Some Properties and Stability Results for Sector-Bounded LTI Systems

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Abstract

This paper presents necessary and sufficient conditions for a linear, time-invariant (LTI) system to be inside sector $[a, b]$ in terms of linear matrix inequalities in its state-space realization matrices, which represents a generalization of similar conditions for bounded $\mathcal{H}_\infty$-norm systems. Further, a weaker definition of LTI systems strictly inside sector $[a, b]$ is proposed, and state-space characterization of such systems is presented. Sector conditions for stability of the negative feedback interconnection of two LTI systems and for stability of LTI systems with feedback nonlinearities are investigated using the Lyapunov function approach. It is shown that the proposed weaker conditions for an LTI system to be strictly inside a sector are sufficient to establish closed-loop stability of these systems.

Introduction

Sector conditions for the stability of the feedback interconnection of general input-output systems were introduced in [1] and further expanded upon in [2,3]. These results were developed in an abstract mathematical setting of general input-output maps using operator-theoretic methods. In this paper we first present a state-space characterization of linear, time-invariant (LTI) systems inside sector $[a, b]$, in terms of three matrix relations in its system matrices, which is referred to as the sector-boundedness lemma, since it represent a generalization of the bounded realness lemma for bounded $\mathcal{H}_\infty$-norm systems [4,5]. Moreover, this state-space characterization of sector-bounded LTI systems can be equivalently expressed in terms of linear matrix inequalities, as shown in Theorem 1. Using these state-space characterizations, analysis of sector bounds on stable LTI systems can be performed with algebraic Riccati equations as well as convex programming techniques for the solution of linear matrix inequalities [6]. Next, a new definition is proposed for LTI systems strictly inside a sector, which is weaker than the corresponding definition in previous literature [1,2], and extends the notion of strict bounded realness [4,5] as well as strict positive realness [7]. State-space characterization of LTI systems strictly inside a sector is presented in Theorem 2. The next result

in Theorem 3 demonstrates that the weaker definition of systems strictly inside a sector is adequate to establish sector conditions for closed-loop stability of the negative feedback interconnection of two LTI systems [1,2], thus extending the previously available results. Theorem 4 presents a similar extension for sector conditions to guarantee stability of LTI systems with feedback nonlinearities. Moreover, these results are established using straightforward Lyapunov function techniques rather than the abstruse operator theoretic methods. Sector conditions for closed-loop stability are seen to be a generalization of the small gain theorem, and the ensuing discussion shows that these conditions may be used for synthesis of less conservative robust controllers, as opposed to the small gain theorem-based controllers.

Before proceeding, we present the notation used in the developments of this paper. $\mathrm{Herm}\{M\}$ is the Hermitian part of matrix M, that is, $\mathrm{Herm}\{M\} = \frac{1}{2}(M + M^*)$, where the superscript * denotes the conjugate transpose of the matrix; and M^T is the transpose of the matrix, M. A stable linear time-invariant system, Σ, is represented in state-space form as $\dot{x} = Ax + Bu$, $y = Cx + Du$, where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$ and $y \in \mathbf{R}^m$, with $n \geq m$. The square transfer function matrix of Σ, is $G(s) = C(sI - A)^{-1}B + D$, and the quadruple $[A, B, C, D]$ is a state-space representation of $G(s)$. $G^\sim(s) \triangleq G^T(-s) = B^T(-sI - A^T)^{-1}C^T + D^T$; thus, the quadruple $[-A^T, -C^T, B^T, D^T]$ is a state-space representation of $G^\sim(s)$. A square matrix transfer function, $\phi(s)$ is said to be parahermitian if $\phi^\sim(s) = \phi(s)$. The next section discusses the conditions for sector boundedness of LTI systems.

