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# SOME PROPERTIES OF ANALYTIC FUNCTIONS ASSOCIATED WITH FRACTIONAL $q$-CALCULUS OPERATORS

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*Abstract.* By applying a fractional $q$-calculus operator, we define the subclasses $\mathcal{S}_n^\alpha(\lambda,\beta,b,q)$ and $\mathcal{G}_n^\alpha(\lambda,\beta,b,q)$ of normalized analytic functions with complex order and negative coefficients. Among the results investigated for each of these function classes, we derive their associated coefficient estimates, radii of close-to-convexity, starlikeness and convexity, extreme points, and growth and distortion theorems.

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## 1. INTRODUCTION AND DEFINITIONS

Here, in this paper, we denote by $\mathcal{A}(n)$ the class of functions of the following normalized form:

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \qquad (n \in \mathbb{N};\ \mathbb{N} := \{1,2,3,\cdots\}), \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U}$ centered at the origin ($z = 0$) in the complex $z$-plane. We write $\mathcal{A}(1) = \mathcal{A}$. We also denote by $\mathcal{T}(n)$ the subclass of $\mathcal{A}(n)$ consisting of functions of the form:

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \qquad (a_k \geqq 0;\ k \geqq n+1;\ n \in \mathbb{N}). \tag{1.2}$$

In our investigation, we make use of various operators of $q$-calculus and fractional $q$-calculus. For this purpose, we refer the reader to the various definitions, notations and conventions, which are considerably detailed in our earlier paper (see, for details, [22]; see also [8]).

For a fixed $\mu \in \mathbb{C}$, a set $\mathbb{D}$ is called a $\mu$-geometric set if and only if both $z \in \mathbb{D}$ and $\mu z \in \mathbb{D}$. For a function $f$ defined on a $q$-geometric set, we make use of Jackson's $q$-derivative and $q$-integral $(0 < q < 1)$ of a function on a subset of $\mathbb{C}$, which are already introduced in several earlier investigations (see, for example, [2], [4], [6], [8], [9], [10], [14], [15], [16], [17], [21], [22] and [25]).

Now, for a complex-valued function $f(z)$, we introduce the fractional $q$-calculus operators as follows (see, for example, [12] and [13]; see also [1]).

**Definition 1** (Fractional $q$-integral operator)**.** The fractional $q$-integral operator $I^{\lambda}_{q,z}$ of order $\lambda$ is defined, for a function $f(z)$, by

$$I^{\lambda}_{q,z} f(z) = D^{-\lambda}_{q,z} f(z) = \frac{1}{\Gamma_q(\lambda)} \int_0^z (z - tq)_{\lambda-1} f(t) d_q t \qquad (\lambda > 0), \tag{1.3}$$

where the function $f(z)$ is analytic in a simply-connected region of the complex $z$-plane containing the origin. Here, and elsewhere in this paper, the $q$-binomial $(z - tq)_{\lambda-1}$ is given by

$$\begin{aligned} (z-tq)_{\lambda-1} &= z^{\lambda-1} \prod_{k=0}^{\infty} \left[ \frac{1-(tqz^{-1})q^k}{1-(tqz^{-1})q^{\lambda+k-1}} \right] \\ &= z^{\lambda}\, {}_1\Phi_0(q^{1-\lambda}; \text{—}; q, tq^{\lambda} z^{-1}). \end{aligned} \tag{1.4}$$

*Remark* 1. The $q$-hypergeometric series ${}_1\Phi_0(\lambda; \text{—}; q, z)$ is known to be single-valued when $|\arg(z)| < \pi$ (see, for example, [8]). Therefore, the $q$-binomial $(z - tq)_{\lambda-1}$ in (1.4) is single-valued when

$$\left| \arg\left(-tq^{\lambda} z^{-1}\right) \right| < \pi, \qquad \left| \frac{tq^{\lambda}}{z} \right| < 1 \text{ and } |\arg(z)| < \pi.$$

**Definition 2** (Fractional $q$-derivative operator)**.** The fractional $q$-derivative operator $D^{\lambda}_{q,z}$ of order $\lambda$ $(0 \leqq \lambda < 1)$ is defined, for a function $f(z)$, by

$$D^{\lambda}_{q,z} f(z) = D_{q,z} I^{1-\lambda}_{q,z} f(z) = \frac{1}{\Gamma_q(1-\lambda)} D_q \int_0^z (z - tq)_{-\lambda} f(t) d_q t, \tag{1.5}$$

where $f(z)$ is suitably constrained and the multiplicity of $(z - tq)_{-\lambda}$ is removed as in Definition 1.

**Definition 3** (Extended fractional $q$-derivative operator)**.** Under the hypotheses of Definition 2, for a function $f(z)$, the fractional $q$-derivative of order $\lambda$ is defined by

$$D^{\lambda}_{q,z} f(z) = D^{m}_{q,z} I^{m-\lambda}_{q,z} f(z) \qquad (m - 1 \leqq \lambda < 1;\ m \in \mathbb{N}). \tag{1.6}$$

Clearly, we have

$$D^{\lambda}_{q,z} z^n = \frac{\Gamma_q(n+1)}{\Gamma_q(n+1-\lambda)} z^{n-\lambda} \qquad (\lambda \geqq 0;\ n > -1).$$

