

## SOME PROPERTIES OF A SEQUENCE ANALOGOUS TO EULER NUMBERS

ZHI-HONG SUN

(Received 28 April 2012; accepted 6 May 2012; first published online 12 June 2012)

### Abstract

Let  $\{U_n\}$  be given by  $U_0 = 1$  and  $U_n = -2 \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{2k} U_{n-2k}$  ( $n \geq 1$ ), where  $\lfloor \cdot \rfloor$  is the greatest integer function. Then  $\{U_n\}$  is analogous to the Euler numbers and  $U_{2n} = 3^{2n} E_{2n}(\frac{1}{3})$ , where  $E_m(x)$  is the Euler polynomial. In a previous paper we gave many properties of  $\{U_n\}$ . In this paper we present a summation formula and several congruences involving  $\{U_n\}$ .

2010 *Mathematics subject classification*: primary 11A07; secondary 11B68.

*Keywords and phrases*: congruence, summation formula, Euler number.

### 1. Introduction

The Euler numbers  $\{E_n\}$  and Euler polynomials  $\{E_n(x)\}$  are defined by

$$E_0 = 1, \quad E_n = - \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{2k} E_{n-2k} \quad (n \geq 1),$$
$$E_n(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (2x-1)^{n-2k} E_{2k} \quad (n \geq 0),$$

where  $\lfloor x \rfloor$  is the greatest integer not exceeding  $x$ . In [8] the author introduced and studied the sequence  $\{U_n\}$  (similar to Euler numbers) given by

$$U_0 = 1, \quad U_n = -2 \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{2k} U_{n-2k} \quad (n \geq 1). \quad (1.1)$$

Since  $U_1 = 0$ , by induction  $U_{2n-1} = 0$  for  $n \geq 1$ . The first few values of  $U_{2n}$  are as follows:

$$U_2 = -2, \quad U_4 = 22, \quad U_6 = -602, \quad U_8 = 30742, \quad U_{10} = -2\,523\,002,$$
$$U_{12} = 303\,692\,662, \quad U_{14} = -50\,402\,079\,002, \quad U_{16} = 11\,030\,684\,333\,782.$$

The author is supported by the Natural Sciences Foundation of China (grant no. 10971078).

© 2012 Australian Mathematical Publishing Association Inc. 0004-9727/2012 \$16.00

Let  $(a/p)$  be the Legendre symbol. In [8], the author proved that for any prime  $p > 3$ ,

$$\sum_{k=1}^{\lfloor 2p/3 \rfloor} \frac{(-1)^{k-1}}{k} \equiv 3p \left(\frac{p}{3}\right) U_{p-3} \pmod{p^2}. \tag{1.2}$$

The Bernoulli numbers  $\{B_n\}$  and Bernoulli polynomials  $\{B_n(x)\}$  are given by

$$B_0 = 1, \quad \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0 \quad (n \geq 2) \quad \text{and} \quad B_n(x) = \sum_{k=0}^n \binom{n}{k} B_k x^{n-k} \quad (n \geq 0).$$

By [8, p. 217],

$$B_{p-2}(\tfrac{1}{3}) \equiv 6U_{p-3} \pmod{p} \quad \text{for any prime } p > 3. \tag{1.3}$$

In [4] Mattarei and Tauraso proved that for any prime  $p > 3$ ,

$$\sum_{k=0}^{p-1} \binom{2k}{k} \equiv \left(\frac{p}{3}\right) - \frac{p^2}{3} B_{p-2}(\tfrac{1}{3}) \pmod{p^3}.$$

Thus,

$$\sum_{k=0}^{p-1} \binom{2k}{k} \equiv \left(\frac{p}{3}\right) - 2p^2 U_{p-3} \pmod{p^3} \quad \text{for any prime } p > 3.$$

Suppose that  $p$  is a prime of the form  $3k + 1$  and so  $4p = L^2 + 27M^2$  for some integers  $L$  and  $M$ . Assume that  $L \equiv 1 \pmod{3}$ . From (1.3) and [3, Theorem 6]

$$\left(\frac{\frac{2(p-1)}{3}}{\frac{p-1}{3}}\right) \equiv \left(-L + \frac{p}{L} + \frac{p^2}{L^3}\right) (1 + p^2 U_{p-3}) \equiv -L + \frac{p}{L} + p^2 \left(\frac{1}{L^3} - LU_{p-3}\right) \pmod{p^3}.$$

In Section 2 we prove a summation formula involving  $U_n$ ; see Theorem 2.1. Let  $\mathbb{N}$  be the set of positive integers. If  $n \in \mathbb{N}$  and  $2^\alpha \mid n$ , in [8] the author determined  $U_{2n} \pmod{2^{\alpha+7}}$ . In Section 3 we prove that

$$3U_{2n} \equiv -3072n^4 + 4608n^3 + 2240n^2 + 1680n + 2 \pmod{2^{\alpha+14}} \quad \text{for } n \geq 7.$$

For  $k, m, b \in \mathbb{N}$  with  $2 \mid b$ , in Section 3 we also show that

$$U_{2^m k+b} \equiv U_b + 2^{b+1} \pmod{2^{\min\{b,m\}+3}}.$$

Let  $k, m \in \mathbb{N}$  and  $b \in \{0, 2, 4, \dots\}$ . From [8, Theorem 4.3] we have  $U_{k\varphi(3^m)+b} \equiv U_b \pmod{3^m}$ , where  $\varphi(n)$  is Euler’s totient function. In Section 4 we prove a congruence for  $U_{k\varphi(3^m)+b} - U_b \pmod{3^{m+4}}$  for  $m \geq 3$ ; see Theorem 4.2. In Section 5 we prove a congruence for  $E_{k\varphi(3^m)+b} - (3^b + 1)E_b \pmod{3^{m+4}}$  for  $m \geq 3$ ; see Theorem 5.2.