Sector-Bounded LTI Systems

A memoryless, time-varying nonlinearity, $\psi(y, t)$, is said to be inside sector $[a, b]$ if $(\psi - by)^T(\psi - ay) \leq 0$ for all $y \in \mathbf{R}^m$ [1]. Geometrically, these sector conditions imply that the graph of the nonlinearity lies within a conical region in the $\mathbf{R}^m \times \mathbf{R}^m$ input-output space for all time, t. For $m = 1$, Figure 1 shows a nonlinearity, $\psi(y, t)$, inside sector $[a, b]$; the graph of $\psi(y, t)$ must lie in the shaded region within two lines of slopes a and b. The concept of sector-boundedness extends to linear time-invariant systems by defining an LTI system, Σ, to be inside sector $[a, b]$ if $\mathrm{Herm}\{[G(j\omega) - aI]^*[G(j\omega) - bI]\} \leq 0$, for all $\omega \in \mathbf{R}$ [10]. The Nyquist plot of an LTI system, inside a sector $[a, b]$, $b > a$, lies within a circle in the frequency plane, whose

center is at $[(a+b)/2, j0]$ and has a radius of $(b-a)/2$. Note that a square, bounded, real system, that is, a system satisfying $\| G(s) \|_\infty \leq 1$, is inside sector $[-1, 1]$; and its Nyquist plot lies within a unit circle centered at the origin. For example, $G(s) = 15/(s+3)(s+5)$ is inside sector $[-0.4, 1.0]$ and its Nyquist plot lies within the corresponding circle, shown in Figure 2. Note that $\| G(s) \|_\infty \leq 1$, so that its Nyquist plot also lies within the unit circle centered at the origin, as shown in Figure 2.

The bounded realness lemma [4,5] gives a state-space characterization of $\mathcal{H}_\infty$-norm bounded LTI systems; and these conditions may equivalently be expressed as linear matrix inequalities [11]. Theorem 1 below presents the extension of these state-space characterizations to LTI systems inside sector $[a, b]$.

Theorem 1: Given a stable LTI system Σ : $\dot{x} = Ax + Bu$, $y = Cx + Du$, where the quadruple $[A, B, C, D]$ is a minimal realization of the transfer function matrix, $G(s) = C(sI - A)^{-1}B + D$, the following statements are equivalent.

(i) Σ is inside sector $[a, b]$.
(ii) There exist real matrices $P = P^T > 0$, L and W which satisfy

$$PA + A^T P = -C^T C - L^T L,$$
$$PB = C^T(\alpha I - D) - L^T W,$$
$$D^T D - \alpha\left(D + D^T\right) + abI = -W^T W,$$

where $\alpha = (a+b)/2$.
(iii) There exists a real matrix $P = P^T > 0$ which satisfies the linear matrix inequality

$$\begin{bmatrix} C^T C + PA + A^T P & PB - C^T(\alpha I - D) \\ B^T P - (\alpha I - D)^T C & D^T D - \alpha(D + D^T) + abI \end{bmatrix} \leq 0.$$

The proof of this theorem is presented in the Appendix. Note that, for $a = -1$ and $b = 1$, the conditions in (ii) are equivalent to the bounded realness lemma [4,5], and the linear matrix inequality in (iii) corresponds to an LMI condition for norm-bounded systems [6,11]. These state-space characterizations of sector-bounded systems allow the use of reliable numerical algorithms for solution of linear matrix inequalities and algebraic Riccati inequalities, to determine sector-bounds for a given LTI system from its state-space realizations. The existence of a symmetric, positive definite solution to the linear matrix inequality (LMI) of condition (iii) may be established using convex programming techniques as in [6]. Alternately, if $\tilde{R} = D^T D - \alpha\left(D + D^T\right) + abI < 0$, the LMI can be equivalently expressed as the algebraic Riccati inequality

$$A^T P + PA - \left[PB - C^T(\alpha I - D)\right] \tilde{R}^{-1}$$
$$\left[PB - C^T(\alpha I - D)\right]^T + C^T C \leq 0,$$

using a Schur complements identity. Positive definite solutions of this algebraic Riccati inequality may be obtained from a solution of the corresponding algebraic Riccati equation and using the results on comparative solutions of these equations [11, 12].

