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SOME PROPERTIES OF 6-DIMENSIONAL K-SPACES

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§1. Introduction.

It is a well-known conjecture that a compact Riemannian manifold M with constant scalar curvature R is isometric to a sphere if it admits a one-parameter group of proper conformal transformations. One of the purposes of the present paper is to prove that the above conjecture is true in a 6-dimensional compact K-space without assuming that the scalar curvature is constant (cf. Theorem 5. 2). We shall also prove that a 6-dimensional compact simply connected K-space is necessarily isometric to a sphere if it admits a proper projective Killing vector (cf. Theorem 5. 4).

To prove the so-called sphere theorems, we need sometimes the decomposition theorems of conformal or projective Killing vectors in a Riemannian manifold. In this respect, we have already the following theorems in Einstein spaces:

THEOREM A. (Lichnerowicz [2]) *If an Einstein space M with non vanishing scalar curvature admits an infinitesimal conformal motion defined by v^i, that is, if we have $\mathcal{L}_v g_{ji}=2\rho g_{ji}$, ρ being a function, then the vector v^i is decomposed into*

$$v^i=p^i+\zeta^i,$$

where p^i is a Killing vector and ζ^i is a gradient vector defining an infinitesimal conformal motion.

THEOREM B. (Yano [13]) *If an Einstein space M with non vanishing scalar curvature admits an infinitesimal projective motion v^i, that is, if we have $\mathcal{L}_v\{{}^{\,h}_{ji}\}=\rho_j\delta_i^h+\rho_i\delta_j^h$, ρ_i being a covector, then the vector v^i is decomposed into*

$$v^i=p^i+\zeta^i,$$

where p^i is a Killing vector and ζ^i is a gradient vector defining an infinitesimal projective motion.

In the present paper, we shall show that the decomposition Theorems A and B hold in a 6-dimensional compact K-space which is not necessarily assumed to be an Einstein space (cf. Theorem 5. 3 and 6. 2).

In §2, some properties of K-spaces are recalled and, in §3, some lemmas are proved for the later use. In §4, we shall prove Theorem 4. 1 which is useful in

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the proofs of other theorems. In § 5, we shall study 6-dimensional K-spaces admitting a proper conformal Killing vector and prove some theorems. In § 6, we shall study 6-dimensional K-spaces admitting a proper projective Killing vector and obtain some results. In the last § 7, some properties of extended almost analytic vectors in a 6-dimensional compact K-space will be discussed and some results will be obtained.

§ 2. Preliminaries.

Let M be an n-dimensional ($n>2$) almost Hermitian manifold with Hermitian structure $(F_j{}^i, g_{ji})$, i.e. with an almost complex structure tensor $F_j{}^i$ and a positive definite Riemannian metric tensor g_{ji} satisfying

$$F_l{}^i F_j{}^l = -\delta_j^i, \tag{2.1}$$

$$g_{lt} F_j{}^l F_i{}^t = g_{ji}. \tag{2.2}$$

Then, from (2. 1) and (2. 2), we have

$$F_{ji} = -F_{ij}, \tag{2.3}$$

where $F_{ji} = F_j{}^l g_{li}$.

In an almost Hermitian manifold, we now define the following linear operators

$$O_{ih}^{ml} = \frac{1}{2}(\delta_i^m \delta_h^l - F_i{}^m F_h{}^l), \qquad {}^*O_{ih}^{ml} = \frac{1}{2}(\delta_i^m \delta_h^l + F_i{}^m F_h{}^l).$$

A tensor T_{ji} (resp. $T_j{}^i$) is said to be *pure* in the indices j, i, if it satisfies

$${}^*O_{ji}^{ab} T_{ab} = 0 \qquad (\text{resp.} \quad {}^*O_{jb}^{ai} T_a{}^b = 0),$$

and T_{ji} (resp. $T_j{}^i$) is said to be *hybrid* in the indices j, i, if it satisfies

$$O_{ji}^{ab} T_{ab} = 0 \qquad (\text{resp.} \quad O_{jb}^{ai} T_a{}^b = 0).$$

Then we can easily verify the following properties:

If $T_j{}^i$ is pure (resp. hybrid) in the indices j, i, then we have

$$F_t{}^i T_j{}^t = F_j{}^t T_t{}^i \qquad (\textit{resp.} \quad F_t{}^i T_j{}^t = -F_j{}^t T_t{}^i).$$

If S^{ji} is pure (resp. hybrid) in the indices j, i, then we have

$$F_t{}^j S^{ti} = F_t{}^i S^{jt} \qquad (\textit{resp.} \quad F_t{}^j S^{ti} = -F_t{}^i S^{jt}).$$

Let T_{ji} be pure in the indices j, i. If $S_j{}^i$ is pure (resp. hybrid) in the indices j, i, then $T_{jr} S_i{}^r$ is pure (resp. hybrid) in the indices j, i.

If T_{ji} is pure in the indices j, i and S^{ji} is hybrid in the indices j, i, then we have

$$T_{ji}S^{ji}=0.$$

These four properties will be often used in the sequel

If an almost Hermitian structure satisfies

$$\nabla_j F_{ih}+\nabla_i F_{jh}=0, \tag{2. 4}$$

where ∇_j denotes the operator of covariant differentiation with respect to Christoffel symbols $\{{}^{h}_{ji}\}$ formed with g_{ji}, then the manifold is called a *K-space* or a *Tachibana space*. Thus, from (2. 4), we have easily

$$\nabla_j F_i{}^j=0 \tag{2. 5}$$

in a K-space.