## 2. A summation formula involving $U_n(x)$

For  $n = 0, 1, 2, \dots$  let

$$U_n(x) = \sum_{r=0}^n \binom{n}{r} U_r x^{n-r} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} U_{2k} x^{n-2k}. \quad (2.1)$$

The first few  $U_n(x)$  are as follows:

$$\begin{aligned} U_0(x) &= 1, & U_1(x) &= x, & U_2(x) &= x^2 - 2, \\ U_3(x) &= x^3 - 6x, & U_4(x) &= x^4 - 12x^2 + 22, \\ U_5(x) &= x^5 - 20x^3 + 110x, & U_6(x) &= x^6 - 30x^4 + 330x^2 - 602. \end{aligned}$$

By [8, Theorem 2.3],

$$U_n(x-1) - U_n(x) + U_n(x+1) = x^n, \quad (2.2)$$

$$U_n(x) + U_n(x+3) = (x+1)^n + (x+2)^n, \quad (2.2)$$

$$U_n(x+3) - U_n(x-3) = (x+2)^n + (x+1)^n - (x-1)^n - (x-2)^n. \quad (2.3)$$

Taking  $a_n = U_n(x)$  and  $b_n = x^n$  in [8, Theorem 2.2],

$$x^n = 2 \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} U_{n-2k}(x) - U_n(x).$$

That is,

$$U_n(x) = x^n - 2 \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{2k} U_{n-2k}(x).$$

Since

$$\begin{aligned} \int_a^b U_n(x) dx &= \sum_{k=0}^n \binom{n}{k} U_k \int_a^b x^{n-k} dx = \sum_{k=0}^n \binom{n}{k} U_k \frac{x^{n-k+1}}{n-k+1} \Big|_a^b \\ &= \frac{1}{n+1} \sum_{k=0}^{n+1} \binom{n+1}{k} U_k x^{n+1-k} \Big|_a^b, \end{aligned}$$

we see that

$$\int_a^b U_n(x) dx = \frac{U_{n+1}(b) - U_{n+1}(a)}{n+1}. \quad (2.4)$$

This together with (2.3) yields

$$\int_{a-3}^{a+3} U_n(x) dx = \frac{(a+2)^{n+1} + (a+1)^{n+1} - (a-1)^{n+1} - (a-2)^{n+1}}{n+1}.$$

Since  $U_n(0) = U_n$ , by (2.4),

$$U_n(x) = U_n + n \int_0^x U_{n-1}(t) dt.$$

Let  $m, n \in \mathbb{N}$ . From [1] we have the following well-known summation formulas:

$$\sum_{k=0}^{m-1} k^n = \frac{B_{n+1}(m) - B_{n+1}}{n + 1} \quad \text{and} \quad \sum_{k=0}^{m-1} (-1)^k k^n = \frac{E_n(0) - (-1)^m E_n(m)}{2}.$$

We now present the following similar result.

**THEOREM 2.1.** *Let  $m, n \in \mathbb{N}$  and*

$$S_n(m) = (m - 1)^n + (m - 2)^n - (m - 4)^n - (m - 5)^n + (m - 7)^n + (m - 8)^n - \dots,$$

where the term  $a^n$  vanishes when  $a \leq 0$ . Then

$$S_n(m) = \begin{cases} U_n(m) - (-1)^{m/3} U_n & \text{if } 3 \mid m, \\ U_n(m) - (-1)^{\lfloor (m+1)/3 \rfloor} U_n/2 & \text{if } 3 \nmid m \text{ and } 2 \mid n, \\ U_n(m) - (-1)^{\lfloor m/3 \rfloor} U_n(1) & \text{if } 3 \nmid m \text{ and } 2 \nmid n. \end{cases}$$

**PROOF.** Using (2.2), we see that

$$\begin{aligned} & (m - 1)^n + (m - 2)^n - (m - 4)^n - (m - 5)^n + (m - 7)^n + (m - 8)^n \\ & - \dots - (-1)^{\lfloor m/3 \rfloor} \left( \left( m - 3 \left\lfloor \frac{m}{3} \right\rfloor + 2 \right)^n + \left( m - 3 \left\lfloor \frac{m}{3} \right\rfloor + 1 \right)^n \right) \\ & = (U_n(m) + U_n(m - 3)) - (U_n(m - 3) + U_n(m - 6)) + (U_n(m - 6) + U_n(m - 9)) \\ & \quad - \dots - (-1)^{\lfloor m/3 \rfloor} \left( U_n \left( m - 3 \left\lfloor \frac{m}{3} \right\rfloor + 3 \right) + U_n \left( m - 3 \left\lfloor \frac{m}{3} \right\rfloor \right) \right) \\ & = U_n(m) - (-1)^{\lfloor m/3 \rfloor} U_n \left( m - 3 \left\lfloor \frac{m}{3} \right\rfloor \right). \end{aligned}$$

Thus,

$$S_n(m) = \begin{cases} U_n(m) - (-1)^{m/3} U_n(0) & \text{if } 3 \mid m, \\ U_n(m) - (-1)^{\lfloor m/3 \rfloor} U_n(1) & \text{if } 3 \mid m - 1, \\ (-1)^{\lfloor m/3 \rfloor} \cdot 1 + U_n(m) - (-1)^{\lfloor m/3 \rfloor} U_n(2) & \text{if } 3 \mid m - 2. \end{cases}$$

Clearly  $U_n(0) = U_n$ . By (2.2) and (2.1), we have  $U_n(-1) + U_n(2) = 1$  and so  $U_n(2) = 1 - U_n(-1) = 1 - (-1)^n U_n(1)$ . If  $2 \mid n$ , using (1.1) we see that

$$U_n(1) = \sum_{k=0}^{n/2} \binom{n}{2k} U_{2k} = \sum_{k=0}^{n/2} \binom{n}{2k} U_{n-2k} = U_n - \frac{1}{2} U_n = \frac{1}{2} U_n$$

and so

$$U_n(2) = 1 - U_n(1) = 1 - \frac{1}{2} U_n.$$

Now putting all the above together, we deduce the result. □

**COROLLARY 2.2.** For  $m \in \mathbb{N}$ ,

$$S_2(m) = \begin{cases} m^2 - 2 + 2(-1)^{m/3} & \text{if } 3 \mid m, \\ m^2 - 2 + (-1)^{\lfloor(m+1)/3\rfloor} & \text{if } 3 \nmid m, \end{cases}$$

$$S_3(m) = \begin{cases} m^3 - 6m & \text{if } 3 \mid m, \\ m^3 - 6m + 5(-1)^{\lfloor m/3 \rfloor} & \text{if } 3 \nmid m, \end{cases}$$

$$S_4(m) = \begin{cases} m^4 - 12m^2 + 22(1 - (-1)^{m/3}) & \text{if } 3 \mid m, \\ m^4 - 12m^2 + 11(2 - (-1)^{\lfloor(m+1)/3\rfloor}) & \text{if } 3 \nmid m. \end{cases}$$

**COROLLARY 2.3.** For  $n \in \mathbb{N}$ ,

$$U_{2n} = \frac{2}{3} \left( 2^{2n} + 3^{2n} - \sum_{k=1}^n \binom{2n}{2k} 4^{2k} U_{2n-2k} \right)$$

$$= \frac{2}{3} \left( 7^{2n} + 6^{2n} - 4^{2n} - 3^{2n} + 1 - \sum_{k=1}^n \binom{2n}{2k} 8^{2k} U_{2n-2k} \right).$$

