Some Properties of Γ-Limits of Integral Functionals (*) (**).

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Summary. – *We prove some integral representation theorems for the Γ limit in L^1 of sequences of the form $\int_\Omega f_h(x, Du)\,dx$ in coercivity and bounded growth hypothesis which are optimal as it is checked with examples. These results are utilized to describe the L^1 lower semicontinuous envelope of a given functional. We consider also the stability of Γ limits with respect to obstacle type perturbations and prove the homogenization formulas in conditions more generals of those already considered by several authors.*

0. – Introduction.

A classical problem in continuum physics consists in going back from the microscopic material models to an evaluation of the connected macroscopic quantities which are measurables.

An example of this kind of problems which POISSON (1) has delt with and which is still a standard subject in electro-magnetism's treatises (2) is the following (homogenization):

Does the sequence ($h \in N$)

$$\operatorname*{Inf}_{u \in C^1} \left\{ \int_Y a_{ij}(hx)\, D_i u\, D_j u\, dx + \int_Y fu\, dx + \lambda \int_Y u^2\, dx \right\} \tag{0.1}$$

(where Y is the unit cube of R^n, $f \in L^1$, a_{ij} are measurable Y-periodic functions and $a_{ij}(x) z_i z_j \geqslant 0$, $\lambda > 0$) converge as $h \to \infty$ to the value

$$\operatorname*{Inf}_{u \in C^1} \left\{ q_{ij} \int_Y D_i u\, D_j u\, dx + \int_Y fu\, dx + \lambda \int_Y u^2\, dx \right\} \tag{0.2}$$

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(1) S. POISSON, *Second mémoire sur la thèorie du magnetisme*, Mém. de l'Acad. de France, 5 (1822).

(2) JACKSON, *Classical Electrodynamics*, ch. 4, New York, 1962.

where q_{ij} (independent from f) are suitable non negative constants? And how is it possible to calculate the q_{ij}? That is to say, do the energies associated to the microscopic problems converge towards the energy of a « similar » macroscopic problem?

A problem of the following kind is still classical [3]: does the sequence

$$\text{(0.3)} \qquad \operatorname*{Inf}_{u\in C^1}\left\{\int_Y a_{ij,h}(x)\,D_iuD_ju\,dx+\int_Y fu\,dx+\lambda\int_Y u^2\,dx\right\}$$

(where $a_{ij,h}$ tends to infinity or to zero as $h\to\infty$ around a surface $\Gamma\subset Y$) converge as $h\to\infty$ to the value

$$\text{(0.4)} \qquad \operatorname*{Inf}_{u\in C^1}\left\{\int_Y q_{ij}(x)\,D_iu\,D_ju\,dx+\int_\Gamma r_{ij}(x)\,D_iuD_ju\,d\mu+\int_Y fu\,dx+\lambda\int_Y u^2\,dx\right\}$$

where μ is a suitable measure on Γ?

The analysis of the L^1-semicontinuity problems for functionals of the type

$$\int_\Omega a_{ij}(x)\,D_iu\,D_ju\,dx$$

with $a_{ij}(x)$ not uniformly elliptic can be framed in the preceeding context.

A first method to study this kind of problems has been introduced by SPAGNOLO [51], [52], SANCHEZ-PALENCIA [41], [42], [43], BABUŠKA [2], [3], DE GIORGI-SPAGNOLO [20], TARTAR [56], [57], BENSOUSSAN-LIONS-PAPANICOLAU [6], [7].

It consists in studying the convergence in L^1 of the solutions of Euler's equations associated to (0.3), (0.4). A new method has been introduced by DE GIORGI in 1974 in [16]. This work has been the starting point for a wide theory. The central idea consists in introducing a convergence notion, the Γ-convergence, for the functionals associated to (0.3), (0.4) which generally implies the convergence of infima.

The preceeding can be divided in a series of intermediate problems.