Now, by using the operator $D_{q,z}^{\lambda}$, we define (for $-\infty < \lambda < 2$, $0 < q < 1$ and $z \in \mathbb{U}$,) a $q$-differintegral operator $\Omega_{q,z}^{\lambda} : \mathcal{T}(n) \to \mathcal{T}(n)$ as follows (see [12] and [13]):

$$\Omega_{q,z}^{\lambda} f(z) = \frac{\Gamma_q(2-\lambda)}{\Gamma_q(\lambda)} z^{\lambda} D_{q,z}^{\lambda} f(z) = z - \sum_{k=n+1}^{\infty} A_q(\lambda,k) a_k z^k \tag{1.7}$$

where

$$A_q(\lambda,k) = \frac{\Gamma_q(k+1)\Gamma_q(2-\lambda)}{\Gamma_q(2)\Gamma_q(k+1-\lambda)} \tag{1.8}$$

and $D_{q,z}^{\lambda} f(z)$ in (1.7) represents, respectively, the fractional $q$-integral of $f(z)$ of order $\lambda$ $(-\infty < \lambda < 0)$ and the fractional $q$-derivative of $f(z)$ of order $\lambda$ $(0 \leqq \lambda < 2)$ (see, for details, [7, 18–20]). We note that some interesting special and limit cases of (1.7) were investigated in the earlier works by Owa and Srivastava [11] and by Srivastava and Owa (see [23] and [24]).

*Remark* 2. From (1.3), (1.7) and (1.8), we find that

$$\begin{aligned}\Omega_{q,z}^{-\lambda} f(z) &= \frac{\Gamma_q(2+\lambda)}{\Gamma_q(2)} z^{-\lambda} D_{q,z}^{-\lambda} f(z) = \frac{\Gamma_q(2+\lambda)}{\Gamma_q(2)} z^{-\lambda} I_{q,z}^{\lambda} f(z) \\ &= z - \sum_{k=n+1}^{\infty} A_q(-\lambda,k) a_k z^k,\end{aligned} \tag{1.9}$$

where

$$A_q(-\lambda,k) = \frac{\Gamma_q(k+1)\Gamma_q(2+\lambda)}{\Gamma_q(2)\Gamma_q(k+1+\lambda)} \qquad (\lambda > 0;\ 0 < q < 1). \tag{1.10}$$

**Definition 4.** A function $f(z) \in \mathcal{T}(n)$ is said to be in the function class:

$$\mathcal{S}_n^{\alpha}(\lambda,\beta,b,q) \qquad (\lambda < 2;\ 0 \leqq \alpha \leqq 1;\ 0 < q < 1;\ \beta > 0;\ b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\})$$

if it satisfies the following condition:

$$\left| \frac{1}{b} \left( \frac{(1-\alpha) z D_q\big(\Omega_{q,z}^{\lambda} f(z)\big) + \alpha z D_q\Big(z D_q\big(\Omega_{q,z}^{\lambda} f(z)\big)\Big)}{(1-\alpha)\Omega_{q,z}^{\lambda} f(z) + \alpha z D_q\big(\Omega_{q,z}^{\lambda} f(z)\big)} - 1 \right) \right| < \beta. \tag{1.11}$$

Some of the interesting particular cases of the function class $\mathcal{S}_n^{\alpha}(\lambda,\beta,b,q)$ are being recorded below:

(i) $\mathcal{S}_n^{\alpha}(\lambda,1,b,q) = \mathcal{S}_n^{\alpha}(\lambda,b,q)$ (see [12]);

(ii) $\mathcal{S}_n^{\alpha}(0,\beta,b,q) = \mathcal{S}_n^{\alpha}(\beta,b,q)$, where

$$\mathcal{S}_n^{\alpha}(\beta,b,q) := \Bigg\{ f : f \in T(n) \qquad \text{and}$$

$$\left|\frac{1}{b}\left(\frac{(1-\alpha)zD_q f(z)+\alpha z^2 D_q^2 f(z)}{(1-\alpha)f(z)+\alpha z D_q f(z)}-1\right)\right|<\beta\Bigg\}.$$

(iii) $\lim_{q\to 1-} \mathcal{S}_n^\alpha(\beta,b,q)=\mathcal{S}_n(b,\alpha,\beta)$ (see [3]);

(iv) $\mathcal{S}_n^0(\lambda,\beta,b,q)=\mathcal{S}_n^*(\lambda,\beta,b,q)$, where

$$\mathcal{S}_n(b,\alpha,\beta):=\left\{f: f\in\mathcal{T}(n) \quad \text{and} \quad \left|\frac{1}{b}\left(\frac{zD_q\big(\Omega_{q,z}^\lambda f(z)\big)}{\Omega_{q,z}^\lambda f(z)}-1\right)\right|<\beta\right\};$$

(v) $\lim_{q\to 1-} \mathcal{S}_n^*(\lambda,\beta,b,q)=\mathcal{K}_n(\lambda,b,\beta)$ (see [5] *with* $p=1$);

(vi) $\mathcal{S}_n^1(\lambda,\beta,b,q)=\mathcal{C}_n(\lambda,\beta,b,q)$, where

$$\mathcal{C}_n(\lambda,\beta,b,q):=\left\{f: f\in\mathcal{T}(n) \quad \text{and} \quad \left|\frac{1}{b}\left(\frac{zD_q^2\big(\Omega_{q,z}^\lambda f(z)\big)}{D_q\big(\Omega_{q,z}^\lambda f(z)\big)}-1\right)\right|<\beta\right\}.$$

**Definition 5.** A function $f(z)\in\mathcal{T}(n)$ is in the function class

$$\mathcal{G}_n^\alpha(\lambda,\beta,b,q) \;\; (\lambda<2;\ 0\leqq\alpha\leqq 1;\ 0<q<1;\ b\in\mathbb{C}^*;\ \beta>0)$$

if it satisfies the following condition:

$$\left|\frac{1}{b}\left(D_q\big(\Omega_{q,z}^\lambda f(z)\big)+\alpha z D_q^2\big(\Omega_{q,z}^\lambda f(z)\big)-1\right)\right|<\beta. \tag{1.12}$$