**PROOF.** Taking  $m = 4, 8$  in Theorem 2.1 and replacing  $n$  with  $2n$ , we see that

$$3^{2n} + 2^{2n} = U_{2n}(4) + \frac{1}{2}U_{2n}$$

and

$$7^{2n} + 6^{2n} - 4^{2n} - 3^{2n} + 1 = U_{2n}(8) + \frac{1}{2}U_{2n}.$$

Since

$$U_{2n}(x) = \sum_{r=0}^n \binom{2n}{2r} U_{2r} x^{2n-2r} = U_{2n} + \sum_{k=1}^n \binom{2n}{2k} x^{2k} U_{2n-2k},$$

from the above we deduce the result. □

### 3. Congruences for $U_{2n} \pmod{2^{14}}$ and $U_{2^m k+b} \pmod{2^{\min(b,m)+3}}$

Suppose that  $n \in \{3, 4, 5, \dots\}$ . From [8, Theorem 4.1 and Corollary 4.1] we know that

$$U_{2n} \equiv -16n - 42 \pmod{2^7}.$$

Moreover, if  $n$  is even and  $2^\alpha \mid n$ , then

$$U_{2n} \equiv 48n + \frac{2}{3} \pmod{2^{\alpha+7}}. \tag{3.1}$$

Let  $p$  be a prime and let  $\text{ord}_p m$  be the greatest integer  $\alpha$  such that  $p^\alpha \mid m$ . If  $p^s \leq n < p^{s+1}$ , then

$$\begin{aligned} \text{ord}_p n! &= \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \dots + \left\lfloor \frac{n}{p^s} \right\rfloor < \frac{n}{p} + \frac{n}{p^2} + \dots + \frac{n}{p^s} + \dots \\ &= n \cdot \frac{1/p}{1 - 1/p} = \frac{n}{p - 1}. \end{aligned} \tag{3.2}$$

**LEMMA 3.1.** *Suppose that  $n \in \mathbb{N}$ ,  $n \geq 5$  and  $2^\alpha \mid n$ . Then*

$$3U_{2n} + 2^7n(2n - 1)U_{2n-2} \equiv 2(7^{2n} + 6^{2n} - 4^{2n} - 3^{2n} + 1) + 2^{16}n^2(n - 1) - 23 \cdot 2^{13}n(n - 1) + 7 \cdot 2^{15}n(n - 1)^3 \pmod{2^{\alpha+19}}.$$

**PROOF.** For  $k \geq 4$  we see that  $6(k - 3) > k \geq \text{ord}_2 k$  and so  $8^{2k}/k = 2^{18} \cdot 2^{6(k-3)}/k \equiv 0 \pmod{2^{19}}$ . For  $3 \leq k \leq n - 1$ , by (1.1) we have  $2 \mid U_{2n-2k}$ . Thus, for  $k \geq 3$ ,

$$2 \binom{2n}{2k} 8^{2k} U_{2n-2k} = 2n \binom{2n-1}{2k-1} \frac{8^{2k}}{k} U_{2n-2k} \equiv 0 \pmod{2^{\alpha+20}}.$$

Hence, by Corollary 2.3,

$$3U_{2n} \equiv 2 \left( 7^{2n} + 6^{2n} - 4^{2n} - 3^{2n} + 1 - \binom{2n}{2} 8^2 U_{2n-2} - \binom{2n}{4} 8^4 U_{2n-4} \right) \pmod{2^{\alpha+20}}. \tag{3.3}$$

Since  $U_{2n-4} \equiv -16(n - 2) - 42 = -16n - 10 \pmod{2^7}$  and

$$2 \binom{2n}{4} 8^4 = 2^{12}n(n - 1) \frac{4(n - 1)^2 - 1}{3} \equiv 0 \pmod{2^{\alpha+12}},$$

we see that

$$\begin{aligned} 2 \binom{2n}{4} 8^4 U_{2n-4} &\equiv 2^{12}n(n - 1) \frac{4(n - 1)^2 - 1}{3} (-16n - 10) \\ &= -2^{16}n^2(n - 1) \frac{4(n - 1)^2 - 1}{3} - 2^{13}n(n - 1) \frac{5(4(n - 1)^2 - 1)}{3} \\ &\equiv -2^{16}n^2(n - 1) \cdot 3(4(n - 1) - 1) - 23 \cdot 2^{13}n(n - 1)(4(n - 1)^2 - 1) \\ &\equiv -2^{16}n^2 \cdot 3(4(n - 1) - (n - 1)) + 23 \cdot 2^{13}n(n - 1) - 23 \cdot 2^{15}n(n - 1)^3 \\ &\equiv -2^{16}n^2(n - 1) + 23 \cdot 2^{13}n(n - 1) - 7 \cdot 2^{15}n(n - 1)^3 \pmod{2^{\alpha+19}}. \end{aligned}$$

Hence, by (3.3) and the fact  $2 \binom{2n}{2} 8^2 U_{2n-2} = 2^7n(2n - 1)U_{2n-2}$  we deduce the result.  $\square$

**THEOREM 3.2.** *Let  $n \in \mathbb{N}$  with  $n \geq 7$  and  $2^\alpha \mid n$ . Then*

$$3U_{2n} \equiv -3072n^4 + 4608n^3 + 2240n^2 + 1680n + 2 \pmod{2^{\alpha+14}}.$$

**PROOF.** Since  $U_{2n-2} \equiv -16(n - 1) - 42 \pmod{2^7}$ , by Lemma 3.1,

$$\begin{aligned} 3U_{2n} + 2^7n(2n - 1)(-16(n - 1) - 42) \\ \equiv 2(7^{2n} + 6^{2n} - 4^{2n} - 3^{2n} + 1) - 23 \cdot 2^{13}n(n - 1) \pmod{2^{\alpha+14}}. \end{aligned}$$

As  $2n \geq \alpha + 13$ ,

$$6^{2n} \equiv 4^{2n} \equiv 0 \pmod{2^{\alpha+13}}.$$

We also note that  $2^7 n(2n - 1)(16(n - 1) + 42) = 2^8(16n^3 + 18n^2 - 13n)$ . Now, from the above we deduce that

$$3U_{2n} \equiv 2^8(16n^3 + 18n^2 - 13n) + 2(7^{2n} - 3^{2n} + 1) - 2^{13}n(n - 1) \pmod{2^{\alpha+14}}. \quad (3.4)$$

For  $k \geq 5$  we see that  $4k - 14 > k \geq \text{ord}_2 k$ . Thus,  $2^{4k}/k \equiv 0 \pmod{2^{14}}$  for  $k \geq 4$ . Hence

$$\begin{aligned} 7^{2n} &= (1 + 48)^n = 1 + \sum_{k=1}^n n \binom{n-1}{k-1} \frac{48^k}{k} \\ &\equiv 1 + 48 \binom{n}{1} + 48^2 \binom{n}{2} + 48^3 \binom{n}{3} \pmod{2^{\alpha+14}}. \end{aligned}$$