Given a Caratheodory function $f(x,z)\geqslant 0$, set for A open set in R^n

$$\text{(0.5)} \qquad F(A,u)=\begin{cases}\displaystyle\int_A f(x,Du)\,dx & u\in C^1(R^n)\\ +\infty & u\in L^1_{\text{loc}}-C^1(R^n)\,.\end{cases}$$

One has to find:

a) the minimal assumptions under which from any sequence of functionals (0.5) it is possible to find a subsequence Γ-converging (Theo. 1.11);

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(3) JACKSON, ch. 1.

b) the natural assumptions under which the Γ-limit of a sequence of functionals (0.5) is a measure with respect to A;

c) the natural assumptions under which the Γ-limit of a sequence of functionals (0.5) can be represented as

$$\int_A f(x, Du)\, dx\,;$$

d) an explicit calculation method of Γ-limits of particular sequences of functionals (0.5);

e) under which conditions the Γ-convergence implies the convergence of the infima of associated variational problems (Theo. 1.12).

In this work we examine essentially points *b*), *c*), *d*).

In § 1 we list the notations, definitions and preliminary results to be used in the paper.

In § 2 we prove the coincidence between two kinds of limits, the $\Gamma(C^0)$ and $\Gamma(C^0_0)$: result which will be utilized in order to obtain the behaviour of

$$F(\Omega, u) = \Gamma(N, C^0(\Omega)^-) \lim_{\substack{h\to\infty \\ v\to u}} \int_\Omega f_h(x, Dv)\, dx \quad (0.6)$$

[4]

as a function of Ω.

In § 3 we continue to study this dependence and prove that assuming

$$\begin{cases} 0 \leqslant f_h(x, z) \leqslant a_h(x)(1+|z|^p)\,, \quad p \geqslant 1,\ a_h \in L^1_{\text{loc}},\ f_h(x,\cdot) \text{ convex} \\ \int_\Omega a_h\, dx \to \int_\Omega a\, dx\,, \quad \forall\Omega \text{ bounded open set such that } \int_{\partial\Omega} a\, dx = 0 \end{cases} \quad (0.7)$$

then, for such Ω and $\forall u \in C^1$ the functional (0.6) has the form

$$F(\Omega, u) = \int_\Omega f(x, Du)\, dx\,. \quad (0.8)$$

Furthermore we prove that if the last of (0.7) is satisfied for any Borel set Ω then (0.8) holds for every bounded open set and it is also equal to

$$\Gamma(N, L^1(\Omega)^-) \lim_{\substack{h\to\infty \\ v\to u}} \int_\Omega f_h(x, Dv)\, dx\,.$$

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(4) See theo. 1.10.

This result had been obtained first by DE GIORGI [16] in the case

$$\begin{cases} |z| \leq f_h(x, z) \leq a(1 + |z|) , & a \in R^+ \\ |f_h(x, z) - f_h(x, z')| \leq a(|z - z'|) \end{cases} \tag{0.9}$$

and then in [48] in the hypothesis that $f_h^{1/p}$ satisfy (0.9).

Afterwards, several papers came out concerning the problem of generalizing the results of [16], [48] by weakening the coercivity or the boundedness assumptions.

In the case $p = 2$, $f_h(x, z) = a_{ij,h}(x) z_i z_j$ some results have been obtained in [29], [30] and, for $p = 1$ in [11].

In § 4 we study some L^1-lower semicontinuity properties of an integral as

$$\int_\Omega f(x, Du)\, dx \tag{0.10}$$

where

$$\begin{cases} 0 \leq f(x, z) \leq a(x)(1 + |z|^p) \\ f(x, \cdot) \quad \text{convex} \end{cases} \tag{0.11}$$

which may fail to be L^1 lower semicontinuous, but we show that we can associate an integral to it

$$\int_\Omega f_0(x, Du)\, dx$$

which is the maximum L^1-lower semicontinuous functional on Lip(Ω) not greater then (0.10). We give a formula for f_0.

In § 5 we consider stability properties of the Γ-convergence under multiplication of the integrands for a continuous function or addition of obstacles perturbations.