For linear, time-invariant systems, the notion of systems strictly inside a given sector in [1, 2] is more restrictive than

necessary to establish certain stability results. Following the approach in [4,5] for strictly bounded real systems and for strictly positive real systems [7], we propose the following definition of an LTI system to be strictly inside sector $[a, b]$.

Definition: A stable LTI system, $G(s)$, is said to be strictly inside sector $[a, b]$ if $\text{Herm}\{[G(j\omega) - aI]^*[G(j\omega) - bI]\} < 0$, for all $|\omega| < \infty$.

This definition for systems strictly inside a sector is weaker than that in previous literature [1, 2], which corresponds to $\text{Herm}\{[G(j\omega) - aI]^*[G(j\omega) - bI]\} \leq -\epsilon I$, for all $\omega \in \mathbf{R}$, with $\epsilon > 0$. Note that, with $a = -1$ and $b = 1$, the new definition of LTI systems strictly inside sector $[-1, 1]$ reduces to the condition for strict, bounded realness of the system, that is, $\| G(s) \|_\infty < 1$, as in [4,5]; whereas, under the previous definition, LTI systems strictly inside sector $[-1, 1]$ corresponds to LTI system satisfying $\| G(s) \|_\infty \leq 1 - \epsilon$, for $\epsilon > 0$. For example, a transfer function, $\frac{s+1}{s+2}$, is strictly inside sector $[-1, 1]$ with the new definition, but is not so under the previous definition. Theorems 3 and 4 show that the new (weaker) definition for systems satisfying sector conditions in a strict sense is adequate for stability results described therein. The following theorem characterizes LTI systems strictly inside sector $[a, b]$ in terms of its state-space realization, and is proved in the Appendix.

Theorem 2: A stable LTI system Σ : $\dot{x} = Ax + Bu$, $y = Cx + Du$, where the quadruple $[A, B, C, D]$ is a minimal realization of the transfer function matrix $G(s) = C(sI - A)^{-1}B + D$, is strictly inside sector $[a, b]$ if and only if there exist real matrices $P = P^T > 0$, L and W which satisfy

$$PA + A^T P = -C^T C - L^T L,$$
$$PB = C^T(\alpha I - D) - L^T W,$$
$$abI - \alpha\left(D + D^T\right) + D^T D = -W^T W,$$

where $\alpha = (a+b)/2$, and that the quadruple $[A, B, L, W]$ is a minimal realization of the transfer function $V(s) = L(sI - A)^{-1}B + W$, which does not have transmission zeros on the imaginary axis (that is, $\text{rank}\,[V(j\omega)] = m$, for all $|\omega| < \infty$).

Stability Results

Negative feedback interconnection of LTI systems, as shown in Figure 3, are known to be stable if the systems satisfy certain sector conditions given in [1-3]. This result is repeated here to demonstrate that the weaker conditions to characterize systems strictly inside a given sector are adequate for stability of the closed-loop system, and to present a direct Lyapunov function-based proof for these results.

Theorem 3: Consider two stable, LTI systems Σ_i, $i = 1, 2$, with minimal state-space realizations, $\dot{x}_i = A_i x_i + B_i u_i$, $y_i = C_i x_i + D_i u_i$, interconnected in the standard negative feedback configuration, shown in Figure 3. If Σ_1 is inside sector $[a, b]$, with $b > 0 > a$, and Σ_2 is inside the sector $\left[-\frac{1}{b}, -\frac{1}{a}\right]$, then the origin is a Lyapunov stable equilibrium point of the closed-loop system. If either Σ_1 or Σ_2 satisfies the sector conditions in a strict sense, then the origin is an asymptotically stable equilibrium point.