For  $k \geq 7$  we have  $3k - 14 \geq k \geq \text{ord}_2 k$ . Thus  $2^{3k}/k \equiv 0 \pmod{2^{14}}$  for  $k \geq 5$ . Hence

$$\begin{aligned} 3^{2n} &= (1 + 8)^n = 1 + \sum_{k=1}^n n \binom{n-1}{k-1} \frac{8^k}{k} \\ &\equiv 1 + 8 \binom{n}{1} + 8^2 \binom{n}{2} + 8^3 \binom{n}{3} + 8^4 \binom{n}{4} \pmod{2^{\alpha+14}}. \end{aligned}$$

Therefore,

$$\begin{aligned} 7^{2n} - 3^{2n} &\equiv (48 - 8) \binom{n}{1} + (48^2 - 8^2) \binom{n}{2} + (48^3 - 8^3) \binom{n}{3} - 8^4 \binom{n}{4} \\ &\equiv 40n + 1120(n^2 - n) - 768n(n - 1)(n - 2) - 1536n(n - 1)(n - 2)(n - 3) \\ &= -2^9 \cdot 3n^4 + 2^8 \cdot 33n^3 - 2^5 \cdot 421n^2 + 6600n \\ &\equiv -2^9 \cdot 3n^4 + 2^8 n^3 + 2^5 \cdot 91n^2 - 1592n \pmod{2^{\alpha+13}}. \end{aligned}$$

This together with (3.4) yields

$$\begin{aligned} 3U_{2n} &\equiv 2^8(16n^3 + 18n^2 - 13n) + 2(-2^9 \cdot 3n^4 + 2^8 n^3 \\ &\quad + 2^5 \cdot 91n^2 - 1592n + 1) - 2^{13}n(n - 1) \\ &= -2^{10} \cdot 3n^4 + 2^9 \cdot 9n^3 + 2^6 \cdot 35n^2 + 1680n + 2 \\ &= -3072n^4 + 4608n^3 + 2240n^2 + 1680n + 2 \pmod{2^{\alpha+14}}. \end{aligned}$$

This proves the theorem. □

**LEMMA 3.3.** *Let  $k, m, b \in \mathbb{N}$  with  $2 \mid b$ . Then*

$$U_{2^m k + b} - U_b \equiv \frac{2^{b+1}}{9} - \frac{2}{3} \sum_{r=1}^{(b/2)-1} \binom{b}{2r} 2^{2r} (U_{2^m k + b - 2r} - U_{b - 2r}) \pmod{2^{m+3}}.$$

**PROOF.** From [8, (4.1)],

$$U_{2n} = \frac{2}{3} \left( 1 - \sum_{r=1}^n \binom{2n}{2r} 2^{2r} U_{2n-2r} \right).$$

Thus,

$$U_b = \frac{2}{3} \left( 1 - \sum_{r=1}^{b/2} \binom{b}{2r} 2^{2r} U_{b-2r} \right)$$

and

$$U_{2^m k+b} = \frac{2}{3} \left( 1 - 2^{2^m k+b} U_0 - \sum_{r=1}^{2^{m-1} k+(b/2)-1} (2^m k+b) \cdots \right. \\ \left. \times (2^m k+b-2r+1) \cdot \frac{2^{2r}}{(2r)!} U_{2^m k+b-2r} \right).$$

By (3.2),  $2^{2r}/(2r)! \equiv 0 \pmod{2}$  for  $r \geq 1$ . By (1.1),  $2 \mid U_{2n}$  for  $n \geq 1$ . We also have  $2^m k+b \geq m+2$ . Thus, from the above we deduce that

$$U_{2^m k+b} \equiv \frac{2}{3} \left( 1 - \sum_{r=1}^{2^{m-1} k+(b/2)-1} b(b-1) \cdots (b-2r+1) \frac{2^{2r}}{(2r)!} U_{2^m k+b-2r} \right) \\ \equiv \frac{2}{3} \left( 1 - \sum_{r=1}^{b/2} \binom{b}{2r} 2^{2r} U_{2^m k+b-2r} \right) \pmod{2^{m+3}}.$$

Therefore,

$$U_{2^m k+b} - U_b \equiv -\frac{2}{3} \sum_{r=1}^{b/2} \binom{b}{2r} 2^{2r} (U_{2^m k+b-2r} - U_{b-2r}) \pmod{2^{m+3}}.$$

By (3.1),  $U_{2^m k} \equiv 48 \cdot 2^{m-1} k + \frac{2}{3} \equiv \frac{2}{3} \pmod{2^{m+2}}$  for  $2^m k \geq 6$ . When  $2^m k = 2$  or  $4$  we also have  $U_{2^m k} \equiv \frac{2}{3} \pmod{2^{m+2}}$ . Thus

$$U_{2^m k+b} - U_b \equiv -\frac{2^{b+1}}{3} (U_{2^m k} - 1) - \frac{2}{3} \sum_{r=1}^{(b/2)-1} \binom{b}{2r} 2^{2r} (U_{2^m k+b-2r} - U_{b-2r}) \\ \equiv \frac{2^{b+1}}{9} - \frac{2}{3} \sum_{r=1}^{(b/2)-1} \binom{b}{2r} 2^{2r} (U_{2^m k+b-2r} - U_{b-2r}) \pmod{2^{m+3}},$$

as required. □

**THEOREM 3.4.** *Let  $k, m \in \mathbb{N}$ .*

(i) *If  $b \in \{2, 4, 6, \dots\}$ , then*

$$U_{2^m k+b} \equiv U_b + 2^{b+1} \pmod{2^{\min\{b,m\}+3}}.$$

(ii) *We have*

$$U_{2^m k+2} \equiv -\frac{10}{9} \pmod{2^{m+3}} \quad \text{and} \quad U_{2^m k+4} \equiv \frac{34}{3} \pmod{2^{m+3}}.$$



(iii) If  $b \in \{4, 6, 8, \dots\}$  and  $b \leq m - 2$ , then

$$U_{2^m k+b} \equiv U_b + 2^{b+1}(4b + 5) \pmod{2^{b+5}}.$$

**PROOF.** If  $b \in \{2, 4, 6, \dots\}$ , by Lemma 3.3,

$$\begin{aligned} U_{2^m k+b} - U_b - \frac{2^{b+1}}{9} &\equiv -\frac{2}{3} \sum_{r=1}^{(b/2)-1} \binom{b}{2r} 2^{2r} \left( U_{2^m k+b-2r} - U_{b-2r} - \frac{2^{b-2r+1}}{9} \right) \\ &\quad - \frac{2}{3} \cdot \frac{2^{b+1}}{9} \sum_{r=1}^{(b/2)-1} \binom{b}{2r} \\ &= -\frac{2}{3} \sum_{r=1}^{(b/2)-1} \binom{b}{2r} 2^{2r} \left( U_{2^m k+b-2r} - U_{b-2r} - \frac{2^{b-2r+1}}{9} \right) \\ &\quad - \frac{2^{b+2}}{27} (2^{b-1} - 2) \pmod{2^{m+3}}. \end{aligned} \tag{3.5}$$

