIC	BIC	CAIC	HQIC
HCEE	-458.5402	923.0805	930.9258	923.3279	926.2565
EW	-458.7600	923.5201	931.3654	923.7675	926.6961
EE	-460.8964	925.7928	931.0231	925.9153	927.9102
GE	-463.7324	931.4648	936.6951	931.5873	933.5822
WE	-466.0029	938.0058	945.8512	938.2532	941.1818
Chen	-467.0598	938.1196	943.3499	938.2421	940.2370
MW	-469.4255	944.8511	952.6964	945.0985	948.0271

We have displayed the graph of goodness-of-fit of HCEE distribution and some selected distributions are in Figure 5.

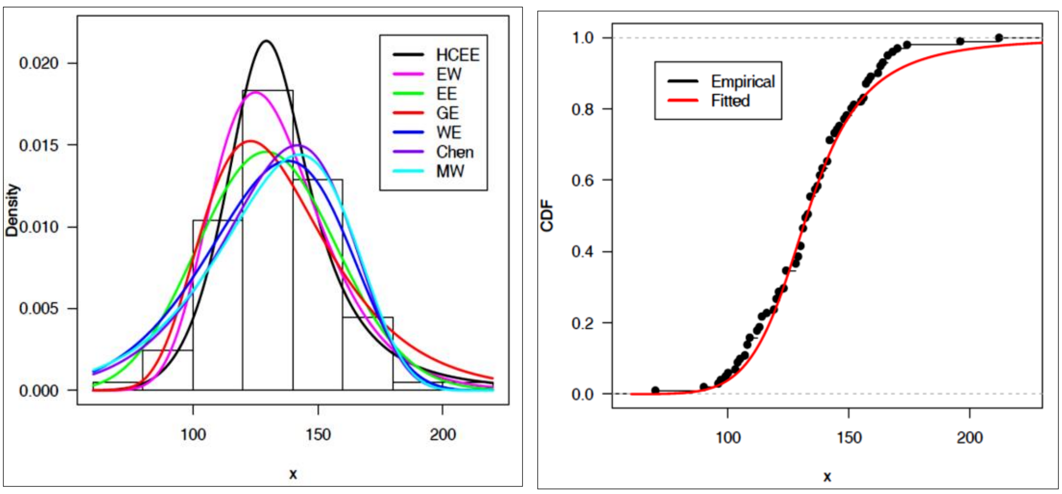

Fig 5: The Histogram and the density function of fitted distributions (left panel) and Empirical distribution function with estimated distribution function (right panel) of HCEE distribution.

To compare the goodness-of-fit of the HCEE distribution with other competing distributions, we have also displayed the value of Kolmogorov-Simnorov (KS), the Anderson-Darling (AD) and the Cramer-Von Mises (CVM) statistic in Table 4. It is observed that the HCEE distribution has the minimum value of the test statistic and higher *p*-value thus we conclude that the HCEE distribution gets quite better fit and more consistent and reliable results from others taken for comparison.

Table 4: The goodness-of-fit statistics and their corresponding p-value

Model	*KS(p-value)*	*AD(p-value)*	*CVM(p-value)*
HCEE	0.0642(0.7999)	0.0758(0.7177)	0.6866(0.5697)
EW	0.1105(0.1698)	0.1762(0.3190)	0.9031(0.4121)
EE	0.1226(0.0959)	0.3927(0.0753)	2.3444(0.0600)
GE	0.1066(0.2014)	0.3112(0.1257)	2.0724(0.0840)
WE	0.1174(0.1234)	0.3796(0.0817)	2.5899(0.0446)
Chen	0.1102(0.1718)	0.2960(0.1386)	2.0769(0.0835)
MW	0.1107(0.1682)	0.3691(0.0871)	2.5820(0.0450)

Conclusion

In this article, a new distribution named half Cauchy extended exponential distribution is presented. A broad study of some statistical characteristics of the new distribution like the derivation of precise expressions for its hazard rate function, survival function, the quantile function and skewness and kurtosis are presented. Three well-known estimation methods namely maximum likelihood estimation (MLE), Cramer-Von-Mises estimation (CVME), and least-square estimation (LSE) methods are used to estimate the parameter and we found that the MLEs are relatively better than LSE and CVM methods. The curves of the PDF of the proposed distribution have shown that it can have various shapes like increasing-decreasing and right skewed and flexible for modeling real-life data. Also, the graph of the hazard function is monotonically increasing or constant or reverse j-shaped according to the value of the model parameters. The applicability and suitability of the half Cauchy extended exponential distribution has been evaluated by considering a real-life dataset and the results exposed that the proposed distribution is much flexible as compared to some other fitted distributions.

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