We choose to note the following special case of the function class $\mathcal{G}_n^\alpha(\lambda,\beta,b,q)$:

(i) $\mathcal{G}_n^\alpha(0,\beta,b,q)=\mathcal{G}_n^\alpha(\beta,b,q)$, where

$$\mathcal{G}_n^\alpha(\beta,b,q)=\left\{f: f\in\mathcal{T}(n) \quad \text{and} \quad \left|\frac{1}{b}\left(D_q f(z)+\alpha z D_q^2 f(z)-1\right)\right|<\beta\right\};$$

(ii) $\mathcal{G}_n^\alpha(\lambda,1,b,q)=\mathcal{R}_n^\alpha(\lambda,b,q)$ (see [13]);

(iii) $\mathcal{G}_n^\alpha(0,\beta,b,q)=\mathcal{R}_n(\alpha,\beta,b,q)$ (see [13]);

(iv) $\lim_{q\to 1-} \mathcal{G}_n^\alpha(0,\beta,b,q)=\mathcal{R}_n(\alpha,\beta,b)$ (see [3]).

For each of the above-defined general function classes $\mathcal{S}_n^\alpha(\lambda,\beta,b,q)$ and $\mathcal{G}_n^\alpha(\lambda,\beta,b,q)$ of analytic functions with complex order and negative coefficients, we propose here to investigate the associated coefficient estimates, radii of close-to-convexity, starlikeness and convexity, extreme points, and growth and distortion theorems.

## 2. PROPERTIES OF THE FUNCTION CLASSES $\mathcal{S}_n^{\alpha}(\lambda,\beta,b,q)$ AND $\mathcal{G}_n^{\alpha}(\lambda,\beta,b,q)$

Henceforth in this paper, unless otherwise mentioned, we assume that $\lambda < 2$, $0 \leqq \alpha \leqq 1$, $0 < q < 1$, $b \in \mathbb{C}^*$, $\beta > 0$, $[\lambda]_q$ denotes the basic (or $q$-) number defined by

$$[\lambda]_q = \frac{1-q^{\lambda}}{1-q} \qquad (|q| < 1), \tag{2.1}$$

which readily yields

$$[\lambda]_q = \frac{1-q^{\lambda}}{1-q} \to \lambda \qquad (q \to 1-),$$

$A_q(\lambda,k)$ is given by (1.8), $f(z)$ is in the form (1.2) and $z \in \mathbb{U}$.

**Theorem 1.** *The function $f(z) \in \mathcal{S}_n^{\alpha}(\lambda,\beta,b,q)$ if and only if*

$$\sum_{k=n+1}^{\infty} \big([k]_q + \beta|b| - 1\big)[1+\alpha([k]_q-1)]A_q(\lambda,k)a_k \leqq \beta|b|. \tag{2.2}$$

*Proof.* Let $f(z) \in \mathcal{S}_n^{\alpha}(\lambda,\beta,b,q)$. Then, in view of (1.11) and (1.7), we readily find that

$$\Re\left(\frac{-\sum\limits_{k=n+1}^{\infty}[1+\alpha([k]_q-1)]([k]_q-1)A_q(\lambda,k)a_k z^{k-1}}{1-\sum\limits_{k=n+1}^{\infty}[1+\alpha([k]_q-1)]A_q(\lambda,k)a_k z^{k-1}}\right) > -\beta|b|. \tag{2.3}$$

Setting $z = r$ $(0 \leqq r < 1)$ in (2.3), we observe that the expression in the denominator of the left-hand side of (2.3) is positive for $r = 0$ and also for $0 < r < 1$. Thus, if we let $r \to 1-$ through real values, (2.3) would lead us to (2.2).

Conversely, let (2.2) hold true and $|z| = 1$. We then find that

$$\left|\frac{(1-\alpha)zD_q\big(\Omega_{q,z}^{\lambda}f(z)\big)+\alpha zD_q\Big(zD_q\big(\Omega_{q,z}^{\lambda}f(z)\big)\Big)}{(1-\alpha)\Omega_{q,z}^{\lambda}f(z)+\alpha zD_q\big(\Omega_{q,z}^{\lambda}f(z)\big)}-1\right|$$

$$\leqq \frac{\beta|b|\{1-\sum\limits_{k=n+1}^{\infty}[1+\alpha([k]_q-1)]A_q(\lambda,k)a_k\}}{1-\sum\limits_{k=n+1}^{\infty}[1+\alpha([k]_q-1)]A_q(\lambda,k)a_k} = \beta|b|.$$

Hence, by the *Maximum Modulus Theorem*, we conclude that $f(z) \in \mathcal{S}_n^{\alpha}(\lambda,\beta,b,q)$, which completes the proof of Theorem 1. □

The following corollary follows easily from Theorem 1.

**Corollary 1.** *Let* $f(z) \in \mathcal{S}_n^\alpha(\lambda,\beta,b,q)$*. Then*

$$a_k \leqq \frac{\beta\,|b|}{([k]_q+\beta\,|b|-1)[1+\alpha([k]_q-1)]A_q(\lambda,k)} \qquad (k \geqq n+1). \tag{2.4}$$

*The result is sharp for the function* $f(z)$ *given (for* $(k \geqq n+1)$ *by*

$$f(z) = z - \frac{\beta\,|b|}{([k]_q+\beta\,|b|-1)[1+\alpha([k]_q-1)]A_q(\lambda,k)} z^k. \tag{2.5}$$

Putting $\beta = 1$ in Theorem 1, we have Corollary 2 below.