Now, in a K-space, let $R_{kji}{}^h$ and $R_{ji}=R_{tji}{}^t$ be the Riemannian curvature and Ricci tensors respectively. Then we have the following identities (cf. Tachibana [7], [9]):

$$*O_{ji}^{ab}\nabla_a F_{bh}=0, \tag{2. 6}$$

$$F_{hk}\nabla^t\nabla_t F_j{}^h=R_{kj}-R^*{}_{jk}, \qquad \text{or} \qquad \nabla^t\nabla_t F_j{}^h=F^{hl}(R_{jl}-R^*{}_{lj}), \tag{2. 7}$$

where $\nabla^t=g^{ta}\nabla_a$ and $R^*{}_{ji}=(1/2)F^{ab}R_{abti}F_j{}^t$;

$$O_{ji}^{ab}R_{ab}=0, \qquad O_{ji}^{ab}R^*{}_{ab}=0, \tag{2. 8}$$

$$R^*{}_{ji}=R^*{}_{ij}, \tag{2. 9}$$

$$\nabla_j F_{tl}(\nabla_i F^{tl})=R_{ji}-R^*{}_{ji}, \tag{2. 10}$$

where $F^{ji}=F_t{}^i g^{tj}$; and

$$(\nabla_j F_{ih})\nabla^j F^{ih}=R-R^*=\text{constant}>0, \tag{2. 11}$$

where $R=g^{ji}R_{ji}$ and $R^*=g^{ji}R^*{}_{ji}$.

In a K-space, we have

$$\frac{1}{2}\nabla_i R^*=\nabla^j R^*{}_{ji}$$

(cf. Sawaki [4]). Thus we have

$$\nabla^k(R_{ik}-R^*{}_{ik})=\frac{1}{2}\nabla_i(R-R^*)=0, \tag{2. 12}$$

because, in general, we have

$$\frac{1}{2}\nabla_i R=\nabla^j R_{ji}$$

in any Riemannian manifold.

§ 3. Some lemmas.

In general, it is well-known that the differential form

$$\hat{K} = \hat{K}_{ji} dx^j \wedge dx^i$$

is closed in any almost Hermitian manifold, where

$$\hat{K}_{ji} = 2R_{jir}{}^t F_t{}^r - F_s{}^t \nabla_j F_r{}^s \cdot \nabla_i F_t{}^r,$$

which is called the generalized Chern-form (cf. Tachibana [9]).

In a K-space, taking account of (2. 6) and (2. 10), we have

$$\hat{K}_{ji} = F_j{}^r (5R^*{}_{ri} - R_{ri}), \tag{3. 1}$$

from which, we have

LEMMA 3. 1. *In a K-space, the relation*

$$(R_{ji} - R^*{}_{ji})(5R^{*ji} - R^{ji}) = 0 \tag{3. 2}$$

holds.

Proof. Since, by (2. 8), $\hat{K}_{ji}$ is hybrid and $\nabla^h F^{ji}$ is pure in indices j, i, we have

$$\nabla_h F^{ji} \cdot \hat{K}_{ji} = 0, \tag{3. 3}$$

from which, applying ∇^h,

$$\nabla^h \nabla_h F^{ji} \cdot \hat{K}_{ji} + \nabla^h F^{ji} \cdot \nabla_h \hat{K}_{ji} = 0. \tag{3. 4}$$

In (3. 4), as $\hat{K}_{ji}$ is closed and $\nabla^h F^{ji}$ is skew-symmetric with respect to all indices, the second term vanishes. Hence, taking account of (2. 7) and (3. 1), we get (3. 2).

LEMMA 3. 2. *In a K-space, the relation*

$$R^*{}_{ji}(R^{ji} - R^{*ji}) = R_{kjih} O_{ts}^{ih} R^{kjts} \tag{3. 5}$$

holds.

Proof. Since $F^{kj} R_{kjih}$ is hybrid and $\nabla^t F^{ih}$ is pure in indices i, h, we get

$$F^{kj} R_{kjih} \nabla^t F^{ih} = 0, \tag{3. 6}$$

from which, applying ∇_t,

$$\nabla_t F^{kj} \cdot R_{kjih} \nabla^t F^{ih} + F^{kj} \nabla_t R_{kjih} \cdot \nabla^t F^{ih} + F^{kj} R_{kjih} \nabla_t \nabla^t F^{ih} = 0. \tag{3. 7}$$

In the next step, from (2. 1), we have

$$\nabla^i (F_t{}^j F^{th}) = 0, \tag{3. 8}$$

from which, applying ∇^k,

$$\nabla^k\nabla^i F_t{}^j\cdot F^{th}+\nabla^i F_t{}^j\cdot\nabla^k F^{th}+\nabla^k F_t{}^j\cdot\nabla^i F^{th}+F_t{}^j\nabla^k\nabla^i F^{th}=0. \tag{3.9}$$

On the other hand, since $R_{[kji]h}=0$, we have

$$R_{kjih}(\nabla^k F_t{}^j\cdot\nabla^i F^{th}+\nabla^j F_t{}^i\cdot\nabla^k F^{th}+\nabla^i F_t{}^k\cdot\nabla^j F^{th})=3\,!\,R_{[kji]h}(\nabla^k F_t{}^j\cdot\nabla^i F^{th})=0,$$

from which and (2. 4),

$$\begin{aligned} R_{kjih}\nabla^k F_t{}^j\cdot\nabla^i F^{th}&=-R_{kjih}\nabla^j F_t{}^i\cdot\nabla^k F^{th}+R_{kjih}\nabla^i F_t{}^j\cdot\nabla^k F^{th}\\ &=2R_{kjih}\nabla^i F_t{}^j\cdot\nabla^k F^{th}. \end{aligned} \tag{3.10}$$