Hence,

$$\begin{aligned} U_{2^m k+b} - U_b - \frac{2^{b+1}}{9} \\ \equiv -\frac{2}{3} \sum_{r=1}^{(b/2)-1} \binom{b}{2r} 2^{2r} \left( U_{2^m k+b-2r} - U_{b-2r} - \frac{2^{b-2r+1}}{9} \right) \pmod{2^{\min\{b,m\}+3}}. \end{aligned}$$

Therefore, for  $b = 2$ ,  $U_{2^m k+b} - U_b - 2^{b+1}/9 \equiv 0 \pmod{2^{\min\{b,m\}+3}}$ . We now prove (i) by induction on  $b$ . Suppose that the congruence

$$U_{2^m k+b-2r} - U_{b-2r} - \frac{2^{b-2r+1}}{9} \equiv 0 \pmod{2^{\min\{m,b-2r\}+3}}$$

holds for  $r = 1, 2, \dots, (b/2) - 1$ . As

$$2 \cdot 2^{2r} \cdot 2^{\min\{m,b-2r\}+3} \equiv \begin{cases} 2 \cdot 2^{2r} \cdot 2^{b-2r+3} \equiv 0 \pmod{2^{b+3}} & \text{if } b \leq m, \\ 2 \cdot 2^{m+1} \cdot 2^3 \equiv 0 \pmod{2^{m+3}} & \text{if } b > m \text{ and } 2r > m, \\ 2 \cdot 2^{2r} \cdot 2^{m-2r+3} \equiv 0 \pmod{2^{m+3}} & \text{if } b > m \geq 2r, \end{cases}$$

from the above and induction we deduce  $U_{2^m k+b} - U_b - 2^{b+1}/9 \equiv 0 \pmod{2^{\min\{b,m\}+3}}$ . This yields (i).

We now consider (ii). Putting  $b = 2$  in Lemma 3.3, we see that

$$U_{2^m k+2} \equiv U_2 + \frac{2^3}{9} = -2 + \frac{8}{9} = -\frac{10}{9} \pmod{2^{m+3}}.$$

Taking  $b = 4$  in Lemma 3.3 and then applying the above, we deduce that

$$U_{2^m k+4} - U_4 \equiv \frac{2^5}{9} - \frac{2}{3} \binom{4}{2} \cdot 2^2 (U_{2^m k+2} - U_2) \equiv \frac{32}{9} - 16 \cdot \frac{8}{9} = -\frac{32}{3} \pmod{2^{m+3}}$$

and so  $U_{2^m k+4} \equiv U_4 - \frac{32}{3} = 22 - \frac{32}{3} = \frac{34}{3} \pmod{2^{m+3}}$ . This proves (ii).

Finally, we consider (iii). Assume that  $2 \leq b \leq m - 2$ . By (i), for  $1 \leq r \leq (b/2) - 1$ ,

$$U_{2^m k+b-2r} - U_{b-2r} - \frac{2^{b-2r+1}}{9} \equiv 0 \pmod{2^{b-2r+3}}.$$

Thus, it follows from (3.5) that

$$U_{2^m k+b} - U_b - \frac{2^{b+1}}{9} \equiv -\frac{2^{b+2}}{27}(2^{b-1} - 2) \equiv 2^{2b+1} - 2^{b+3} \pmod{2^{b+4}}.$$

Using this and (3.5),

$$\begin{aligned} U_{2^m k+b} - U_b - \frac{2^{b+1}}{9} &\equiv -\frac{2}{3} \sum_{r=1}^{(b/2)-1} \binom{b}{2r} 2^{2r} (2^{2(b-2r)+1} - 2^{b-2r+3}) - \frac{2^{b+3}}{27} (2^{b-2} - 1) \\ &= -\frac{2^{b+2}}{3} \sum_{r=1}^{(b/2)-1} \binom{b}{2r} 2^{b-2r} + \frac{2^{b+4}}{3} \sum_{r=1}^{(b/2)-1} \binom{b}{2r} - \frac{2^{2b+1}}{27} + \frac{2^{b+3}}{27} \\ &\equiv \begin{cases} -\frac{2^{b+2}}{3} \binom{b}{b-2} \cdot 2^2 + \frac{2^{b+4}}{3} (2^{b-1} - 2) - 2^{b+3} \\ \equiv 2^{b+3}(b-1) \pmod{2^{b+5}} & \text{if } b > 2, \\ 0 \pmod{2^{b+5}} & \text{if } b = 2 \end{cases} \end{aligned}$$

and therefore for  $b > 2$ ,

$$U_{2^m k+b} - U_b \equiv \frac{2^{b+1}}{9} + 2^{b+3}(b-1) \equiv 2^{b+1}(4b+5) \pmod{2^{b+5}}.$$

This proves (iii). The proof is now complete. □

**COROLLARY 3.5.** *Let  $k, m, b \in \mathbb{N}$  with  $2 \mid b$ . Then  $U_{2^m k+b} \equiv U_b \pmod{2^{\min(b,m)+1}}$ .*

**PROOF.** This is immediate from Theorem 3.4(i). □

#### 4. A congruence for $U_{k\varphi(3^m)+b} \pmod{3^{m+4}}$

In [6] the author proved that for  $k, m \in \mathbb{N}$ ,  $m \geq 4$  and  $b \in \{0, 2, 4, \dots\}$ ,

$$E_{2^m k+b} - E_b \equiv \begin{cases} 5 \cdot 2^m k \pmod{2^{m+4}} & \text{if } b \equiv 0, 6 \pmod{8}, \\ -3 \cdot 2^m k \pmod{2^{m+4}} & \text{if } b \equiv 2, 4 \pmod{8}. \end{cases}$$

A generalisation to Euler polynomials was given in [7, Theorem 3.3].

From the proof of [8, Theorem 4.2] we have the following lemma.