**Corollary 2.** *Let* $f(z) \in \mathcal{S}_n^\alpha(\lambda,b,q)$*. Then*

$$\sum_{k=n+1}^{\infty} ([k]_q+|b|-1)[1+\alpha([k]_q-1)]A_q(\lambda,k)a_k \leqq |b|\,.$$

**Corollary 3.** *Let* $f(z) \in \mathcal{S}_n^\alpha(\lambda,b,q)$*. Then*

$$a_k \leqq \frac{|b|}{([k]_q+|b|-1)[1+\alpha([k]_q-1)]A_q(\lambda,k)} \qquad (k \geqq n+1).$$

*The result is sharp for the function* $f(z)$ *given by*

$$f(z) = z - \frac{|b|}{([k]_q+|b|-1)[1+\alpha([k]_q-1)]A_q(\lambda,k)}\, z^k \qquad (k \geqq n+1).$$

It is not difficult to prove the following results. The details involved are being left as an exercise for the interested reader.

**Theorem 2.** *The function* $f(z) \in \mathcal{G}_n^\alpha(\lambda,\beta,b,q)$ *if and only if*

$$\sum_{k=n+1}^{\infty} [k]_q\big[1+\alpha([k]_q-1)\big]A_q(\lambda,k)a_k \leqq \beta\,|b|\,. \tag{2.6}$$

**Corollary 4.** *Let* $f(z) \in \mathcal{G}_n^\alpha(\lambda,\beta,b,q)$*. Then*

$$a_k \leqq \frac{\beta\,|b|}{[k]_q\big[1+\alpha([k]_q-1)\big]A_q(\lambda,k)}. \tag{2.7}$$

*The result is sharp for the function* $f(z)$ *given by*

$$f(z) = z - \frac{\beta\,|b|}{[k]_q\big[1+\alpha[k]_q-1\big]A_q(\lambda,k)}\, z^k \qquad (k \geqq n+1). \tag{2.8}$$

We now state (without proof) Theorem 3 below.

**Theorem 3.** *If* $b_1, b_2 \in \mathbb{C}^*$ *and* $|b_1| < |b_2|$*, then*

$$\mathcal{S}_n^\alpha(\lambda,\beta,b_1,q) \subset \mathcal{S}_n^\alpha(\lambda,\beta,b_2,q).$$

The following result can indeed be proven along the lines which we have already indicated above.

**Theorem 4.** *If $b_1, b_2 \in \mathbb{C}^*$ and $|b_1| < |b_2|$, then*

$$\mathcal{G}_n^{\alpha}(\lambda,\beta,b_1,q) \subset \mathcal{G}_n^{\alpha}(\lambda,\beta,b_2,q). \tag{2.9}$$

## 3. Extreme points for the function classes $\mathcal{S}_n^{\alpha}(\lambda,\beta,b,q)$ and $\mathcal{G}_n^{\alpha}(\lambda,\beta,b,q)$

In this section, we first prove the following result.

**Theorem 5.** *Let $f_n(z) = z$ and*

$$f_k(z) = z - \frac{\beta\,|b|}{([k]_q + \beta\,|b| - 1)[1+\alpha([k]_q - 1)]A_q(\lambda,k)} z^k \tag{3.1}$$

$$(k \geqq n+1).$$

*Then the function $f(z)$ is in the class $\mathcal{S}_n^{\alpha}(\lambda,\beta,b,q)$ if and only if it can be expressed in the following form*:

$$f(z) = \sum_{k=n}^{\infty} \mu_k f_k(z), \tag{3.2}$$

*where*

$$\sum_{k=n}^{\infty} \mu_k = 1 \qquad \text{and} \qquad \mu_k \geqq 0.$$

*Proof.* By assuming (3.2) to hold true, if we appropriately apply Theorem 1, it is not difficult to conclude that $f(z) \in \mathcal{S}_n^{\alpha}(\lambda,\beta,b,q)$.

Conversely, upon leting $f(z) \in \mathcal{S}_n^{\alpha}(\lambda,\beta,b,q)$, if we set

$$\mu_k = \frac{([k]_q + \beta\,|b| - 1)[1+\alpha([k]_q - 1)]A_q(\lambda,k)}{\beta\,|b|} a_k \qquad (k \geqq n+1)$$

and

$$\mu_n = 1 - \sum_{k=n+1}^{\infty} \mu_k,$$

we can easily see that $f(z)$ can be expressed in the form (3.2). This completes the proof of Theorem 5. □

**Corollary 5.** *The extreme points of the function class $\mathcal{S}_n^{\alpha}(\lambda,\beta,b,q)$ are the functions $f_n(z) = z$ and $f_k(z)$ $(k \geqq n+1)$ given by* (3.1).

Similarly, we can prove the following theorem.