In § 6 we obtain some results of Γ-convergence for functionals depending also on u

$$\int_\Omega f(x, u, Du)\, dx .$$

In § 7 we give the homogenization theorem, which generalize those of [2], [28], [56] because no regularity assumptions of $f(x, z)$ are made.

At the end, in § 8 we indicate some counterexample to the preceeding theorems.

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1. – Definitions, notations and preliminary results.

For any Ω open set in R^n let us indicate with:

- $C^m(\Omega)$ the set of the functions with continuous derivatives on Ω up to the order $m \in N$.
- $C_0^m(\Omega)$ the set of $u \in C^m(\Omega)$ with compact support in Ω.
- $\mathcal{M}(\Omega)$ the space of measurable functions on Ω.
- $L^p_{\text{loc}}(\Omega)$ the space of $u \in \mathcal{M}(\Omega)$ such that $|u|^p$ is locally integrable on $\Omega (p \geq 1)$.
- $L^p(\Omega)$ the space of functions $u \in \mathcal{M}(\Omega)$ such that $|u|^p$ is integrable on Ω.
- $H^{1,p}(\Omega)$ the Sobolev space of functions $u \in L^p(\Omega)$ such that $Du \in [L^p(\Omega)]^n$.
- $\text{Lip}_{\text{loc}}(\Omega)$ the space of locally liptschitzian functions on Ω.
- $BV_{\text{loc}}(\Omega)$ the space of L^1_{loc} functions whose distributional derivatives are measures [0].

For any $u \in BV_{\text{loc}}$ we indicate with $|Du|$ the total variation measure of $Du = (\partial u/\partial x_1, \dots, \partial u/\partial x_n)$ and for any Borel set $B \subseteq \Omega$ we will indicate with

$$\int_B |Du|$$

the value of $|Du|$ on B.

We recall that for any open set $A \subseteq \Omega$

$$\int_A |Du| = \sup \left\{ \sum_{i=1}^n \int_A u D_i g_i \, dx : g_i \subset C_0^\infty(A), \ \sum_{i=1}^n g_1^2 \leq 1 \right\}$$

and denote with $BV(\Omega)$ the space of BV_{loc} functions such that:

$$\|u\|_{BV(\Omega)} = \int_\Omega |u| \, dx + \int_\Omega |Du| < \infty .$$

It is well known [0] that $BV(\Omega)$ is not a normal space of distributions, but it can be proved the following useful

THEOREM 1.1. – *Let $u \in BV(\Omega)$, then there exists $(u_h) \subset C^\infty(\Omega)$ such that $u_h \to u$ in $L^1(\Omega)$ and $\int_\Omega |Du_h| \, dx \to \int_\Omega |Du|$.*

It is also well known that $BV(\Omega) \subset L^1(\Omega)$ with compact injection.

In the following we indicate with Ap_n the family of all bounded open sets of R^n and introduce $\forall \Omega \in Ap_n$ several metrics or extended metrics on $\text{Lip}_{\text{loc}} = \text{Lip}_{\text{loc}}(R^n)$.

With an abuse of notation we will indicate with $C_0^0(\Omega)$ the extended metric

$$(u, v) \in \mathrm{Lip}^2_{\mathrm{loc}} \to \begin{cases} \sup\limits_{x\in\Omega} |u(x) - v(x)| & \text{if } spt(u) \subset\subset \Omega \\ +\infty & \text{otherwise} \end{cases}$$

with $C^0(\Omega)$ the metric

$$(u, v) \in \mathrm{Lip}^2_{\mathrm{loc}} \to \sup_{x\in\Omega} |u(x) - v(x)| ,$$

with $L^p(\Omega)$ the metric

$$(u, v) \in \mathrm{Lip}^2_{\mathrm{loc}} \to \Big(\int_\Omega |u - v|^p \, dx\Big)^{1/p} .$$

We recall some notions of measure theory, for which see [15], [19].