Moreover, by $R_{k[jih]}=0$, we have

$$R_{kjih}(F^{tj}\nabla^k\nabla^i F_t{}^h+F^{ti}\nabla^k\nabla^h F_t{}^j+F^{th}\nabla^k\nabla^j F_t{}^i)=3\,!\,R_{k[jih]}F^{tj}\nabla^k\nabla^i F_t{}^h=0,$$

from which,

$$\begin{aligned} R_{kjih}(F^{tj}\nabla^k\nabla^i F_t{}^h)&=R_{kjih}(F^{th}\nabla^k\nabla^i F_t{}^j-F^{th}\nabla^k\nabla^j F_t{}^i)\\ &=2R_{kjih}F^{th}\nabla^k\nabla^i F_t{}^j. \end{aligned} \tag{3.11}$$

Therefore, transvecting (3. 9) with R_{kjih} and making use of (3. 10) and (3. 11), we get

$$\frac{3}{2}R_{kjih}\nabla^k F_t{}^j\cdot\nabla^i F^{th}=-3R_{kjih}F^{th}\nabla^k\nabla^i F_t{}^j,$$

from which,

$$R_{kjih}\nabla^k F_t{}^j\cdot\nabla^i F^{th}=2R_{kjih}F^{th}\nabla^k\nabla^j F_t{}^i,$$

or, by Ricci's identity,

$$\begin{aligned} R_{kjih}\nabla^k F_t{}^j\cdot\nabla^i F^{th}&=-R_{kjih}F^{ht}(\nabla^k\nabla^j F_t{}^i-\nabla^j\nabla^k F_t{}^i)\\ &=-R_{kjih}(R^{kjih}-R^{kjts}F_t{}^i F_s{}^h). \end{aligned} \tag{3.12}$$

Thus, substituting (3. 12) into (3. 7), we have

$$-R_{kjih}(R^{kjih}-R^{kjts}F_t{}^i F_s{}^h)+F^{kj}\nabla_t R_{kjih}\cdot\nabla^t F^{ih}+F^{kj}R_{kjih}\nabla_t\nabla^t F^{ih}=0. \tag{3.13}$$

Since $\nabla^t F^{ih}$ is skew-symmetric with respect to all indices and $\nabla_{[t}R_{|kj|ih]}=0$ in (3. 13), the second term of the left hand side vanishes. Thus (3. 13) becomes

$$F^{kj}R_{kjih}\nabla_t\nabla^t F^{ih}=R_{kjih}(R^{kjih}-R^{kjts}F_t{}^i F_s{}^h). \tag{3.14}$$

Thus, making use of (2. 7), we have (3. 5), from (3. 14).

LEMMA 3. 3. *In a K-space, the relation*

$$(R_{ji}-R^*{}_{ji})(R^{ji}-R^{*ji})=4R_{kjih}O_{ts}^{ih}R^{kjts} \tag{3. 15}$$

holds.

Proof. Equation (3. 2) reduces to

$$(R_{ji}-R^*{}_{ji})(R^{ji}-R^{*ji})=4R^*{}_{ji}(R^{ji}-R^{*ji}). \tag{3. 16}$$

Substituting (3. 16) into (3. 5), we get immediately (3. 15).
We now put, for arbitrarily given constants a and b,

$$\begin{aligned} T_{kjih}=R_{kjih}-a\{g_{kh}(R_{ji}-R^*{}_{ji})-g_{jh}(R_{ki}-R^*{}_{ki}) \\ +g_{ji}(R_{kh}-R^*{}_{kh})-g_{ki}(R_{jh}-R^*{}_{jh})\}+b(R-R^*)(g_{kh}g_{ji}-g_{jh}g_{ki}) \end{aligned} \tag{3. 17}$$

and

$$U_{kjih}=O_{ih}^{ts}T_{kjts}. \tag{3. 18}$$

Then we have

LEMMA 3. 4. *In a K-space, we have*

$$\begin{aligned} U_{kjih}U^{kjih}=R^{kjih}O_{ih}^{ts}R_{kjts}+2\{(n-4)a^2-2a\}(R_{ji}-R^*{}_{ji})(R^{ji}-R^{*ji}) \\ +\{2a^2+2b-4(n-2)ab+n(n-2)b^2\}(R-R^*)^2, \end{aligned} \tag{3. 19}$$

for arbitrary constants a and b.

Proof. First we have

$$\begin{aligned} O_{ih}^{ts}T_{kjts}=O_{ih}^{ts}R_{kjts}-\frac{a}{2}\{g_{kh}(R_{ji}-R^*{}_{ji})-g_{jh}(R_{ki}-R^*{}_{ki}) \\ +g_{ji}(R_{kh}-R^*{}_{kh})-g_{ki}(R_{jh}-R^*{}_{jh})\}+\frac{a}{2}\{F_i{}^tF_{hk}(R_{jt}-R^*{}_{jt}) \\ -F_i{}^tF_{hj}(R_{kt}-R^*{}_{kt})+F_{ij}F_h{}^s(R_{ks}-R^*{}_{ks})-F_{ik}F_h{}^s(R_{js}-R^*{}_{js})\} \\ +\frac{b}{2}(R-R^*)(g_{kh}g_{ji}-g_{jh}g_{ki}-F_{ij}F_{hk}+F_{ik}F_{hj}) \end{aligned} \tag{3. 20}$$

and, hence

$$\begin{aligned} U_{kjih}U^{kjih}=T^{kjih}O_{ih}^{ts}T_{kjts} \\ =R^{kjih}O_{ih}^{ts}T_{kjts}-ag^{kh}(R^{ji}-R^{*ji})O_{ih}^{ts}T_{kjts} \\ +ag^{jh}(R^{ki}-R^{*ki})O_{ih}^{ts}T_{kjts}-ag^{ji}(R^{kh}-R^{*kh})O_{ih}^{ts}T_{kjts} \end{aligned} \tag{3. 21}$$