2

A straightforward Lyapunov function based proof to Theorem 3 is presented in the Appendix. Note that substituting $a = -1$ and $b = 1$ in Theorem 3 leads to a small gain theorem for stability of the feedback interconnection of LTI systems shown in Figure 3. The importance of this result stems from the fact that it can be used for less conservative robust controllers as opposed to those based on the small gain theorem. While characterizing uncertainty by the $\mathcal{H}_\infty$ norm of the uncertain plant, the frequency response of the plant must lie within a circle in the frequency plane, whose center must be at the origin. Tighter bounds for the uncertainty can be achieved by allowing the center of this circle to move along the real axis, as is provided by the sector boundedness condition. With tighter bounds on the uncertainty in terms of sector conditions, the result in Theorem 3 characterizes a larger set of compensators which provide robust stability, and consequently leads to synthesis of less conservative robust compensators. For example, by Theorem 3, stable closed-loop system is achieved with feedback around the plant $G(s)$ in Figure 2, for all compensators, $H(s)$, whose frequency response is within the circle corresponding to sector $[-1.0, 2.5]$. This circle is shown in Figure 4, along with a unit circle centered at the origin, within which the frequency response must lie to guarantee stability by the small gain theorem. Thus, sector conditions for stability characterize a larger set of compensators that provide closed-loop stability. Specifically, a compensator $H(s) = 15/s^2 + 4s + 8$, whose Nyquist plot is shown in Figure 4, would lead to a stable closed-loop system by Theorem 3; however, the small gain theorem cannot be used to establish closed-loop stability. Synthesis of compensators within a given sector $[a, b]$ can be performed using techniques suggested in [10].

Another stability result in sector theory considers the stability of linear systems with memoryless nonlinearities in the feedback loop [13, 14]. Let a minimal state-space realization of a stable LTI system be Σ : $\dot{x} = Ax + Bu$, $y = Cx + Du$, and let $\psi(y, t)$ be a memoryless nonlinearity, which may be time-varying. The problem is to obtain sufficient conditions for the stability of the negative feedback interconnection of the LTI system and the nonlinearity, as shown in Figure 5. Time-varying, nonlinear closed-loop equations of this system are $\dot{x} = Ax - B\psi(y, t)$, $y = Cx - D\psi(y, t)$. Theorem 4 below shows that under mild conditions on the nonlinearity, the new definition of LTI systems satisfying sector conditions in a strict sense is adequate to establish asymptotic stability of the closed-loop system above.

Theorem 4: Consider the negative feedback interconnection of a stable LTI system Σ : $\dot{x} = Ax + Bu$, $y = Cx + Du$, where the quadruple $[A, B, C, D]$ is a minimal realization, and a memoryless, time-varying, nonlinearity, $\psi(y, t)$ (as shown in Figure 5). If the nonlinearity, $\psi(y, t)$, is locally Lipschitz in y, uniformly Lipschitz in t, and belongs to sector $[a, b]$, with $b > 0 > a$, then the origin is a Lyapunov stable equilibrium point of the closed-loop system if Σ is inside sector $\left[-\frac{1}{b}, -\frac{1}{a}\right]$. If Σ is strictly inside sector $\left[-\frac{1}{b}, -\frac{1}{a}\right]$, then the origin is a globally asymptotically stable equilibrium point of the closed-loop system.

The proof of this theorem is given in the Appendix. Note that the conditions on the nonlinearity $\psi(y, t)$ are not very restrictive and are needed for well-posed closed-loop system as well. Moreover, global asymptotic stability of the closed-loop is ensured when the nonlinearity satisfies the sector condition for all $y \in \mathbf{R}^m$. However, if the nonlinearity satisfies the sector

condition in some local neighborhood of the origin, then this theorem guarantees (local) asymptotic stability about the origin. Theorem 4 presents a direct approach to this problem rather than applying loop transformations to convert sector $[a, b]$ into sector $[0, \infty)$ nonlinearities [11, 12]. Similar to Theorem 3, weaker constraints for LTI systems satisfying sector conditions in a strict sense make these results more general than those presented previously in the literature.

The results presented in this paper consider LTI systems inside sector $[a, b]$ with $b > 0 > a$. Corresponding results for a larger class of sectors can be developed with a similar approach [15]. For example, the negative feedback interconnection of an LTI system, Σ_1, inside sector $[0, b]$, for $b > 0$, with another LTI system, Σ_2, such that $(\Sigma_2 + bI)$ is strictly positive real, is stable, as expected from [1,2].