For any $U, V \in Ap_n$ we use the notation $U \subset\subset V$ to mean that there exists $K \in \mathcal{K}_n$ ($\mathcal{K}_n$ is the family of all compact sets in R^n) such that $U \subseteq K \subseteq V$.

DEFINITION 1.2. – *A set $\mathcal{U} \subseteq Ap_n$ is called «rich» if for any family $(A_t)_{t\in]0,1[}$ with $A_t \in Ap_n$ satisfying*

$$t < s \Rightarrow A_t \subset\subset A_s ,$$

the set $\{t, A_t \notin \mathcal{U}\}$ is at most countable; it is said «dense» if $\forall \Omega_1, \Omega_2 \in Ap_n$ with $\Omega_1 \subset\subset \Omega_2$ there exists $A \in \mathcal{U}$ such that $\Omega_1 \subset\subset A \subset\subset \Omega_2$.

If α: $Ap_n \to [0, \infty]$ is increasing and $\alpha(\emptyset) = 0$, we set $\forall U \in Ap_n$

$$\alpha_-(U) = \sup_{V\subset\subset U} \alpha(V) \qquad \alpha_+(U) = \inf_{U\subset\subset V} \alpha(V) .$$

DEFINITION 1.3. – *We say that α is regular iff*

$$\alpha_-(U) = \alpha(U) = \alpha_+(U) , \qquad \forall U \in Ap_n .$$

DEFINITION 1.4. – *We say that α is subadditive (respectively additive) on Ap_n iff for $U, U', V, W \in Ap_n$:*

$$U \subseteq V \cup W \Rightarrow \alpha(U) \leqslant \alpha(V) + \alpha(W)$$

(resp. $V \cap W = 0$, $U \subseteq V \cup W \subseteq U' \Rightarrow \alpha(U) \leqslant \alpha(V) + \alpha(W) \leqslant \alpha(U')$).

DEFINITION 1.5. – *We say that α is a measure on Ap_n if $\alpha = \alpha_-$, and it can be extended to a measure on the σ-algebra generated by Ap_n.*

Let α_h be a sequence of increasing non negative set functions on Ap_n such that $\alpha_h(\emptyset) = 0$:

DEFINITION 1.6. – *We say that α_h converges weakly to α iff*

$$(\liminf_h \alpha_h)_- = \alpha = (\limsup_h \alpha_h)_-$$

or, equivalently, iff there exists $\mathcal{U} \subseteq Ap_n$, $\mathcal{U}$ rich, such that

$$\liminf_h \alpha_h(A) = \alpha(A) = \limsup_h \alpha_h(A) \qquad \forall A \in \mathcal{U}.$$

We will use the following

PROPOSITION 1.7. – *The sequence α_h of finite mesures on Ap_n converges weakly to α iff*

$$\alpha(A) = \lim_h \alpha_h(A) \qquad \forall A \in Ap_n \text{ such that } \mu(\partial A) = 0\,,$$

(where μ is the measure induced by α on the σ-algebra generated by Ap_n), or equivalently iff

$$\int_{R^n} f\, d\alpha_h \to \int_{R^n} f\, d\alpha\,, \qquad \forall f \in C_0^0(R^n)\,.$$

We shall consider another kind of convergence for sequences of Radon measures on the bounded Borel sets $\mathcal{B}_n$ of R^n of the form

$$\alpha_h(B) = \int_B a_h\, dx\,, \qquad a_h \in L^1_{\text{loc}}\,.$$

DEFINITION 1.8. – *We say that the sequence α_h is pointwise converging to $\alpha(B) = \int_B a\, dx$ iff*

$$\forall B \in \mathcal{B}_n\,, \qquad \int_B a_h\, dx \to \int_B a\, dx\,. \tag{1.1}$$

REMARK 1.9. – The Vitali-Hahn-Saks theorem assures the equiboundedness and equiabsolutely continuity of $\int_B a_h\, dx$ if (1.1) is satisfied.