$$+ag^{ki}(R^{jh}-R^{*jh})O_{ih}^{ts}T_{kjts}+b(R-R^{*})g^{kh}g^{ji}O_{ih}^{ts}T_{kjts}$$

$$-b(R-R^{*})g^{jh}g^{ki}O_{ih}^{ts}T_{kjts}.$$

Taking account of (2. 8) and (3. 20), we shall now calculate each term of the right hand side of (3. 21). The first term reduces to

$$
\begin{aligned}
&R^{kjih}O_{ih}^{ts}T_{kjts}\\
&=R^{kjih}O_{ih}^{ts}R_{kjts}-\frac{a}{2}\{R^{ji}(R_{ji}-R^{*}{}_{ji})+R^{ki}(R_{ki}-R^{*}{}_{ki})\\
&\quad+R^{kh}(R_{kh}-R^{*}{}_{kh})+R^{jh}(R_{jh}-R^{*}{}_{jh})\}+\frac{a}{2}\{R^{*jt}(R_{jt}-R^{*}{}_{jt})\\
&\quad+R^{*kt}(R_{kt}-R^{*}{}_{kt})+R^{*ks}(R_{ks}-R^{*}{}_{ks})+R^{*js}(R_{js}-R^{*}{}_{js})\}\\
&\quad+\frac{b}{2}(R-R^{*})(2R-2R^{*})\\
&=R^{kjih}O_{ih}^{ts}R_{kjts}-2a(R_{ji}-R^{*}{}_{ji})(R^{ji}-R^{*ji})+b(R-R^{*})^{2}.
\end{aligned}
\tag{3. 22}
$$

The second term reduces to

$$
\begin{aligned}
&-ag^{kh}(R^{ji}-R^{*ji})O_{ih}^{ts}T_{kjts}\\
&=-\frac{a}{2}(R^{ji}-R^{*ji})(R_{ji}-R^{*}{}_{ji})+\frac{a^{2}}{2}\{n(R^{ji}-R^{*ji})(R_{ji}-R^{*}{}_{ji})\\
&\quad-(R^{ji}-R^{*ji})(R_{ji}-R^{*}{}_{ji})+(R-R^{*})^{2}-(R^{ji}-R^{*ji})(R_{ji}-R^{*}{}_{ji})\}\\
&\quad-\frac{a^{2}}{2}\{(R^{ji}-R^{*ji})(R_{ji}-R^{*}{}_{ji})+(R^{ji}-R^{*ji})(R_{ji}-R^{*}{}_{ji})\}\\
&\quad-\frac{ab}{2}\{n(R-R^{*})^{2}-(R-R^{*})^{2}-(R-R^{*})^{2}\}\\
&=\frac{(n-4)a^{2}-a}{2}(R_{ji}-R^{*}{}_{ji})(R^{ji}-R^{*ji})+\frac{1}{2}\{a^{2}-(n-2)ab\}(R-R^{*})^{2}.
\end{aligned}
\tag{3. 23}
$$

Since T_{kjih} is skew-symmetric in the indices k, j and also i, h, the third, fourth and fifth terms are all equal to the second term, that is,

$$-ag^{kh}(R^{ji}-R^{*ji})O_{ih}^{ts}T_{kjts}=ag^{jh}(R^{ki}-R^{*ki})O_{ih}^{ts}T_{kjts} \tag{3. 24}$$

$$=-ag^{ji}(R^{kh}-R^{*kh})O_{ih}^{ts}T_{kjts}=ag^{ki}(R^{jh}-R^{*jh})O_{ih}^{ts}T_{kjts}.$$

The last two terms are equal to each other and each of term reduces to

$$b(R-R^*)g^{kh}g^{ji}O_{ih}^{ts}T_{kjts}=-b(R-R^*)g^{jh}g^{ki}O_{ih}^{ts}T_{kjts}$$

(3. 25)

$$=\left\{\frac{b}{2}-(n-2)ab+\frac{n(n-2)}{2}b^2\right\}(R-R^*)^2.$$

Substituting (3. 22), (3. 23), (3, 24) and (3. 25) into (3. 21), we have (3. 19).

Substituting (3. 15) into (3. 19), we obtain

$$U_{kjih}U^{kjih}=A(R_{ji}-R^*{}_{ji})(R^{ji}-R^{*ji})+B(R-R^*)^2, \tag{3. 26}$$

with

$$A=2(n-4)a^2-4a+\frac{1}{4}, \tag{3. 27}$$

$$B=2a^2+2b-4(n-2)ab+n(n-2)b^2, \tag{3. 28}$$

where a and b are arbitrary constants appearing in (3. 17).

If we consider the case $A+nB=0$, then we have, from (3. 26),

LEMMA 3. 5. *If, in a K-space, the arbitrary constants a and b satisfy the condition $A+nB=0$, then we have*

$$U_{kjih}U^{kjih}=A\left(R_{ji}-R^*{}_{ji}-\frac{R-R^*}{n}g_{ji}\right)\left(R^{ji}-R^{*ji}-\frac{R-R^*}{n}g^{ji}\right). \tag{3. 29}$$

We now note that there exist real numbers a and b satisfying the condition $A+nB=0$, if and only if $n\leqq 6$.

§ 4. Some theorems.

