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SOME PROPERTIES OF FC-GROUPS WHICH OCCUR AS AUTOMORPHISM GROUPS

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ABSTRACT. We prove that if G is a group such that $\operatorname{Aut} G$ is a countably infinite torsion FC-group, then $\operatorname{Aut} G$ contains an infinite locally soluble, normal subgroup and hence a nontrivial abelian normal subgroup. It follows that a countably infinite subdirect product of nontrivial finite groups, of which only finitely many have nontrivial abelian normal subgroups, is not the automorphism group of any group.

We are concerned with the question: What classes of torsion groups can occur as the full group of automorphisms $\operatorname{Aut} G$ of a group G? Robinson [**1**] has shown that if $\operatorname{Aut} G$ is a Černikov group (a finite extension of a radicable abelian group with the minimal condition), then $\operatorname{Aut} G$ is finite. He has also shown that if $\operatorname{Aut} G$ is a nilpotent torsion group, then $\operatorname{Aut} G$ has finite exponent.

The case where G is a group such that $\operatorname{Aut} G$ is a countable torsion FC-group (finite conjugate) was examined in a previous paper [**2**]. It was shown that if G is a group such that $\operatorname{Aut} G$ is a countable torsion FC-group, then $\operatorname{Aut} G$ has finite exponent if either (1) $\operatorname{Aut} G$ has min-2 or (2) $\pi(\operatorname{Aut} G)$ is finite, where $\pi(H)$ is the set of all primes dividing the order of some torsion element of H. In addition, an example of a countable torsion FC-group of infinite exponent which occurred as an automorphism group was given to show that the theorem could not be improved. This example contains a nontrivial abelian normal subgroup. The question arises: Can we find an example which has no nontrivial abelian normal subgroups? We will answer this question in the negative.

THEOREM. *Let G be a group such that* $\operatorname{Aut} G$ *is a countably infinite periodic FC-group. Then either*

(a) $\operatorname{Aut} G$ *contains an infinite abelian normal subgroup N, or*

(b) $\operatorname{Aut} G$ *contains an infinite, locally soluble, normal $\{2,3\}$-subgroup of bounded exponent and finite index.*

In either case, $\operatorname{Aut} G$ *contains a nontrivial abelian normal subgroup.*

PROOF. Let $Q = G/C \cong \operatorname{Inn} G$, where C is the center of G, and let T be the torsion subgroup of C. It was proven in [**2**] that Q and T_p are finite for all primes p.

Let $q = |Q|$ and let p be any prime which does not divide $2q$. Since T_p is finite, we have $C = C_1 \times T_p$. It is well known that since $|Q|$ and $|T_p|$ are relatively prime, G splits over T_p. It follows that there exists a group G_1 containing C_1 such that

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$G = G_1 \times T_p$. Clearly, T_p is characteristic in G. Hence we have the short exact sequence

$$\mathrm{Hom}(G_1, T_p) \rightarrowtail \mathrm{Aut}\, G \twoheadrightarrow \mathrm{Aut}\, G_1 \times \mathrm{Aut}\, T_p.$$

Let I be the set of all primes p not dividing $2q$ such that $T_p \neq 1$.

Case 1: *I is infinite*. Define $M_p = \mathrm{Hom}(G_1, T_p) \lhd \mathrm{Aut}\, G$. Assume that $p \in I$. If $M_p \neq 1$, define $N_p = M_p$. If $M_p = 1$, then $\mathrm{Aut}\, G \cong \mathrm{Aut}\, G_1 \times \mathrm{Aut}\, T_p$ and define $N_p = Z(\mathrm{Aut}\, T_p) \leqslant \mathrm{Aut}\, G$. Since the inversion automorphism on T_p is contained in $Z(\mathrm{Aut}\, T_p)$, the group N_p is nontrivial. It is easily shown that $N = \langle N_p | p \in I \rangle$ is an abelian normal subgroup of $\mathrm{Aut}\, G$ which is infinite.

Case 2: *I is finite*. It follows that T is finite and hence $C = F \times T$ for some torsion-free group F. In the proof of Lemma 7 and Theorem A in **[2]**, it was shown that under these circumstances there exists a normal subgroup N of $\mathrm{Aut}\, G$ such that $\mathrm{Aut}\, G/N$ is finite and N is a $\{2, 3\}$-group of finite exponent. Clearly, N is a locally soluble, normal subgroup of $\mathrm{Aut}\, G$. If N is finite, then $\mathrm{Aut}\, G$ is finite. However, since $\mathrm{Aut}\, G$ is infinite, it must have an infinite locally soluble, normal subgroup N. Since $\mathrm{Aut}\, G$ is a periodic FC-group, it is locally finite and normal. Hence $\mathrm{Aut}\, G$ contains a finite normal subgroup which is soluble and therefore a nontrivial abelian normal subgroup. □

COROLLARY 1. *Let G be a group such that* $\mathrm{Aut}\, G$ *is a countably infinite periodic FC-group. If either* $\mathrm{Aut}\, G$ *has infinite exponent or if it has no elements of order* 2 *or* 3, *then* $\mathrm{Aut}\, G$ *contains an infinite abelian normal subgroup.*

COROLLARY 2. *If among the countably infinite sequence of nontrivial finite groups F_i there are only finitely many with a nontrivial soluble normal subgroup, then no subdirect product of the F_i can be an automorphism group.*

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