Conclusions

A state-space characterization of stable linear time-invariant systems inside sector $[a, b]$ in terms of a sector boundedness lemma and linear matrix inequalities has been presented. Moreover, weaker conditions for LTI systems to be strictly inside a sector are introduced. These weaker conditions are shown to be adequate for establishing sector conditions for (1) stability of the negative feedback interconnection of LTI systems, as well as, (2) the stability of LTI systems with sector-bounded feedback nonlinearities. Furthermore, a straightforward proof for these results is presented using the Lyapunov function approach, in contrast to the operator-theoretic methods used in previous works. Finally, it is argued, using the interpretation of sector-boundedness of LTI systems in terms of circles in the frequency plane, that less conservative robust controllers may be synthesized with these results, as opposed to those based on the small gain theorem.

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Appendix

Proof of Theorem 1:

(i) $\Rightarrow$ (ii): Note that $\phi(s) \triangleq G^\sim(s)G(s) - \alpha[G(s) + G^\sim(s)] + abI$, where $\alpha = (a + b)/2$, is parahermitian. If the linear system is inside sector $[a, b]$, then, by definition, $\phi(j\omega) \leq 0$, for all $\omega \in \mathbf{R}$. From the spectral factorization theorem [8, 9], there exists a stable transfer function, $V(s)$, with no transmission zeros in the open right-half plane, such that

$$\phi(s) = -V^\sim(s)V(s). \qquad (1)$$

If the quadruple $[A_1, B_1, C_1, D_1]$ is a minimal realization of $V(s)$, then $[-A_1^T, -C_1^T, B_1^T, D_1^T]$ is a minimal realization for $V^\sim(s)$, and a minimal realization for the product $V^\sim(s)V(s)$ is given by

$$\left[\begin{pmatrix} A_1 & 0 \\ -C_1^T C_1 & -A_1^T \end{pmatrix}, \begin{pmatrix} B_1 \\ -C_1^T D_1 \end{pmatrix}, (D_1^T C_1 \quad B_1^T), D_1^T D_1 \right].$$

Let $X_1 = X_1^T > 0$ be the observability grammian, that is, X_1 satisfies the Lyapunov equation

$$A_1^T X_1 + X_1 A_1 + C_1^T C_1 = 0. \qquad (2)$$

Applying the state transformation $\begin{bmatrix} I & 0 \\ X_1 & I \end{bmatrix}$, another minimal realization for the product $V^\sim(s)V(s)$ is given by

$$[\begin{pmatrix} A_1 & 0 \\ 0 & -A_1^T \end{pmatrix}, \begin{pmatrix} B_1 \\ -X_1 B_1 - C_1^T D_1 \end{pmatrix}, \qquad (3)$$
$$(B_1^T X_1 + D_1^T C_1 \quad B_1^T), \ D_1^T D_1].$$

Repeating this process with $G(s)$, which has a minimal realization $[A, B, C, D]$, a minimal realization of the product $G^\sim(s)G(s)$, is given by

$$[\begin{pmatrix} A & 0 \\ 0 & -A^T \end{pmatrix}, \begin{pmatrix} B \\ -XB - C^T D \end{pmatrix},$$
$$(B^T X + D^T C \quad B^T), \ D^T D],$$

where $X = X^T > 0$ is the symmetric positive definite observability grammian of $G(s)$, that is, X satisfies

$$A^T X + XA + C^T C = 0. \qquad (4)$$

Therefore, $\phi(s)$ has a minimal realization

$$[\begin{pmatrix} A & 0 \\ 0 & -A^T \end{pmatrix}, \begin{pmatrix} B \\ -XB + C^T(\alpha I - D) \end{pmatrix},$$
$$(B^T X - (\alpha I - D)^T C \quad B^T), \qquad (5)$$
$$D^T D - \alpha\left(D + D^T\right) + abI]$$