When $n=6$, there exist constants a and b, for example $a=1/2$, $b=1/8$, satisfying $A+nB=0$. If we put $n=6$, $a=1/2$ and $b=1/8$, then we have, from (3. 29),

$$U_{kjih}U^{kjih}=-\frac{3}{4}\left(R_{ji}-R^*{}_{ji}-\frac{R-R^*}{6}g_{ji}\right)\left(R^{ji}-R^{*ji}-\frac{R-R^*}{6}g^{ji}\right)$$

and hence

$$\left(R_{ji}-R^*{}_{ji}-\frac{R-R^*}{6}g_{ji}\right)\left(R^{ji}-R^{*ji}-\frac{R-R^*}{6}g^{ji}\right)=0,$$

from which, we have

THEOREM 4. 1. *In a 6-dimensional K-space, we have*

(4. 1) $$R_{ji}-R^*_{ji}=\frac{R-R^*}{6}g_{ji}.$$

Next, we have

THEOREM 4. 2. (Gray [1]) *There does not exist a 4-dimensional K-space, which is a Kähler space.*

Proof. In (3. 27) and (3. 28), if we put $n=4$, $a=1/8$ and $b=-1/16$, then we get

$$A=-\frac{1}{4}, \qquad B=0.$$

Hence, from (3. 26) with $n=4$, $a=1/8$ and $b=-1/16$, we have

$$R_{ji}-R^*_{ji}=0,$$

and, transvecting with g^{ji},

$$R-R^*=0,$$

from which and (2. 11),

$$\nabla_j F_{ih}=0.$$

This means that our manifold is Kählerian [12].

We shall now prove some theorems concerning curvatures of K-spaces.

THEOREM 4. 3. *In a 6-dimensional K-space, the scalar curvature is a positive constant.*

Proof. Substituting (4. 1) into (3. 2), we have

(4. 2) $$\frac{1}{6}(R-R^*)(5R^*-R)=0,$$

from which and $R\neq R^*$, it follows that

(4. 3) $$5R^*-R=0.$$

On the other hand, by (2. 12), we have

$$\nabla_i R=\nabla_i R^*.$$

Thus, taking account of (4. 3), we have

$$\nabla_i R=0,$$

which means that R is constant. Moreover, since (4. 2) can be written as

$$5(R-R^*)^2=4R(R-R^*)$$

and $R-R^*>0$ because of (2. 11), we see $R>0$.

THEOREM 4. 4. *In an Einstein K-space, the scalar curvature is a positive constant.*

Proof. From (3. 2), we have

$$5(R_{ji}-R^*_{ji})(R^{ji}-R^{*ji})=4R_{ji}(R^{ji}-R^{*ji}),$$

and, substituting $R_{ji}=(R/n)g_{ji}$,

$$(R_{ji}-R^*_{ji})(R^{ji}-R^{*ji})=\frac{4R}{5n}(R-R^*).$$

Consequently, by virtue of (2. 11), we have $R>0$.

THEOREM 4. 5. *In a 6-dimensional Einstein K-space, the generalized Chern-form $\hat{K}$ vanishes.*

Proof. From (3. 2), we have

$$(5R^*_{ji}-R_{ji})(5R^{*ji}-R^{ji})=4R_{ji}(5R^{*ji}-R^{ji})$$

and, substituting $R_{ji}=(R/6)g_{ji}$,

$$(5R^*_{ji}-R_{ji})(5R^{*ji}-R^{ji})=\frac{2}{3}R(5R^*-R).$$

On the other hand, we have, from (4. 3),

$$5R^*_{ji}-R_{ji}=0.$$

Thus we have

$$\hat{K}_{ji}=F_j{}^h(5R^*_{hi}-R_{hi})=0.$$

§ 5. Conformal Killing vectors.

A vector v^i is, by definition, a conformal Killing vector if it satisfies

$$\mathcal{L}_v g_{ji}=\nabla_j v_i+\nabla_i v_j=2\rho g_{ji}, \tag{5. 1}$$

where $\mathcal{L}_v$ is the operation of Lie derivation with respect to v^i, ρ being a scalar. For a conformal Killing vector, we have already the following

THEOREM 5. 1. (Sawaki and Takagi [6]) *If a compact K-space M with constant scalar curvature R of dimension $n>2$ such that*

$$R_{ji}-R^*_{ji}=kg_{ji} \qquad (k=constant) \tag{5. 2}$$

admits an infinitesimal nonhomothetic conformal transformation (*i.e. conformal Killing vector*) $\mathcal{L}_v g_{ji}=2\rho g_{ji}$, $\rho\neq$*constant, then M is isometric to a sphere.*

Thus, from Theorems 4. 1, 4. 3 and 5. 1, we have immediately

THEOREM 5. 2. *If a* 6-*dimensional compact K-space M admits a proper conformal Killing vector* v^i: $\mathcal{L}_v g_{ji}=2\rho g_{ji}$, $\rho\neq$*constant, then M is isometric to a sphere.*

We have here the following decomposition theorem:

THEOREM 5. 3. *In a* 6-*dimensional compact K-space, a conformal Killing vector* v^i *is decomposed into*

$$v^i=p^i+F_r{}^i q^r, \tag{5. 3}$$

where p^i *and* q^r *are both Killing vectors.*

Proof. In a compact K-space with constant scalar curvature R of dimension $n>2$ which satisfies (5. 2), a conformal Killing vector v^i is decomposed into

$$v^i=p^i+\eta^i, \tag{5. 4}$$

where p^i is a Killing vector and η^i is a conformal Killing vector which is gradient (cf. Sawaki and Takagi [6]). Thus, taking account of Theorems 4. 1 and 4. 3, we see that in a 6-dimensional compact K-space, in which the condition (5. 2) is not necessarily assumed, a conformal Killing vector v^i is decomposed into (5. 4). If we now put

$$q^i=-F_r{}^i\eta^r, \tag{5. 5}$$

then, applying ∇^j to (5. 5), we have

$$\nabla^j q^i+\nabla^i q^j=-(\nabla^j F_r{}^i+\nabla^i F_r{}^j)\eta^r-(F_r{}^i\nabla^j\eta^r+F_r{}^j\nabla^i\eta^r). \tag{5. 6}$$

Since $\nabla^j\eta^r=\rho g^{jr}$, we have, from (2. 3) and (2. 4),

$$\nabla^j q^i+\nabla^i q^j=0,$$

which means that q^i is a Killing vector. Substituting $\eta^i=F_r{}^i q^r$ obtained from (5. 5) into (5. 4), we have (5. 3).