**LEMMA 4.1.** For  $n \in \mathbb{N}$ ,

$$2^{2n}U_{2n} = \sum_{k=0}^n \binom{2n}{2k} 3^{2k} E_{2k}.$$

**THEOREM 4.2.** Let  $k, m \in \mathbb{N}$ ,  $m \geq 3$  and  $b \in \{0, 2, 4, \dots\}$ . Then

$$U_{k\varphi(3^m)+b} - U_b \equiv \begin{cases} 3^m k(9b - 40) \pmod{3^{m+4}} & \text{if } 3 \mid b, \\ -3^m k \cdot 22 \pmod{3^{m+4}} & \text{if } 3 \mid b - 1, \\ -3^m k(9b - 32) \pmod{3^{m+4}} & \text{if } 3 \mid b - 2. \end{cases}$$

**PROOF.** By (3.2),  $\text{ord}_3(2r)! \leq r - 1$ . Thus, for  $r \geq 3$ ,

$$2r - \text{ord}_3(2r)! \geq 2r - (r - 1) = r + 1 \geq 4$$

and so  $3^{2r}/(2r)! \equiv 0 \pmod{3^4}$ . Clearly  $3^2 \mid b(b - 1) \cdots (b - 2r + 1)$  for  $r \geq 3$ . Thus, for  $r \geq 3$ ,

$$\begin{aligned} & (k\varphi(3^m) + b)(k\varphi(3^m) + b - 1) \cdots (k\varphi(3^m) + b - 2r + 1) \\ & \equiv b(b - 1) \cdots (b - 2r + 1) + k\varphi(3^m) \sum_{i=0}^{2r-1} \frac{b(b - 1) \cdots (b - 2r + 1)}{b - i} \\ & \equiv b(b - 1) \cdots (b - 2r + 1) \pmod{3^m}. \end{aligned}$$

Hence,

$$\binom{k\varphi(3^m) + b}{2r} 3^{2r} \equiv \binom{b}{2r} 3^{2r} \pmod{3^{m+4}} \quad \text{for } r \geq 3. \tag{4.1}$$

Since  $E_0 = 1$ ,  $E_2 = -1$  and  $E_4 = 5$ , using (4.1) and Lemma 4.1 we see that

$$\begin{aligned} 2^{k\varphi(3^m)+b} U_{k\varphi(3^m)+b} &= 1 - 9 \binom{k\varphi(3^m) + b}{2} + 5 \cdot 3^4 \binom{k\varphi(3^m) + b}{4} \\ &\quad + \sum_{r=3}^{(k\varphi(3^m)+b)/2} \binom{k\varphi(3^m) + b}{2r} 3^{2r} E_{2r} \\ &\equiv -\frac{9}{2} (4 \cdot 3^{2m-2} k^2 + 2 \cdot 3^{m-1} k(2b - 1)) + \frac{3^3 \cdot 5}{8} (k^4 \varphi(3^m)^4 \\ &\quad + (4b - 6)k^3 \varphi(3^m)^3 + (6b^2 - 18b + 11)k^2 \varphi(3^m)^2 \\ &\quad + (4b^3 - 18b^2 + 22b - 6)k\varphi(3^m)) + \sum_{r=0}^{b/2} \binom{b}{2r} 3^{2r} E_{2r} \\ &\equiv -2 \cdot 3^{2m} k^2 - (2b - 1)3^{m+1} k - (4b^3 + 4b - 6)3^{m+2} k \\ &\quad + 2^b U_b \pmod{3^{m+4}}. \end{aligned}$$

By (3.2),  $\text{ord}_3 r \leq \text{ord}_3 r! < r/2$ . Thus, for  $r \geq 4$ ,

$$2r - \text{ord}_3 r > 2r - \frac{r}{2} = \frac{3r}{2} \geq 6$$

and so

$$\varphi(3^{m-1}) \frac{9^r}{r} = 2 \cdot 3^{m-2} \cdot \frac{3^{2r}}{r} \equiv 0 \pmod{3^{m+4}}.$$

Hence,

$$\begin{aligned} 2^{k\varphi(3^m)} - 1 &= (1 - 9)^{k\varphi(3^{m-1})} - 1 = \sum_{r=1}^{k\varphi(3^{m-1})} \binom{k\varphi(3^{m-1})}{r} (-9)^r \\ &= \binom{k\varphi(3^{m-1})}{1} (-9) + \binom{k\varphi(3^{m-1})}{2} (-9)^2 + \binom{k\varphi(3^{m-1})}{3} (-9)^3 \\ &\quad + \sum_{r=4}^{k\varphi(3^{m-1})} k\varphi(3^{m-1}) \binom{k\varphi(3^{m-1}) - 1}{r-1} \frac{(-9)^r}{r} \\ &\equiv -9k\varphi(3^{m-1}) + 81 \frac{k\varphi(3^{m-1})(k\varphi(3^{m-1}) - 1)}{2} \\ &\quad + \frac{k\varphi(3^{m-1})(k\varphi(3^{m-1}) - 1)(k\varphi(3^{m-1}) - 2)}{6} (-9)^3 \\ &\equiv 3^m k(16 + 2 \cdot 3^m k) \pmod{3^{m+4}}. \end{aligned}$$

Thus,

$$\begin{aligned} &2^{k\varphi(3^m)+b} U_{k\varphi(3^m)+b} - 2^b U_b \\ &\equiv (1 + 3^m k(16 + 2 \cdot 3^m k)) 2^b U_{k\varphi(3^m)+b} - 2^b U_b \\ &= 2^b (U_{k\varphi(3^m)+b} - U_b) + 3^m k(16 + 2 \cdot 3^m k) 2^b U_{k\varphi(3^m)+b} \pmod{3^{m+4}}. \end{aligned}$$

By Lemma 4.1,

$$2^{2n} U_{2n} \equiv E_0 + \binom{2n}{2} 3^2 E_2 = 1 - 9n(2n - 1) \pmod{81}.$$

Thus,

$$2^{k\varphi(3^m)+b} U_{k\varphi(3^m)+b} \equiv 1 - \frac{9}{2} (k\varphi(3^m) + b)(k\varphi(3^m) + b - 1) \equiv 1 - 9 \binom{b}{2} \pmod{81}$$

and so

$$\begin{aligned} 2^b U_{k\varphi(3^m)+b} &\equiv \frac{1 - 9 \binom{b}{2}}{2^{k\varphi(3^m)}} \\ &\equiv \frac{1 - \frac{9}{2} b(b - 1)}{1 + 3^m k(16 + 2 \cdot 3^m k)} \end{aligned}$$

$$\begin{aligned} &\equiv \left(1 - \frac{9}{2}b(b-1)\right)(1 - 3^m k(16 + 2 \cdot 3^m k)) \\ &\equiv 1 - \frac{9}{2}b(b-1) - 16 \cdot 3^m k \pmod{81}. \end{aligned}$$

