Statistica Sinica **13**(2003), 403-408

SOME PROPERTIES OF BLOCKED AND UNBLOCKED FOLDOVERS OF 2^{k-p} DESIGNS

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Abstract: In this article, we focus on the theoretical properties of the foldover design and the resulting combined design obtained by augmenting an initial design by its foldover. We prove that there are 2^p distinct ways to fold over a 2^{k-p} design. Optimal foldover plans are also discussed. We investigate the impact of the inclusion of a blocking variable to the design. We show that the minimum aberration foldover design with the presence of the blocking effect is the same as the one without blocking.

Key words and phrases: Minimum aberration, optimal foldover, resolution, word length pattern.

1. Introduction

Fractional factorial designs are widely used in engineering and scientific experiments. One consequence of using a fraction of the full factorial design is that some effects are confounded with others. Follow-up experiments may be needed to break the confounding. A useful and simple strategy is to add a foldover of the initial design by reversing the signs of one or more of its factors. Many textbooks (e.g., Box, Hunter and Hunter (1978), Montgomery (2001), Neter, Kutner, Nachishein and Wassserman (1996), Wu and Hamada (2000)) discuss foldover techniques. Following the nomenclature in Li and Lin (2001), we refer a *foldover plan* to a collection of columns (factors) whose signs are to be reversed. The *foldover design* resulting from a foldover plan is also called a *foldover*. When the initial design is augmented by its foldover, it is called a *combined design*. Consider a design with k factors $x_1, \ldots, x_k$. A standard strategy is to construct a foldover design by reversing the signs of all factors. Such a foldover is called a full-foldover, and its foldover plan is denoted by $\gamma = \{1, \ldots, k\}$. It is well-known that the *full-foldover* design is not effective for all designs. For example, the design obtained by combining a resolution IV design and its full foldover is still of resolution IV.

Foldover plans other than full-foldover have also been proposed in the literature. The sign-reversal of one factor was considered in Box, Hunter and Hunter (1978), and Wu and Hamada (2000). Montgomery and Runger (1996) considered reversing the signs of one or two factors. Recently, papers by Li and Mee (2000) and Li and Lin (2001) address the issue of an *optimal foldover*. The papers have different emphases: the former provides a sufficient condition of the existence of an alternative foldover that is better than the full-foldover for resolution III designs; the latter aims to find the optimal foldover for a given fractional factorial design.

In this article we explore the structures and properties of foldover designs. In a 2^k factorial design, denote the treatment combinations by k-dimensional row vectors over the finite field $GF(2)$. Consider a regular 2^{k-p} fractional factorial design A. To avoid trivialities, let A have resolution of at least III. It is well-known (see, e.g., Cheng and Mukerjee (1998)) that A is given by $R(C)$, where C is a $(k-p) \times k$ matrix that is of full row rank over $GF(2)$ such that no two columns of C are proportional to each other, and $R(\cdot)$ stands for the row space of a matrix. A factorial effect (*i.e.*, a main effect or an interaction) can be represented by a k-dimensional nonnull column vector over $GF(2)$. Such a factorial effect, say z, is a defining effect (that is, appears in the defining relation) of A if and only if (Cheng and Mukerjee (1998))

$$Cz = 0. \tag{1}$$

Now consider the foldover design A_γ, where γ is a foldover plan consisting of columns whose signs are reversed. It can be easily seen that the combined design is given by $R(C^*)$, where

$$C^* = \begin{bmatrix} C \\ \xi^T \end{bmatrix}, \tag{2}$$

and ξ is a $k \times 1$ vector over $GF(2)$ with ith element 1 if $i \in \gamma$, and 0 otherwise.

The remainder of the article is organized as following. Section 2 studies some properties of the foldover plans for a 2^{k-p} design. Section 3 investigates the impact of the blocking factor. It is shown that the inclusion of the blocking factor results in the change of the word length pattern of a combined design. However, such a change does not alter the aberration comparisons of two combined designs. Thus, the optimal foldover plans in terms of aberration criterion that were obtained without considering the blocking effect (Li and Lin (2001)) are still optimal in the presence of a blocking factor. The conclusion and future work are discussed in Section 4.