§ 6. Projective Killing vectors.

A vector v^i is, by definition, a projective Killing vector if it satisfies

$$\mathcal{L}_v\left\{ {h \atop j\ i} \right\}=\rho_j\delta_i^h+\rho_i\delta_j^h, \qquad \rho_j=\frac{1}{n+1}\nabla_j\nabla_r v^r,$$

i.e., if it satisfies

(6. 1) $$\nabla_j\nabla_i v^h + R_{rji}{}^h v^r = \rho_j\delta_i^h + \rho_i\delta_j^h.$$

To prove Theorem 6. 2, we need

LEMMA 6. 1. (Takamatsu [10]) *In a compact K-space M with constant scalar curvature R, if p^i is a vector such that $\nabla_j p_i + \nabla_i p_j$ is pure in j, i, and, if r_i is a vector such that $r_i = \nabla_i r$ for a certain scalar r, then we have*

(6. 2) $$\int_M p^i r^j R_{ji} dv = 0,$$

where dv is the volume element of M.

THEOREM 6. 2. *In a 6-dimensional compact K-space, a projective Killing vector v^i is decomposed into*

(6. 3) $$v^i = p^i + r^i,$$

where p^i is a Killing vector and r^i is a gradient projective Killing vector.

Proof. A *K*-space is orientable. Thus, from the theory of harmonic integrals, we see that any vector v^i can be decomposed into

(6. 4) $$v^i = p^i + r^i,$$

where $\nabla_i p^i = 0$ and r^i is a vector such that $r^i = \nabla^i r$ for a certain scalar r. First of all, we shall prove that $\nabla_j p_i + \nabla_i p_j$ is pure in indices j, i. If we put

$$T_{ji} = \nabla_j p_i + \nabla_i p_j + F_j{}^a F_i{}^b(\nabla_a p_b + \nabla_b p_a),$$

then, we have

$$\nabla^i(p^j T_{ji}) = \frac{1}{4} T_{ji}T^{ji} + p^j \nabla^i T_{ji}$$

(6. 5) $$= \frac{1}{4} T_{ji}T^{ji} + p^j\nabla^i(\nabla_j p_i + \nabla_i p_j) + p^j F_j{}^a(\nabla^i F_i{}^b)(\nabla_a p_b + \nabla_b p_a)$$
$$+ p^j(\nabla^i F_j{}^a)F_i{}^b(\nabla_a p_b + \nabla_b p_a) + p^j F_j{}^a F_i{}^b \nabla^i(\nabla_a p_b + \nabla_b p_a),$$

because

$$\frac{1}{4} T_{ji}T^{ji} = (\nabla_j p_i + \nabla_i p_j)\nabla^j p^i + F_j{}^a F_i{}^b \nabla^j p^i(\nabla_a p_b + \nabla_b p_a).$$

The third and fourth terms in the right hand side of (6. 5) vanish, because $\nabla^i F_i{}^b = 0$ and $F_i{}^b \nabla^i F_j{}^a$ is skew-symmetric in the indices a, b. Therefore, (6. 5) turns out to be

(6. 6) $$\nabla^i(p^j T_{ji}) = \frac{1}{4} T_{ji}T^{ji} + p^j\{\nabla^i(\nabla_j p_i + \nabla_i p_j) + F_j{}^a F_i{}^b \nabla^i(\nabla_a p_b + \nabla_b p_a)\}.$$

On the other hand, substituting (6. 4) into (6. 1), we have

$$\nabla_j\nabla_i p^h+\nabla_j\nabla_i r^h+R_{rji}{}^h v^r=\rho_j\delta_i^h+\rho_i\delta_j^h, \tag{6. 7}$$

or

$$\nabla^i\nabla_a p_b=\rho^i g_{ab}+\rho_a\delta_b^i-\nabla^i\nabla_a r_b-R_r{}^i{}_{ab}v^r. \tag{6. 8}$$

Making use of (6. 8), we have

$$\nabla^i\nabla_a p_b+\nabla^i\nabla_b p_a=2\rho^i g_{ab}+\rho_a\delta_b^i+\rho_b\delta_a^i-2\nabla^i\nabla_a r_b. \tag{6. 9}$$

Transvecting (6. 7) with g^{ji} and δ_h^j, we have respectively,

$$\nabla^i\nabla_i p_h+\nabla^i\nabla_i r_h+R_{rh}v^r=2\rho_h, \tag{6. 10}$$

and

$$\nabla_j\nabla_i p^j+\nabla_j\nabla_i r^j-R_{ri}v^r=7\rho_i, \tag{6. 11}$$

From (6. 10) and (6. 11), we have

$$\begin{aligned}\nabla^i\nabla_i p_j+\nabla^i\nabla_j p_i&=2\rho_j-\nabla^i\nabla_i r_j-R_{rj}v^r+7\rho_j-\nabla_i\nabla_j r^i+R_{rj}v^r\\&=9\rho_j-2\nabla^i\nabla_j r_i.\end{aligned} \tag{6. 12}$$

Next, substituting (6. 9) and (6. 12) into (6. 6), we obtain

$$\nabla^i(p^jT_{ji})=\frac{1}{4}T_{ji}T^{ji}-2p^j\nabla^i\nabla_j r_i+10p^j\rho_j-2p^jF_j{}^aF^{ib}\nabla_i\nabla_b r_a. \tag{6. 13}$$