Therefore,

$$\begin{aligned} &-2 \cdot 3^{2m} k^2 - (2b-1)3^{m+1}k - (4b^3 + 4b - 6)3^{m+2}k \\ &\equiv 2^{k\varphi(3^m)+b} U_{k\varphi(3^m)+b} - 2^b U_b \equiv 2^b (U_{k\varphi(3^m)+b} - U_b) + 3^m k(16 + 2 \cdot 3^m k) 2^b U_{k\varphi(3^m)+b} \\ &\equiv 2^b (U_{k\varphi(3^m)+b} - U_b) + 3^m k(16 + 2 \cdot 3^m k) \left(1 - \frac{9}{2}b(b-1) - 16 \cdot 3^m k\right) \\ &\equiv \begin{cases} 2^b (U_{k\varphi(3^m)+b} - U_b) + 3^m k(16 + 9b(b-1)) \pmod{3^{m+4}} & \text{if } m \geq 4, \\ 2^b (U_{k\varphi(3^3)+b} - U_b) + 27k(16 + 9b(b-1)) + 3^6 k^2 \pmod{3^7} & \text{if } m = 3. \end{cases} \end{aligned}$$

This yields

$$2^b (U_{k\varphi(3^m)+b} - U_b) \equiv -3^m k(36b^3 + 9b^2 + 33b - 41) \pmod{3^{m+4}}.$$

If  $3 \mid b$ , then  $2^{-b} = (1 - 9)^{-b/3}$  and so

$$\begin{aligned} U_{k\varphi(3^m)+b} - U_b &\equiv (1 - 9)^{-b/3} (-3^m k)(36b^3 + 9b^2 + 33b - 41) \\ &\equiv -(1 + 3b)(33b - 41)3^m k \equiv (9b - 40)3^m k \pmod{3^{m+4}}. \end{aligned}$$

If  $3 \mid b - 1$ , then  $b^2 \equiv 2b - 1 \pmod{9}$ ,  $b^3 \equiv 1 \pmod{9}$  and  $2^{-b} = 4 \cdot 2^{-b-2} = 4(1 - 9)^{-(b+2)/3}$ . Thus,

$$\begin{aligned} U_{k\varphi(3^m)+b} - U_b &\equiv 4(1 - 9)^{-(b+2)/3} (-3^m k)(36b^3 + 9b^2 + 33b - 41) \\ &\equiv -4(1 + 3(b+2))(36 + 9(2b-1) + 33b - 41)3^m k \\ &= -4(3b+7)(51b-14)3^m k \equiv 8(3b+7)(15b+7)3^m k \\ &\equiv 8(45(2b-1) + 126b + 49)3^m k = 8(216b + 4)3^m k \\ &\equiv 8(216 + 4)3^m k \equiv -22 \cdot 3^m k \pmod{3^{m+4}}. \end{aligned}$$

If  $3 \mid b - 2$ , then  $b^2 \equiv -2b - 1 \pmod{9}$ ,  $b^3 \equiv -1 \pmod{9}$  and  $2^{-b} = 2 \cdot 2^{-b-1} = -2(1 - 9)^{-(b+1)/3}$ . Thus,

$$\begin{aligned} U_{k\varphi(3^m)+b} - U_b &\equiv -2(1 - 9)^{-(b+1)/3} (-3^m k)(36b^3 + 9b^2 + 33b - 41) \\ &\equiv 2(1 + 3(b+1))(-36 + 9(-2b-1) + 33b - 41)3^m k \\ &\equiv 10(3b+4)(3b-1)3^m k \equiv 10(9(-2b-1) + 9b - 4)3^m k \\ &\equiv 10(-9b - 13)3^m k \equiv -(9b - 32)3^m k \pmod{3^{m+4}}. \end{aligned}$$

This completes the proof. □

**5. A congruence for  $E_{k\varphi(3^m)+b} \pmod{3^{m+4}}$**

**LEMMA 5.1.** For  $n \in \mathbb{N}$ ,

$$(3^{2n} + 1)E_{2n} = \sum_{r=0}^n \binom{2n}{2r} 2^{2n-2r+1} 3^{2r} E_{2r}.$$

**PROOF.** By [7, Theorem 2.1 and Lemma 2.1],

$$\frac{1}{2}(3^{2n} + 1)E_{2n} = \sum_{r=0}^n \binom{2n}{2r} (1 - 3)^{2n-2r} 3^{2r} E_{2r},$$

as required. □

Let  $k, m \in \mathbb{N}$  and  $b \in \{0, 2, 4, \dots\}$ . From [2, p. 231] or [5, Corollary 7.1],

$$E_{k\varphi(3^m)+b} \equiv (3^b + 1)E_b \pmod{3^m}.$$

Now we prove the following stronger congruence.

**THEOREM 5.2.** Let  $k, m \in \mathbb{N}$ ,  $m \geq 3$  and  $b \in \{0, 2, 4, \dots\}$ . Then

$$E_{k\varphi(3^m)+b} - (3^b + 1)E_b \equiv \begin{cases} (9b + 20)3^m k \pmod{3^{m+4}} & \text{if } 3 \mid b, \\ -16 \cdot 3^m k \pmod{3^{m+4}} & \text{if } 3 \mid b - 1, \\ (-9b + 11)3^m k \pmod{3^{m+4}} & \text{if } 3 \mid b - 2. \end{cases}$$

**PROOF.** As  $\varphi(3^m) \geq m + 4$ , using Lemma 5.1 and (4.1) we see that

$$\begin{aligned} E_{k\varphi(3^m)+b} &\equiv (3^{k\varphi(3^m)+b} + 1)E_{k\varphi(3^m)+b} \\ &= \sum_{r=0}^{(k\varphi(3^m)+b)/2} \binom{k\varphi(3^m) + b}{2r} 2^{k\varphi(3^m)+b-2r+1} 3^{2r} E_{2r} \\ &= 2^{k\varphi(3^m)+b+1} E_0 + \binom{k\varphi(3^m) + b}{2} 2^{k\varphi(3^m)+b-1} 3^2 E_2 \\ &\quad + \binom{k\varphi(3^m) + b}{4} 2^{k\varphi(3^m)+b-3} 3^4 E_4 \\ &\quad + \sum_{r=3}^{b/2} \binom{b}{2r} 2^{k\varphi(3^m)+b-2r+1} 3^{2r} E_{2r} \pmod{3^{m+4}}. \end{aligned}$$