2. Properties of Foldover Plans

We now prove that there are 2^p distinct ways to fold over a 2^{k-p} design. Note that the result is discussed in Li and Mee (2000) and Li and Lin (2001). The former offers an informal argument, and the latter gives a rigorous but long proof. By using the notations introduced in the last section, we obtain a simple and brief proof.

Theorem 2.1. *For a 2^{k-p} design with p generators, there are 2^p distinct ways to generate a foldover design.*

Proof. Because the $(k-p) \times k$ matrix C has full row rank, there exists a $p \times k$ matrix G over $GF(2)$ such that the $k \times k$ matrix $\begin{bmatrix} C \\ G \end{bmatrix}$ is nonsingular. Denoting the rows of G by $g_1^T, \ldots, g_p^T$, then ξ in (2) must be of the form

$$\xi = C^T h + \sum_{j=1}^{p} m_j g_j, \tag{3}$$

where h is a $(k-p) \times 1$ vector over $GF(2)$ and each m_j is 0 or 1. By the definition of G, it is clear from (2) and (3) that the distinct possibilities for $R(C^*)$ correspond to the 2^p distinct choices of $(m_1, \ldots, m_p)$. Thus, there exist 2^p distinct combined designs, or equivalently, there are 2^p distinct ways to generate a foldover design.

Note that when $m_1 = \cdots = m_p = 0$ in (3), the foldover design is the same as A itself. This trivial case is hereafter left out of consideration.

It is well-known that the combined design after foldover has only half the words of the initial design. The following theorem gives a necessary and sufficient condition for judging whether a word should be canceled or stay in the combined design.

Theorem 2.2. *Denote the combined design of A and its foldover design A_γ by A^*, where γ is a foldover plan. Then a defining effect (or word) of A, say z, stays in A^* if and only if z has an even number of factors that are included in γ.*

Proof. By (2), z is a defining effect of A^* if and only if

$$Cz = 0 \text{ and } \xi^T z = 0. \tag{4}$$

Thus by (1), a defining effect z of A is also a defining effect of A^* if and only if $\xi^T z = 0$, *i.e.*, if and only if z involves an even number of factors in common with the set of factors that are in γ.

3. Blocking Effect on Foldover Designs

Because of the sequential nature of foldover designs, the first half and the second half of a combined design can be seen as two blocks. If the blocking effect is believed to be important, it should be considered when we select a design and analyze the data. In this section, we study the properties of foldover designs when the blocking factor is included.

First, without loss of generality, assume that the blocking factor x_b takes value 0 for the initial design A and 1 for its foldover. Denote the combined design including the blocking factor by A_b. Then A_b is given by $R(C_b)$ where

$$C_b = [w, C^*]. \tag{5}$$

Here C^* is as in (2) and w is a $(k-p+1) \times 1$ vector with 0 in the first $k-p$ positions and 1 in the last position. The first column of C_b corresponds to the blocking factor, and the other columns are associated with the treatment factors.

By (2) and (5), for any nonnull $(k+1) \times 1$ vector $z_b = (z_0, z^T)^T$, z a $k \times 1$ vector,

$$C_b z_b = 0 \text{ if and only if } Cz = 0 \text{ and } z_0 + \xi^T z = 0. \tag{6}$$

By (1) and (6), $z_b = (z_0, z^T)^T$ is a defining effect of A_b if and only if z is a defining effect of A and $z_0 + \xi^T z = 0$. For a full foldover, each element of ξ is 1. Thus, for an initial design A whose resolution is III or IV, the combined design resulting from a full foldover is a resolution IV design *when the blocking factor is included*.

The next theorem demonstrates that the inclusion of the blocking factor does not change the comparison of two foldover plans in terms of the aberrations of the resulting combined design.