**Theorem 6.** *Let $f_n(z) = z$ and*

$$f_k(z) = z - \frac{\beta\,|b|}{[k]_q[1+\alpha([k]_q - 1)]A_q(\lambda,k)} z^k \qquad (k \geqq n+1). \tag{3.3}$$

*Then the function $f(z)$ is in the class $\mathcal{S}_n^{\alpha}(\lambda,\beta,b,q)$ if and only if it can be expressed in the form given by*

$$f(z)=\sum_{k=n}^{\infty}\mu_k f_k(z), \tag{3.4}$$

*where*

$$\sum_{k=n}^{\infty}\mu_k=1 \qquad \text{and} \qquad \mu_k \geqq 0. \tag{3.5}$$

**Corollary 6.** *The extreme points of the function class $\mathcal{G}_n^{\alpha}(\lambda,\beta,b,q)$ are the functions $f_n(z)=z$ and $f_k(z)$ $(k \geqq n+1)$ given by (3.3).*

## 4. Radii of close-to-convexity, starlikeness and convexity of the function class $\mathcal{S}_n^{\alpha}(\lambda,\beta,b,q)$

**Theorem 7.** *Let $f(z) \in \mathcal{S}_n^{\alpha}(\lambda,\beta,b,q)$. Then $f(z)$ is close-to-convex of order $\rho$ $(0 \leqq \rho < 1)$ in $|z|<r_1$, where*

$$r_1 := \inf_{k \geqq n+1}\left\{\frac{(1-\rho)([k]_q+\beta|b|-1)[1+\alpha([k]_q-1)]A_q(\lambda,k)}{k\beta|b|}\right\}^{\frac{1}{k-1}}. \tag{4.1}$$

*The sharpness of this result is attained for the function $f(z)$ given by (2.5).*

*Proof.* By showing that

$$\left|f'(z)-1\right| \leqq 1-\rho \quad \text{for} \quad |z|<r_1,$$

where $r_1$ is given by (4.1), we readily find that

$$\left|f'(z)-1\right| \leqq 1-\rho,$$

if

$$\sum_{k=n+1}^{\infty}\frac{k}{1-\rho}a_k|z|^{k-1} \leqq 1. \tag{4.2}$$

But, by Theorem 1, it is seen that (4.2) will hold true if (for $k \geqq n+1$)

$$|z| \leqq \left(\frac{(1-\rho)([k]_q+\beta|b|-1)[1+\alpha([k]_q-1)]A_q(\lambda,k)}{k\beta|b|}\right)^{\frac{1}{k-1}}.$$

This completes the proof of Theorem 7. □

By using arguments and analysis similar to those in the proof of Theorem 7, we can analogously derive Theorem 8 and Corollary 7 below.

**Theorem 8.** *Let $f(z) \in \mathcal{S}_n^{\alpha}(\lambda,\beta,b,q)$. Then the function $f(z)$ is starlike of order $\rho$ $(0 \leqq \rho < 1)$ in $|z| < r_2$, where*

$$r_2 := \inf_{k \geqq n+1} \left\{ \frac{(1-\rho)([k]_q + \beta\,|b| - 1)[1+\alpha([k]_q - 1)]A_q(\lambda,k)}{(k-\rho)\beta\,|b|} \right\}^{\frac{1}{k-1}}. \tag{4.3}$$

*The sharpness of this result is attained for the function $f(z)$ given by (2.5).*

**Corollary 7.** *Let $f(z) \in \mathcal{S}_n^{\alpha}(\lambda,\beta,b,q)$. Then the function $f(z)$ is convex of order $\rho$ $(0 \leqq \rho < 1)$ in $|z| < r_3$, where*

$$r_3 := \inf_{k \geqq n+1} \left\{ \frac{(1-\rho)([k]_q + \beta\,|b| - 1)[1+\alpha([k]_q - 1)]A_q(\lambda,k)}{k(k-\rho)\beta\,|b|} \right\}^{\frac{1}{k-1}}.$$

*The sharpness of the result is attained for the function $f(z)$ given by (2.5).*

## 5. Growth and distortion theorems

For convenience in this section, for $k \geqq n+1$, we shall henceforth use the following notations:

$$\sigma_{k,\alpha}(\lambda,\beta,b,q) := ([k]_q + \beta\,|b| - 1)[1+\alpha([k]_q - 1)]A_q(\lambda,k) \tag{5.1}$$

and

$$\phi_{k,\alpha}(\lambda,\beta,b,q) := [k]_q[1+\alpha([k]_q - 1)]A_q(\lambda,k). \tag{5.2}$$

We now prove the following which will be needed in our further investigation in this section.

**Lemma 1.** *The sequence $\{A_q(\lambda,k)\}_{k=n+1}^{\infty}$ is a decreasing sequence in $k$ $(k \geqq n+1)$ for $\lambda < 2$ and $0 < q < 1$.*

*Proof.* It follows from (1.8) and the recurrence relation:

$$\Gamma_q(z+1) = [z]_q\,\Gamma_q(z)$$

that

$$\begin{aligned}\frac{A_q(\lambda,k+1)}{A_q(\lambda,k)} &= \frac{\Gamma_q(k+2)\Gamma_q(k-\lambda+1)}{\Gamma_q(k+1)\Gamma_q(k-\lambda+2)}\\ &= \frac{[k+1]_q\Gamma_q(k+1)\Gamma_q(k-\lambda+1)}{\Gamma_q(k+1)[k-\lambda+1]_q\Gamma_q(k-\lambda+2)} = \frac{[k+1]_q}{[k-\lambda+1]_q}.\end{aligned}$$

It is sufficient to consider the value $k = n+1$. By using the definition (2.1) of the basic (or $q$-) number $[\lambda]_q$ again, we thus find that

$$\frac{A_q(\lambda,k+1)}{A_q(\lambda,k)} = \frac{[n+2]_q}{[n-\lambda+2]_q} = \frac{1-q^{n+2}}{1-q^{n-\lambda+2}} \qquad (0 < q < 1;\ -\infty < \lambda < 2).$$