From (1), (3) and (5), we conclude (as in [14]) that there exists a nonsingular matrix T such that

$$A_1 = T^{-1}AT; \qquad B_1 = T^{-1}B$$
$$B_1^T X_1 + D_1^T C_1 = -\left[B^T X - (\alpha I - D)^T C\right]T \qquad (6)$$
$$D_1^T D_1 = -\left[D^T D - \alpha\left(D + D^T\right) + abI\right]$$

Premultiplying (2) by T^{-T} and postmultiplying by T^{-1} results in

$$\left(T^{-T}A_1^T T^T\right)T^{-T}X_1 T^{-T} + T^{-T}X_1 T^{-1}\left(TA_1 T^{-1}\right)$$
$$= -T^{-T}C_1^T C_1 T^{-1}. \qquad (7)$$

Adding (4) and (7) leads to $PA + A^T P = -C^T C - L^T L$, where $P = X + T^{-T}X_1 T^{-1}$, $P = P^T > 0$, and $L = C_1 T^{-1}$. Setting $D = W_1$ and rearranging the terms in (6) leads to the other two relations in (ii).

(i) $\Leftarrow$ (ii): Substituting $G(j\omega) = C(j\omega I - A)^{-1}B + D$, in $\phi(j\omega) = G^*(j\omega)G(j\omega) - \alpha[G(j\omega) + G^*(j\omega)] + abI$ leads to

$$\phi(j\omega) = B^T\left(-j\omega I - A^T\right)^{-1}C^T C(j\omega I - A)^{-1}B +$$
$$B^T\left(-j\omega I - A^T\right)^{-1}C^T(D - \alpha I) +$$
$$(D - \alpha I)^T C(j\omega I - A)^{-1}B +$$
$$D^T D - \alpha\left(D + D^T\right) + abI.$$

Using the matrix relations in (ii) leads to

$$\phi(j\omega) = -B^T\left(-j\omega I - A^T\right)^{-1}L^T L(j\omega I - A)^{-1}B$$
$$- B^T\left(-j\omega I - A^T\right)^{-1}\left[PA + A^T P\right](j\omega I - A)^{-1}B$$
$$- B^T\left(-j\omega I - A^T\right)^{-1}\left[PB + L^T W\right]$$
$$- \left[B^T P + W^T L\right]C(j\omega I - A)^{-1}B + W^T W.$$

Using the identity

$$B^T\left(-j\omega I - A^T\right)^{-1}\left(PA + A^T P\right)(j\omega I - A)^{-1}B =$$
$$- B^T P(j\omega I - A)^{-1}B - B^T\left(-j\omega I - A^T\right)^{-1}PB,$$

and some algebraic manipulations lead to

$$\phi(j\omega) = -B^T\left(-j\omega I - A^T\right)^{-1}L^T L(j\omega I - A)^{-1}B$$
$$- B^T\left(-j\omega I - A^T\right)^{-1}L^T W$$
$$- W^T L(j\omega I - A)^{-1}B - W^T W$$
$$= -V^*(j\omega)V(j\omega) \leq 0$$

for all $\omega \in \mathbf{R}$, where $V(j\omega) = L(j\omega I - A)^{-1}B + W$.

(ii) $\Leftrightarrow$ (iii): This equivalence follows directly by noting that

$$
\begin{bmatrix} C^T C + PA + A^T P & PB - C^T(\alpha I - D) \\ B^T P - (\alpha I - D)^T C & D^T D - \alpha(D + D^T) + abI \end{bmatrix}
$$
$$
= -\begin{bmatrix} L^T L & L^T W \\ W^T L & W^T W \end{bmatrix} \le 0.
$$

Proof of Theorem 2:

If $\phi(j\omega) < 0$, for all $|\omega| < \infty$, then $\phi(j\omega) = -V^*(j\omega)V(j\omega)$, where $V(s) = L(sI - A)^{-1}B + W$, as in the proof of Theorem 1. Further, since $\phi(j\omega)$ is negative definite on the imaginary axis, $V(j\omega)$ is full rank on the imaginary axis, that is, it does not have transmission zeros on the imaginary axis. Conversely, the proof of Theorem 1 shows that the sector-boundedness lemma leads to $\phi(j\omega) = -V^*(j\omega)V(j\omega)$. Since $V(s)$ is full rank on the imaginary axis, $\phi(j\omega) < 0$, for all $|\omega| < \infty$. Further, it is always possible to select L such that transmission zeros of $V(s)$ are in the open left-half plane [8, 9].