Theorem 3.1. *Consider two foldover plans γ and γ'. Denote the two resulting combined designs without the blocking factor by $A^*(\gamma)$ and $A^*(\gamma')$, respectively. When the blocking factor is included, the two resulting combined designs are denoted by $A_b(\gamma)$ and $A_b(\gamma')$, respectively. Then $A^*(\gamma)$ has less aberration than $A^*(\gamma')$ if and only if $A_b(\gamma)$ has less aberration than $A_b(\gamma')$.*

Proof. By (1), (4) and (6), $z_b = [z_0, z^T]^T$ has i nonzero elements and is a defining effect of A_b if and only if either (i) $z_0 = 0$, z has i nonzero elements and z is a defining effect of A^*, or (ii) $z_0 = 1$, z has $i-1$ nonzero elements and z is a defining effect of A but not of A^*. Thus, $\theta_i = \beta_i + \alpha_{i-1} - \beta_{i-1}$, where θ_i, β_i and α_i are the numbers of defining effects involving i factors in A_b, A^* and A, respectively. The theorem follows immediately.

Theorem 3.1 shows that the blocking factor does not change the comparison of the combined designs in terms of their aberrations. Note that many papers on the foldover designs do not consider the blocking effect directly (e.g., Li and Mee (2000), Li and Lin (2001)). In particular, Li and Lin (2001) provided a catalog of optimal foldover designs using the aberration criterion of the combined design without considering the blocking factor. By Theorem 3.1, these designs are still minimum-aberration designs in the presence of a blocking factor.

We conclude this section by considering the impact of the inclusion of the blocking factor on certain partial foldover plans discussed in the previous sections.

1. If the sign of only one factor is reversed, then all words involving this factor are multiplied by the blocking factor. The interaction effects represented by those words are now confounded with the blocking effect. For example, if the sign of factor x_1 is reversed, the word $x_1x_2x_3$ becomes $x_bx_1x_2x_3$. Hence, the three-factor ($3f$) interaction $x_1x_2x_3$ is confounded with the blocking effect. Moreover, the $2f$ interaction x_1x_2 is confounded with x_3x_b. However, because the blocking factor usually does not interact with other factors, x_3x_b is often considered to be negligible. Therefore, if the initial design is of resolution III, all $2f$ interactions involving x_1 are cleared by the foldover. If the initial design is of resolution IV, all $3f$ interactions involving x_1 are cleared by the foldover. Note that a $2f$ interaction is called *clear* if it is not aliased with any main effects or other two-factor interactions (Wu and Chen (1992)). In our consideration, we extend the definition to allow a clear $2f$ interaction to be confounded with a two-factor interaction involving the blocking factor.

2. Suppose that the signs of two factors, say x_1 and x_2, are reversed. Then, following the above discussion, all $2f$ interactions involving one of the two factors x_1 and x_2 are clear. However, x_1x_2 may not be clear.

3. Suppose that we want to break the confounding between two $2f$ interactions, say x_1x_2 and x_3x_4. By reversing the sign of $x_1x_2x_3x_4$, the word becomes $x_bx_1x_2x_3x_4$. Hence, x_1x_2 is confounded with $x_bx_3x_4$, and x_3x_4 is confounded with $x_bx_1x_2$. Therefore, x_1x_2 and x_3x_4 are no longer confounded.

4. Concluding Remarks and Future Work

In Theorem 3.1, we do not distinguish words involving the blocking factor from those not involving the blocking factor. Because the interactions between the blocking factor and others are less likely to be active and often negligible, Sitter, Chen, Feder (1997) argue that, for two words that are of the same length, the one involving the blocking factor is less important than the one not involving the blocking factor. It can be seen from the proof of Theorem 3.1 that the result still holds for their aberration criterion. The theorem, however, may not hold for some other criteria proposed recently for blocking designs (*e.g.*, Chen and Cheng (1999), Mukerjee and Wu (1999), and Cheng and Wu (2002)).

An interesting question raised by the associate editor is: can these results be extended to s-level regular fractions, where $s(\geq 2)$ is a prime or prime power? It can be easily seen that there are $1 + \{(s^p - 1)/(s - 1)\}$ distinct combined designs—a generalization of Theorem 1. Other generalizations, however, are not trivial and are currently under investigation.

Acknowledgement

We thank the Editor, the associate editor, and the referees for helpful comments. We are especially grateful to the associate editor who provided detailed comments that have led to substantiative improvement of the article.

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(Received May 2001; accepted October 2002)