The sequence $\{A_q(\lambda,k)\}_{k=n+1}^{\infty}$ is a decreasing sequence in $k$ if

$$\frac{A_q(\lambda,k+1)}{A_q(\lambda,k)} < 1 \qquad (k \geqq n+1),$$

that is, if

$$\frac{1-q^{n+2}}{1-q^{n-\lambda+2}} < 1 \qquad (0<q<1;\ -\infty<\lambda<2), \tag{5.3}$$

which implies that $\lambda < 0$. Thus $\{A_q(\lambda,k)\}_{k=n+1}^{\infty}$ is a decreasing sequence in $k$ $(k \geqq n+1)$ for $-\infty<\lambda<2$ and $0<q<1$. □

**Theorem 9.** *Let $f(z) \in \mathcal{S}_n^{\alpha}(\lambda,\beta,b,q)$. Then*

$$|z| - \frac{\beta\,|b|}{\sigma_{n+1,\alpha}(\lambda,\beta,b,q)}\,|z|^{n+1} \leqq |f(z)| \leqq |z| + \frac{\beta\,|b|}{\sigma_{n+1,\alpha}(\lambda,\beta,b,q)}\,|z|^{n+1}. \tag{5.4}$$

*The result is sharp for the function $f(z)$ given by*

$$f(z) = z - \frac{\beta\,|b|}{\sigma_{n+1,\alpha}(\lambda,\beta,b,q)} z^{n+1}. \tag{5.5}$$

*Proof.* Since $f(z) \in \mathcal{S}_n^{\alpha}(\lambda,\beta,b,q)$, in view of Theorem 1, we have

$$\sigma_{n+1,\alpha}(\lambda,\beta,b,q) \sum_{k=n+1}^{\infty} a_k \leqq \sum_{k=n+1}^{\infty} \sigma_{k,\alpha}(\lambda,\beta,b,q) a_k \leqq \beta\,|b|,$$

that is,

$$\sum_{k=n+1}^{\infty} a_k \leqq \frac{\beta\,|b|}{\sigma_{n+1,\alpha}(\lambda,\beta,b,q)}. \tag{5.6}$$

We thus obtain

$$\begin{aligned} |f(z)| &\geqq |z| - \sum_{k=n+1}^{\infty} a_k\,|z|^k \geqq |z| - |z|^{n+1} \sum_{k=n+1}^{\infty} a_k \\ &\geqq |z| - \frac{\beta\,|b|}{\sigma_{n+1,\alpha}(\lambda,\beta,b,q)}\,|z|^{n+1} \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} |f(z)| &\leqq |z| + \sum_{k=n+1}^{\infty} a_k\,|z|^k \leqq |z| + |z|^{n+1} \sum_{k=n+1}^{\infty} a_k \\ &\leqq |z| + \frac{\beta\,|b|}{\sigma_{n+1,\alpha}(\lambda,\beta,b,q)}\,|z|^{n+1}. \end{aligned} \tag{5.8}$$

These last inequalities (5.7) and (5.8) complete the proof of Theorem 9. □

**Corollary 8.** *Under the hypothesis of Theorem 9, the function $f(z)$ is included in a disk with center at the origin and radius $r$ given by*

$$r = 1 + \frac{\beta\,|b|}{\sigma_{n+1,\alpha}(\lambda,\beta,b,q)}.$$

Similarly, we can prove the following distortion theorem for $f(z) \in \mathcal{G}_n^{\alpha}(\lambda,\beta,b,q)$.

**Theorem 10.** *Let $f(z) \in \mathcal{G}_n^{\alpha}(\lambda,\beta,b,q)$ and let $\phi_{k,\alpha}(\lambda,\beta,b,q)$ be given by (5.2). Then*

$$|z| - \frac{\beta\,|b|}{\phi_{n+1,\alpha}(\lambda,\beta,b,q)}\,|z|^{n+1} \leqq |f(z)| \leqq |z| + \frac{\beta\,|b|}{\phi_{n+1,\alpha}(\lambda,\beta,b,q)}\,|z|^{n+1}. \tag{5.9}$$

*The result is sharp for the function $f(z)$ given by*

$$f(z) = z - \frac{\beta\,|b|}{\phi_{n+1,\alpha}(\lambda,\beta,b,q)}z^{n+1}. \tag{5.10}$$

**Corollary 9.** *Under the hypothesis of Theorem 10, the function $f(z)$ is included in a disk with its center at the origin and its radius $r$ given by*

$$r = 1 + \frac{\beta\,|b|}{\phi_{n+1,\alpha}(\lambda,\beta,b,q)}.$$

A further distortion theorem involving the generalized fractional $q$-differintegral operator $\Omega_{q,z}^{\lambda}$ defined by (1.7) is given by the following theorem.

**Theorem 11.** *Let $f(z) \in \mathcal{S}_n^{\alpha}(\lambda,\beta,b,q)$. Then*

$$\begin{aligned} &|z| - \frac{\beta\,|b|}{([n+1]_q + \beta\,|b| - 1)[1+\alpha([n+1]_q - 1)]}\,|z|^{n+1} \\ &\leqq \left|\Omega_{q,z}^{\lambda} f(z)\right| \\ &\leqq |z| + \frac{\beta\,|b|}{([n+1]_q + \beta\,|b| - 1)[1+\alpha([n+1]_q - 1)]}\,|z|^{n+1}. \end{aligned} \tag{5.11}$$