Proof of Theorem 3:

Stability of the closed-loop system is demonstrated using the following Lyapunov function $V(x_1, x_2) = V_1(x_1) - abV_2(x_2)$, where $V_i(x_i) = x_i^T P_i x_i$, with $P_i = P_i^T > 0$ being a positive definite matrix which satisfies the sector-boundedness lemma, for $i = 1, 2$. Note that since $b > 0 > a$, $V(x_1, x_2)$ is a positive definite function of x_1 and x_2. Time derivative of $V(x_1, x_2)$ is $\dot{V}(x_1, x_2) = \dot{V}_1(x_1) - ab\dot{V}_2(x_2)$. By Theorem 1, for $i = 1, 2$, if Σ_i is inside sector $[a_i, b_i]$, then there exist matrices $P_i = P_i^T > 0$, L_i and W_i, which satisfy

$$
\begin{aligned}
P_i A_i + A_i^T P_i &= -C_i^T C_i - L_i^T L_i, \\
P_i B_i &= C_i^T(\alpha_i I - D_i) - L_i^T W_i, \\
D_i^T D_i - \alpha_i\left(D_i + D_i^T\right) + a_i b_i I &= -W_i^T W_i,
\end{aligned} \tag{8}
$$

where $\alpha_i = (a_i + b_i)/2$. Consider $\dot{V}_i(x_i) = \dot{x}_i^T P_i x_i + x_i^T P_i \dot{x}_i = x_i^T(A_i^T P_i + P_i A_i)x_i + u_i^T B_i^T P_i x_i + x_i^T P_i B_i u_i$. Using the first two relations in (8) leads to

$$
\begin{aligned}
\dot{V}_i(x_i) = &-x_i^T L_i^T L_i x_i - x_i^T L_i^T W_i u_i - u_i^T W_i^T L_i x_i \\
&- x_i^T C_i^T C_i x_i - x_i^T C_i^T D_i u_i - u_i^T D_i^T C_i x_i \\
&+ \alpha_i x_i^T C_i^T u_i + \alpha_i u_i^T C_i x_i.
\end{aligned}
$$

Adding and subtracting $u_i^T W_i^T W_i u_i$ for "completing the square," using the last relationship in (8) and some algebraic manipulations, results in

$$
\begin{aligned}
\dot{V}_i(x_i) = &-[L_i x_i + W_i u_i]^T[L_i x_i + W_i u_i] \\
&- (y_i - b_i u_i)^T(y_i - a_i u_i)
\end{aligned}
$$

With these expressions for $\dot{V}_1(x_1)$ and $\dot{V}_2(x_2)$, the time derivative of the Lyapunov function $V(x_1, x_2)$ becomes

$$
\begin{aligned}
\dot{V}(x_1, x_2) = &-(L_1 x_1 + W_1 u_1)^T(L_1 x_1 + W_1 u_1) \\
&- (y_1 - bu_1)^T(y_1 - au_1) \\
&+ ab(L_2 x_2 + W_2 u_2)^T(L_2 x_2 + W_2 u_2) \\
&+ ab(y_2 + u_2/a)^T(y_2 + u_2/b)
\end{aligned}
$$

Using $u_2 = y_1$ and $u_1 = -y_2$ as implied by the negative feedback interconnection, $ab(y_2 + u_2/a)^T(y_2 + u_2/b) = (y_1 - bu_1)^T(y_1 - au_1)$. Therefore,

$$
\begin{aligned}
\dot{V}(x_1, x_2) = &-(L_1 x_1 + W_1 u_1)^*(L_1 x_1 + W_1 u_1) \\
&- (-ab)(L_2 x_2 + W_2 u_2)^*(L_2 x_2 + W_2 u_2) \\
\le \ & 0.
\end{aligned} \tag{9}
$$

Thus, the origin is a Lyapunov stable equilibrium of the closed-loop system.