From the proof of Theorem 4.2 we know that  $3^{2r}/(2r)! \equiv 0 \pmod{3^4}$  for  $r \geq 3$ . By Euler’s theorem,  $2^{k\varphi(3^m)} \equiv 1 \pmod{3^m}$ . Thus, from the above we deduce that

$$\begin{aligned} E_{k\varphi(3^m)+b} &\equiv 2^{k\varphi(3^m)+b+1} - 9(k\varphi(3^m) + b)(k\varphi(3^m) + b - 1)2^{k\varphi(3^m)+b-2} \\ &\quad + \binom{k\varphi(3^m) + b}{4} 2^{b-3} \cdot 81 \cdot 5 + \sum_{r=3}^{b/2} \binom{b}{2r} 2^{b-2r+1} 3^{2r} E_{2r} \end{aligned}$$

$$\begin{aligned}
 &\equiv 2^{b+1}(2^{k\varphi(3^m)} - 1) - 9(k^2\varphi(3^m)^2 + (2b - 1)k\varphi(3^m) + b(b - 1))2^{k\varphi(3^m)+b-2} \\
 &\quad + 9b(b - 1)2^{b-2} + \left( \binom{k\varphi(3^m) + b}{4} - \binom{b}{4} \right) 2^{b-3} \cdot 81 \cdot 5 \\
 &\quad + \sum_{r=0}^{b/2} \binom{b}{2r} 2^{b-2r+1} 3^{2r} E_{2r} \\
 &\equiv 2^{b+1}(2^{k\varphi(3^m)} - 1) - 2^b k^2 \cdot 3^{2m} - 9(2b - 1)2^{b-2} k\varphi(3^m) \\
 &\quad - 9b(b - 1)2^{b-2}(2^{k\varphi(3^m)} - 1) + 5 \cdot 3^3 \cdot 2^{b-6}(4b^3 - 18b^2) \\
 &\quad + 22b - 6)k\varphi(3^m) + (3^b + 1)E_b \\
 &\equiv 2^{b-2}(8 - 9b(b - 1))(2^{k\varphi(3^m)} - 1) - 2^b k^2 \cdot 3^{2m} - (2b - 1)2^{b-1} k \cdot 3^{m+1} \\
 &\quad - 2^{b-2}(2b^3 + 2b - 3)k \cdot 3^{m+2} + (3^b + 1)E_b \pmod{3^{m+4}}.
 \end{aligned}$$

By the proof of Theorem 4.2,

$$2^{k\varphi(3^m)} - 1 \equiv 16k \cdot 3^m + 2k^2 \cdot 3^{2m} \pmod{3^{m+4}}.$$

Thus,

$$\begin{aligned}
 E_{k\varphi(3^m)+b} - (3^b + 1)E_b &\equiv 2^{b-2}3^m k \{ 16(8 - 9b^2 + 9b) - 6(2b - 1) - 9(2b^3 + 2b - 3) \} \\
 &\quad + 2^{b-2}(8 - 9b(b - 1)) \cdot 2k^2 3^{2m} - 2^b k^2 3^{2m} \\
 &\equiv 2^{b-2}(-18b^3 + 18b^2 - 48b - 1)3^m k \pmod{3^{m+4}}.
 \end{aligned}$$

If  $3 \mid b$ , then  $2^{b-2} \equiv -20(1 - 9)^{b/3} \equiv -20(1 - 3b) \pmod{3^4}$ . Thus,

$$\begin{aligned}
 E_{k\varphi(3^m)+b} - (3^b + 1)E_b &\equiv -20(1 - 3b)(-18b^3 + 18b^2 - 48b - 1)3^m k \\
 &\equiv 20(1 - 3b)(1 + 48b)3^m k \\
 &\equiv 20(1 + 45b)3^m k \equiv (9b + 20)3^m k \pmod{3^{m+4}}.
 \end{aligned}$$

If  $3 \mid b - 1$ , then  $b^2 \equiv 2b - 1 \pmod{9}$ ,  $b^3 \equiv 1 \pmod{9}$  and

$$2^{b-2} = -\frac{1}{2}(1 - 9)^{(b-1)/3} \equiv -\frac{1}{2}\left(1 - 9 \cdot \frac{b - 1}{3}\right) = \frac{1}{2}(3b - 4) \pmod{3^4}.$$

Thus,

$$\begin{aligned}
 E_{k\varphi(3^m)+b} - (3^b + 1)E_b &\equiv \frac{1}{2}(3b - 4)(-18b^3 + 18b^2 - 48b - 1)3^m k \\
 &\equiv \frac{1}{2}(3b - 4)(-18 + 18(2b - 1) - 48b - 1)3^m k \\
 &\equiv (3b - 4)(-6b + 22)3^m k \equiv (-18b^2 + 9b - 7)3^m k \\
 &\equiv (-18(2b - 1) + 9b - 7)3^m k \\
 &= (-27b + 11)3^m k \equiv -16 \cdot 3^m k \pmod{3^{m+4}}.
 \end{aligned}$$

If  $3 \mid b - 2$ , then  $b^2 \equiv -2b - 1 \pmod{9}$ ,  $b^3 \equiv -1 \pmod{9}$  and  $2^{b-2} = (1 - 9)^{(b-2)/3} \equiv 1 - 9 \cdot (b - 2)/3 = 7 - 3b \pmod{3^4}$ . Thus,

$$\begin{aligned} E_{k\varphi(3^m)+b} - (3^b + 1)E_b &\equiv (7 - 3b)(-18b^3 + 18b^2 - 48b - 1)3^m k \\ &\equiv (7 - 3b)(18 + 18(-2b - 1) - 48b - 1)3^m k \\ &\equiv (3b - 7)(3b + 1)3^m k \equiv (9(-2b - 1) - 18b - 7)3^m k \\ &\equiv (-9b + 11)3^m k \pmod{3^{m+4}}. \end{aligned}$$

This completes the proof.  $\square$

### References

- [1] H. Bateman, *Higher Transcendental Functions*, Vol. I (McGraw-Hill, New York, 1953).
- [2] K. W. Chen, 'Congruences for Euler numbers', *Fibonacci Quart.* **42** (2004), 128–140.
- [3] J. B. Cosgrave and K. Dilcher, 'Mod  $p^3$  analogues of theorems of Gauss and Jacobi on binomial coefficients', *Acta Arith.* **142** (2010), 103–118.
- [4] S. Mattarei and R. Tauraso, 'Congruences for central binomial sums and finite polylogarithms', *J. Number Theory*, to appear.
- [5] Z. H. Sun, 'Congruences involving Bernoulli polynomials', *Discrete Math.* **308** (2008), 71–112.
- [6] Z. H. Sun, 'Euler numbers modulo  $2^n$ ', *Bull. Aust. Math. Soc.* **82** (2010), 221–231.
- [7] Z. H. Sun, 'Congruences for sequences similar to Euler numbers', *J. Number Theory* **132** (2012), 675–700.
- [8] Z. H. Sun, 'Identities and congruences for a new sequence', *Int. J. Number Theory* **8** (2012), 207–225.

ZHI-HONG SUN, School of Mathematical Sciences,  
Huaiyin Normal University, Huaian, Jiangsu 223001, PR China  
e-mail: [zhihongsun@yahoo.com](mailto:zhihongsun@yahoo.com)