The second and the fourth terms of the right hand side of (6. 13) reduce respectively, by virtue of Ricci's identity, to

$$2p^j\nabla^i\nabla_j r_i=2p^j(-R^i{}_{ji}{}^s r_s)+2p^j\nabla_j\nabla_i r^i=2p^jR_j{}^s r_s+2p^j\nabla_j\nabla_i r^i, \tag{6. 14}$$

and

$$\begin{aligned}2p^jF_j{}^aF^{ib}\nabla_i\nabla_b r_a&=p^jF_j{}^aF^{ib}(\nabla_i\nabla_b r_a-\nabla_b\nabla_i r_a)\\&=-p^jF_j{}^aF^{ib}R_{iba}{}^s r_s\\&=-2R^*{}_{js}r^sp^j.\end{aligned} \tag{6. 15}$$

Thus (6. 13) becomes

$$\nabla^i(p^jT_{ji})=\frac{1}{4}T_{ji}T^{ji}-2(R_{ji}-R^*{}_{ji})p^jr^i-2p^j\nabla_j\nabla_i r^i+10p^j\rho_j. \tag{6. 16}$$

Moreover, substituting (4. 1) and $7\rho_j=\nabla_j\nabla_s r^s$ into (6. 16), we have

$$\nabla^i(p^jT_{ji})=\frac{1}{4}T_{ji}T^{ji}-p^j\left(\frac{R-R^*}{3}r_j+4\rho_j\right), \tag{6. 17}$$

from which, integrating (6. 17) over M,

$$\int_M \left[\frac{1}{4} T_{ji}T^{ji} - p^j \nabla_j \alpha\right] dv = 0, \tag{6. 18}$$

where $\alpha = ((R-R^*)/3)r + 4\rho$. Since $\nabla_i(\alpha p^i) = p^i \nabla_i \alpha + \alpha \nabla_i p^i = p^i \nabla_i \alpha$, (6. 18) becomes

$$\int_M \frac{1}{4} T_{ji}T^{ji} dv = 0, \tag{6. 19}$$

from which it follows that $T_{ji}=0$, which means that $\nabla_j p_i + \nabla_i p_j$ is pure in indices j, i.

To prove that p^i is a Killing vector, we put

$$U_{ji} = \nabla_j p_i + \nabla_i p_j.$$

Then we have

$$\nabla^i(p^j U_{ji}) = \frac{1}{2} U_{ji}U^{ji} + p^j \nabla^i(\nabla_j p_i + \nabla_i p_j). \tag{6. 20}$$

Substituting (6. 12) into (6. 20), we have, by virtue of (6. 14),

$$\begin{aligned} \nabla^i(p^j U_{ji}) &= \frac{1}{2} U_{ji}U^{ji} - 2p^j \nabla^i \nabla_j r_i + 9p^j \rho_j \\ &= \frac{1}{2} U_{ji}U^{ji} - 2p^j r^i R_{ji} + 9p^j \rho_j - 2p^j \nabla_j \nabla_i r^i \\ &= \frac{1}{2} U_{ji}U^{ji} - 2p^j r^i R_{ji} + 9\nabla_j(p^j \rho) - 2\nabla_j(p^j \nabla_i r^i), \end{aligned} \tag{6. 21}$$

from which, integrating (6. 21) over M,

$$\int_M \left[\frac{1}{2} U_{ji}U^{ji} - 2p^j r^i R_{ji}\right] dv = 0. \tag{6. 22}$$

Thus, making use of Lemma 6. 1, we have, from (6. 22),

$$\int_M \frac{1}{2} U_{ji}U^{ji} dv = 0,$$

from which it follows that $U_{ji}=0$, which means that p^i is a Killing vector.

In the last step, we shall prove that r^i is a projective Killing vector. Substituting $v^i = p^i + r^i$ into (6. 1), we have

$$\nabla_j \nabla_i p^h + R_{rji}{}^h p^r + \nabla_j \nabla_i r^h + R_{rji}{}^h r^r = \rho_j \delta_i^h + \rho_i \delta_j^h, \tag{6. 23}$$

but, since p^i is a Killing vector, it satisfies

$$\nabla_j \nabla_i p^h + R_{rji}{}^h p^r = 0.$$

Consequently, we have, from (6. 23),

$$\nabla_j\nabla_i r^h + R_{rji}{}^h r^r = \rho_j\delta_i^h + \rho_i\delta_j^h,$$

which means that r^i is a projective Killing vector.

To prove Theorem 6. 4, we need

THEOREM 6. 3. (Obata [3]) *Let M be a complete simply connected Riemannian manifold of dimension n. In order for M to admit a non-trivial solution ρ for the system of differential equations*

$$\nabla_k\nabla_j\rho_i + c(2\rho_k g_{ji} + \rho_j g_{ik} + \rho_i g_{kj}) = 0, \qquad c = constant > 0, \tag{6. 24}$$

it is necessary and sufficient that M be isometric with a sphere S^n of radius $1/\sqrt{c}$ in the Euclidean $(n+1)-$space.

THEOREM 6. 4. *If a 6-dimensional compact simply connected K-space M admits a proper projective Killing vector v^i, then M is isometric to a sphere.*

Proof. By Theorem 6. 2, v^i is decomposed into

$$v^i = p^i + r^i,$$

where r^i is a gradient projective Killing vector. Then r^i satisfies

$$\nabla_j\nabla_i r_k + R_{sjik} r^s = \rho_j g_{ik} + \rho_i g_{jk}, \qquad \rho_i = \frac{1}{7}\nabla_i\nabla_t r^t. \tag{6. 25}$$

Transvecting (6. 25) with g^{ji}, we have

$$\nabla^j\nabla_j r_k + R_{sk} r^s = 2\rho_k. \tag{6. 26}$$

On the other hand, we have, by Ricci's identity,

$$\begin{aligned}\nabla^i\nabla_i r_k &= \nabla^i\nabla_k r_i \\ &= \nabla_k\nabla^i r_i - R^i{}_{ki}{}^s r_s \\ &= 7\rho_k + R_k{}^s r_s.\end{aligned}$$