*The result is sharp.*

*Proof.* From the above Lemma 1, in conjunction with the equations (5.6) and (1.7), we have

$$\begin{aligned} \left|\Omega_{q,z}^{\lambda} f(z)\right| &\geqq |z| - A_q(\lambda, n+1)\,|z|^{n+1} \sum_{k=n+1}^{\infty} a_k \\ &\geqq |z| - \frac{\beta\,|b|}{([n+1]_q + \beta\,|b| - 1)[1+\alpha([n+1]_q - 1)]}\,|z|^{n+1} \end{aligned} \tag{5.12}$$

and

$$\left|\Omega_{q,z}^{\lambda} f(z)\right| \leqq |z| + A_q(\lambda, n+1)\,|z|^{n+1} \sum_{k=n+1}^{\infty} a_k$$

$$\leqq |z| + \frac{\beta |b|}{([n+1]_q + \beta |b| - 1)[1 + \alpha([n+1]_q - 1)]} |z|^{n+1}. \tag{5.13}$$

The equalities in (5.11) are attained for the function $f(z)$ given by

$$D_{q,z}^{\lambda} f(z) = \frac{\Gamma_q(z) z^{1-\lambda}}{\Gamma_q(2-\lambda)} \cdot \left(1 - \frac{\beta |b|}{([n+1]_q + \beta |b| - 1)[1 + \alpha([n+1]_q - 1)]} |z|^n\right) \tag{5.14}$$

or by the function $f(z)$ given by (5.5). We have thus completed our demonstration of Theorem 11. □

From Theorem 10 and (1.7), we have the following distortion inequality involving the fractional $q$-derivative operator $D_{q,z}^{\lambda}$.

**Corollary 10.** *Let* $f(z) \in \mathcal{S}_n^{\alpha}(\lambda, \beta, b, q)$. *Then*

$$\frac{\Gamma_q(2)}{\Gamma_q(2-\lambda)} |z|^{1-\lambda} \left(1 - \frac{\beta |b|}{([n+1]_q + \beta |b| - 1)[1 + \alpha([n+1]_q - 1)]} |z|^n\right) \leqq \left|D_{q,z}^{\lambda} f(z)\right| \leqq \frac{\Gamma_q(2)}{\Gamma_q(2-\lambda)} |z|^{1-\lambda} \cdot \left(1 + \frac{\beta |b|}{([n+1]_q + \beta |b| - 1)[1 + \alpha([n+1]_q - 1)]} |z|^n\right). \tag{5.15}$$

*The result is sharp for the function* $f(z)$ *given by* (5.5).

Upon setting $\beta = 1$ in Corollary 10, we get the following corollary which provided the *corrected* version of a result obtained by Purohit and Raina [12, Corollary 1].

**Corollary 11.** *Let* $f(z) \in \mathcal{S}_n^{\alpha}(\lambda, b, q)$. *Then*

$$\frac{\Gamma_q(2)}{\Gamma_q(2-\lambda)} |z|^{1-\lambda} \left(1 - \frac{|b|}{([n+1]_q + |b| - 1)[1 + \alpha([n+1]_q - 1)]} |z|^n\right) \leqq \left|D_{q,z}^{\lambda} f(z)\right| \leqq \frac{\Gamma_q(2)}{\Gamma_q(2-\lambda)} |z|^{1-\lambda} \cdot \left(1 + \frac{|b|}{([n+1]_q + |b| - 1)[1 + \alpha([n+1]_q - 1)]} |z|^n\right). \tag{5.16}$$

*The result is sharp for the function* $f(z)$ *given by* (5.5) *with* $\beta = 1$.

Also, in view of (1.9) or by virtue of (1.3), Theorem 10 gives the following distortion inequality involving the fractional $q$-integral operator $I_{q,z}^{\lambda}$.

**Corollary 12.** *Let* $f(z) \in \mathcal{S}_n^{\alpha}(\lambda, \beta, b, q)$. *Then*

$$\frac{\Gamma_q(2)}{\Gamma_q(2+\lambda)} |z|^{1+\lambda} \left(1 - \frac{\beta |b|}{([n+1]_q + \beta |b| - 1)[1 + \alpha([n+1]_q - 1)]} |z|^n\right)$$

$$\leqq \left| I_{q,z}^{\lambda} f(z) \right| \frac{\Gamma_q(2)}{\Gamma_q(2+\lambda)} |z|^{1+\lambda} \cdot \left( 1 + \frac{\beta |b|}{([n+1]_q + \beta |b| - 1)[1+\alpha([n+1]_q - 1)]} |z|^n \right). \tag{5.17}$$

*The result is sharp for the function $f(z)$ given by (5.5).*

Putting $\beta = 1$ in Corollary 12, we have the following result.

**Corollary 13.** *Let $f(z) \in \mathcal{S}_n^{\alpha}(\lambda, b, q)$. Then*

$$\frac{\Gamma_q(2)}{\Gamma_q(2+\lambda)} |z|^{1+\lambda} \left( 1 - \frac{|b|}{([n+1]_q + |b| - 1)[1+\alpha([n+1]_q - 1)]} |z|^n \right) \leqq \left| I_{q,z}^{\lambda} f(z) \right| \leqq \frac{\Gamma_q(2)}{\Gamma_q(2+\lambda)} |z|^{1+\lambda} \cdot \left( 1 + \frac{|b|}{([n+1]_q + |b| - 1)[1+\alpha([n+1]_q - 1)]} |z|^n \right). \tag{5.18}$$

*The result is sharp for the function $f(z)$ given by (5.5) with $\beta = 1$ and $\lambda$ replaced by $-\lambda$.*