When one of the system satisfies the sector conditions in a strict sense, asymptotic stability of the closed-loop is proved using LaSalle's Invariant Set Theorem [13, 14]. Suppose Σ_2 is strictly inside sector $\left[-\frac{1}{b}, -\frac{1}{a}\right]$. We demonstrate, by contradiction, that $x_1 = x_2 = 0$ is the only trajectory for which $\dot{V}(x_1, x_2) = 0$. If $\dot{V}(x_1, x_2) = 0$, then (9) implies that $y_2' = L_2 x_2 + W_2 u_2 = 0$. If $u_2 \ne 0$, then it has to be an exponentially decreasing function of time, since transmission zeros of $V_2(s)$ are in the open left half plane. Stability of the linear systems and the minimality of their realizations imply that x_1 and x_2 are exponentially decreasing functions of time, leading to $\dot{V}(x_1, x_2) < 0$, which is a contradiction. Thus, the origin is an asymptotically stable equilibrium point of the closed-loop system. A similar argument holds if Σ_1 satisfies the sector conditions in a strict sense.

Proof of Theorem 4:

Consider a Lyapunov function, $V(x) = x^T Px$, where $P = P^T > 0$ is a symmetric positive definite matrix which satisfies the sector-boundedness lemma in Theorem 1. From the proof of Theorem 3, we get

$$
\dot{V}(x) = -[Lx + Wu]^T[Lx + Wu] - (y + u/a)^T(y + u/b) \tag{10}
$$

Since the nonlinearity, $\psi(y, t)$, is inside sector $[a, b]$, $(\psi - by)^T(\psi - ay) \le 0$, for all y. The negative feedback condition implies that $u = -\psi$. Thus, we have $(u + by)^T(u + ay) \le 0$, or, $(y + u/a)^T(y + u/b) \ge 0$. Thus, from (10) we have $\dot{V}(x) \le 0$, and the origin is Lyapunov stable.

If Σ satisfies the sector conditions in a strict sense, and $\dot{V}(x) = 0$, then $x = 0$ is the only possible trajectory of the system, by a contradiction argument as in Theorem 3. Thus, by the Invariance Theorems in [14], the closed-loop system is asymptotically stable.
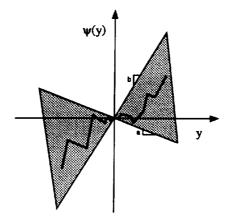
5

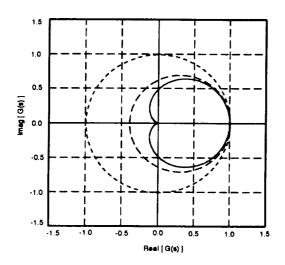Figure 1. Memoryless Nonlinearity Inside Sector [a,b].

Figure 2. Nyquist Plot of G(s) = 15/(s+3)(s+5)
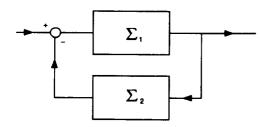Inside Sector [-0.4, 1.0].

Figure 4. Nyquist Plot of H(s) = 15/(s^2 + 4s + 8)
Inside Sector [-1.0, 2.5].

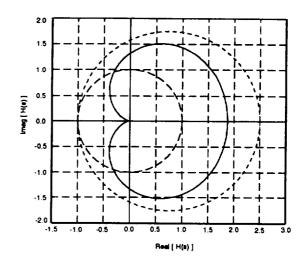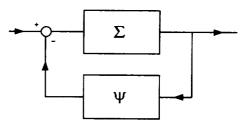Figure 3. Negative feedback interconnection of two
LTI systems, Σ_1 and Σ_2.

Figure 5. Absolute stability framework, with nonlinearity, ψ, in
negative feedback about an LTI system, Σ.

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