Consequently, (6. 26) becomes

$$5\rho_k + 2R_{ks} r^s = 0. \tag{6. 27}$$

Transvecting $F_k{}^j F^{ih}$ with (6. 25), we have

$$F_k{}^j F^{ih}\nabla_j\nabla_i r_h + F_k{}^j F^{ih} R_{sjih} r^s = F_k{}^j F^{ih}\rho_j g_{ih} + F_k{}^j F^{ih}\rho_i g_{jh},$$

from which,

$$-2R^*{}_{kt} r^t = \rho_k, \tag{6. 28}$$

because of $\nabla_i r_h = \nabla_h r_i$ and $F_k{}^j F^{ih} R_{sjih} r^s = -2R^*{}_{kt} r^t$. Adding the both sides of (6. 27) and (6. 28), we have

$$4\rho_k + 2(R_{kt} - R^*{}_{kt}) r^t = 0. \tag{6.29}$$

Substituting (4. 1) into (6. 29), we have

$$4\rho_i + \frac{R - R^*}{3} r_i = 0. \tag{6.30}$$

Applying $\nabla_k \nabla_j$ to (6. 30), we have, by virtue of (2. 12),

$$\nabla_k \nabla_j \rho_i + \frac{R - R^*}{12} \nabla_k \nabla_j r_i = 0. \tag{6.31}$$

Moreover, interchanging i and k in (6. 25) and adding the equation thus obtained to (6. 25), we get

$$2\nabla_j \nabla_i r_k = 2\rho_j g_{ik} + \rho_i g_{jk} + \rho_k g_{ji},$$

or

$$2\nabla_k \nabla_j r_i = 2\rho_k g_{ji} + \rho_j g_{ki} + \rho_i g_{kj}.$$

Substituting the last equation into (6. 31), we obtain

$$\nabla_k \nabla_j \rho_i + \frac{R - R^*}{24} (2\rho_k g_{ji} + \rho_j g_{ik} + \rho_i g_{kj}) = 0. \tag{6.32}$$

Thus, by Theorem 6. 3, we see that M is isometric to a sphere.

§ 7. Extended contravariant almost analytic vectors.

In an almost complex manifold M, a vector v^i is called an extended contravariant almost analytic vector if it satisfies, for a scalar λ,

$$\mathcal{L}_v F_j{}^i + \lambda F_j{}^r N_{rl}{}^i v^l = 0, \tag{7.1}$$

where $N_{rl}{}^i$ is the Nijenhuis tensor, that is, $N_{rl}{}^i = F_r{}^s(\partial_s F_l{}^i - \partial_l F_s{}^i) - F_l{}^s(\partial_s F_r{}^i - \partial_r F_s{}^i)$, (cf. Sawaki and Takamatsu [5]). If $\lambda = 0$, then v^i is a usual contravariant almost analytic vector (cf. Tachibana [7]).

When M is a K-space, for an extended contravariant almost analytic vector v^i corresponding to $\lambda = -1/4$, we have, from (7. 1),

$$\nabla_j v_i - F_j{}^a F_i{}^b \nabla_a v_b = 0, \tag{7.2}$$

where $v_i = g_{ji} v^j$ (cf. Sawaki and Takamatsu [5]). Concerning an extended contravariant almost analytic vector in a K-space, we have proved in [11] the following

LEMMA 7. 1. *In a compact K-space, an extended contravariant almost analytic vector v^i, for a constant λ such that $-3/4 \leqq \lambda \leqq 0$, $\lambda \neq -1/4$, satisfies*

(7. 3) $$\nabla^r\nabla_r v^i+R_t{}^i v^t=0,$$

and can be decomposed into

(7. 4) $$v^i=p^i+r^i,$$

where $\nabla_i p^i=0$ *and* r^i *is a vector such that* $r^i=\nabla^i r$, r *being a certain scalar. Moreover, for the vectors* p^i *and* r^i, *we have respectively*

(7. 5) $$*O_{ab}^{ji}(\nabla^a p^b+\nabla^b p^a)=0,$$

and

(7. 6) $$r^t\nabla_t F_{ji}=0.$$

We shall now prove the following

THEOREM 7. 2. *In a* 6-*dimensional compact K-space, an extended contravariant almost analytic vector* v^i *for a constant* λ *such that* $-3/4\leqq\lambda\leqq 0$, $\lambda\neq -1/4$, *is necessarily isometric.*

Proof. From (7. 6), we have

$$r^s\nabla_s F_{ji}\cdot r^t\nabla_t F^{ji}=0,$$

which reduces to

(7. 7) $$r^s r^t(R_{st}-R^*{}_{st})=0,$$

by (2. 10). We have, from (4. 1) and (7. 7),

$$\frac{R-R^*}{6}r_s r^s=0,$$

from whicn, $r_s=0$. Consequently, we have, by virtue of (7. 3),

$$v^i=p^i.$$

Thus, by $\nabla_i v^i=0$ and (7. 3), we find that v^i is a Killing vector.

To prove Theorem 7. 4, we need

THEOREM 7. 3. (Sawaki and Takagi [6]) *Let M be a compact K-space of dimension* $n>2$ *such that*

$$\frac{1}{n-1}R_{ji}=R^*{}_{ji}.$$

If M admits an extended contravariant almost analytic vector v^i *for* $\lambda=-1/4$, *then M is isometric to a sphere.*

Thus, from Theorems 4. 5 and 7. 3, we have immediately

THEOREM 7.4. *If a 6-dimensional compact Einstein K-space M admits an extended contravariant almost analytic vector v^i for $\lambda=-1/4$, then M is isometric to a sphere.*

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