**Theorem 12.** *Let $f(z) \in \mathcal{G}_n^{\alpha}(\lambda, \beta, b, q)$. Then*

$$|z| - \frac{\beta |b|}{[n+1]_q [1+\alpha([n+1]_q - 1)]} |z|^{n+1} \leqq \left| \Omega_{q,z}^{\lambda} f(z) \right| \leqq |z| + \frac{\beta |b|}{[n+1]_q [1+\alpha([n+1]_q - 1)]} |z|^{n+1}. \tag{5.19}$$

*The result is sharp for the function $f(z)$ given by*

$$D_{q,z}^{\lambda} f(z) = \frac{\Gamma_q(z) z^{1-\lambda}}{\Gamma_q(2-\lambda)} \left( 1 - \frac{\beta |b|}{[n+1]_q [1+\alpha([n+1]_q - 1)]} |z|^n \right) \tag{5.20}$$

*or by the function $f(z)$ given by (5.10).*

Similarly, we can prove the following distortion inequalities for $f(z) \in \mathcal{G}_n^{\alpha}(\lambda, \beta, b, q)$ involving the fractional $q$-derivative operator $D_{q,z}^{\lambda}$ and the fractional $q$-integral operator $I_{q,z}^{\lambda}$.

**Corollary 14.** *Let $f(z) \in G_n^{\alpha}(\lambda, \beta, b, q)$. Then*

$$\frac{\Gamma_q(2)}{\Gamma_q(2-\lambda)} |z|^{1-\lambda} \left( 1 - \frac{\beta |b|}{[n+1]_q [1+\alpha([n+1]_q - 1)]} |z|^n \right) \leqq \left| D_{q,z}^{\lambda} f(z) \right|$$

$$\leqq \frac{\Gamma_q(2)}{\Gamma_q(2-\lambda)}|z|^{1-\lambda}\left(1+\frac{\beta|b|}{[n+1]_q[1+\alpha([n+1]_q-1)]}|z|^n\right). \tag{5.21}$$

*The result is sharp for the function $f(z)$ given by (5.10).*

**Corollary 15.** *Let $f(z) \in \mathcal{G}_n^{\alpha}(\lambda,\beta,b,q)$. Then*

$$\begin{aligned}\frac{\Gamma_q(2)}{\Gamma_q(2+\lambda)}|z|^{1+\lambda}&\left(1-\frac{\beta|b|}{[n+1]_q[1+\alpha([n+1]_q-1)]}|z|^n\right)\\ &\leqq \left|I_{q,z}^{\lambda}f(z)\right|\\ &\leqq \frac{\Gamma_q(2)}{\Gamma_q(2+\lambda)}|z|^{1+\lambda}\left(1+\frac{\beta|b|}{[n+1]_q[1+\alpha([n+1]_q-1)]}|z|^n\right).\end{aligned} \tag{5.22}$$

*The result is sharp for the function $f(z)$ given by (5.10).*

Putting $\beta = 1$ in Corollaries 14 and 15, respectively, we have the following corollaries.

**Corollary 16.** *Let $f(z) \in \mathcal{G}_n^{\alpha}(\lambda,b,q)$. Then*

$$\begin{aligned}\frac{\Gamma_q(2)}{\Gamma_q(2-\lambda)}|z|^{1-\lambda}&\left(1-\frac{|b|}{[n+1]_q[1+\alpha([n+1]_q-1)]}|z|^n\right)\\ &\leqq \left|D_{q,z}^{\lambda}f(z)\right|\\ &\leqq \frac{\Gamma_q(2)}{\Gamma_q(2-\lambda)}|z|^{1-\lambda}\left(1+\frac{|b|}{[n+1]_q[1+\alpha([n+1]_q-1)]}|z|^n\right).\end{aligned} \tag{5.23}$$

*The result is sharp for the function $f(z)$ given by (5.10) with $\beta = 1$.*

**Corollary 17.** *Let $f(z) \in \mathcal{G}_n^{\alpha}(\lambda,\beta,b,q)$. Then*

$$\begin{aligned}\frac{\Gamma_q(2)}{\Gamma_q(2+\lambda)}|z|^{1+\lambda}&\left(1-\frac{|b|}{[n+1]_q[1+\alpha([n+1]_q-1)]}|z|^n\right)\\ &\leqq \left|I_{q,z}^{\lambda}f(z)\right|\\ &\leqq \frac{\Gamma_q(2)}{\Gamma_q(2+\lambda)}|z|^{1+\lambda}\left(1+\frac{|b|}{[n+1]_q[1+\alpha([n+1]_q-1)]}|z|^n\right).\end{aligned} \tag{5.24}$$

*The result is sharp for the function $f(z)$ given by (5.10) with $\beta = 1$.*

*Remark* 3. The results asserted by Corollaries 15 and 16 provide, respectively, the *corrected* versions of the results obtained by Purohit and Raina [12, Corollaries 3 and 4].

*Remark* 4. Putting $\lambda = 0$ in our results, we obtain a number of new results for the function classes $\mathcal{S}_n^{\alpha}(\beta,b,q)$ and $\mathcal{G}_n^{\alpha}(\beta,b,q)$.

## 6. CONCLUSION

In our present investigation, we applied various operators of $q$-calculus and fractional $q$-calculus in the study of two general subclasses $\mathcal{S}_n^{\alpha}(\lambda,\beta,b,q)$ and $\mathcal{G}_n^{\alpha}(\lambda,\beta,b,q)$ of normalized analytic functions with complex order and negative coefficients. For each of these function classes, we have derived their associated coefficient estimates, radii of close-to-convexity, starlikeness and convexity, extreme points, and growth and distortion theorems. Our main results and their new or known consequences are stated and proved as theorems and corollaries.

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