Let (V, τ) be a topological space and F_h: $V \to \bar{R} = R \cup \{-\infty, +\infty\}$ a sequence of functions. Set $\forall u \in V$ (see [10], [17], [18])

$$\Gamma(N^-, \tau^-) \lim_{\substack{h\to\infty \\ v\to u}} F_h(v) = \operatorname{Sup}_{U\in\tau(u)} \liminf_h \operatorname{Inf}_U F_h$$

$$\Gamma(N^+, \tau^-) \lim_{\substack{h\to\infty \\ v\to u}} F_h(v) = \operatorname{Sup}_{U\in\tau(u)} \limsup_h \operatorname{Inf}_U F_h$$

$$\Gamma(N^\pm, \tau^+) \lim_{\substack{h\to\infty \\ u\to v}} F_h(v) = -\Gamma(N^\mp, \tau^-) \lim_{\substack{h\to\infty \\ a\to u}} (-F_h(v))$$

where $\tau(u)$ is the set of open neighborhoods of u.

DEFINITION 1.9. – *The function* $F\colon V \to \bar{R}'$ *is the* $\Gamma(N, \tau^-)$ *limit of* F_h *on* V:

$$F(u) = \Gamma(N, \tau^-) \lim_{\substack{h\to\infty \\ v\to u}} F_h(v) \qquad \forall u \in V$$

iff

$$F(u) = \Gamma(N^-, \tau^-) \lim_{\substack{h\to\infty \\ v\to u}} F_h(v) = \Gamma(N^+, \tau^-) \lim_{\substack{h\to\infty \\ v\to u}} F_h(v) \qquad \forall u \in V. \tag{1.2}$$

The function F *is the* $\Gamma(N, \tau^+)$ *limit of* F_n *iff* (1.2) *holds replacing* $\Gamma(\cdot, \tau^-)$ *by* $\Gamma(\cdot, \tau^+)$.

It is useful in the sequel the following characterization of the Γ limits, true in a space satisfying the 1*rst* countable axiom.

THEOREM 1.10. – *If* (V, τ) *satisfy the first countable axiom, then* (1.2) *is equivalent to the two conditions*

$$u_h \xrightarrow{\tau} u \Rightarrow F(u) \leqslant \liminf_h F_h(u_h) \tag{1.3}$$

$$\forall u \in V \quad \exists u_h \xrightarrow{\tau} u \text{ such that } F(u) = \lim_h F_h(u_h)\,. \tag{1.4}$$

PROOF (see [18]).

A compactness result with respect to the Γ-convergence, which in a different context was given by KURATOWSKY [5] is the following

THEOREM 1.11. – *Let* (V, τ) *be a space with a countable base. Then, for any sequence* $F_h\colon V \to \bar{R}$ *there exists a subsequence which has* $\Gamma(N, \tau^-)$ *limit on* V.

PROOF (see [18]).

We want to recall another result which indicates the connection between Γ-convergence and the convergence of minima of F_h which is due to DE GIORGI-FRANZONI [18].

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[5] KURATOWSKY, *Topology*, Warsaw.

THEOREM 1.12. – *Assume that there exists a compact $K \subset V$ such that $\forall h \in N$*

$$\inf_K F_h = \inf_V F_h .$$

Then if (1.2) *holds we have*

$$\operatorname{Min}_K F = \operatorname{Min}_V F = \lim_h \inf_V F_h .$$

Furthermore if $u_h \in K$ satisfy $F_h(u_h) \to \inf F$ then there exists a subsequence u_{h_r} converging to a minimum point of F.

REMARK 1.13. – If $F_h = G \, \forall h \in N$, then

$$\Gamma(N, \tau^-) \lim_{\substack{h \to \infty \\ v \to u}} F_h(v) = sc^-(\tau)\, G(u)$$

is the greatest lower τ-semicontinuous function on V less than G.

2. – Some technical lemmas.

Let $p \geqslant 1$, $a_h \in L^1_{\text{loc}}(R^n)$ and $f_h(x, z)$ verify for any $h \in N$

$$\left\{ \begin{array}{ll} 0 \leqslant f_h(x, z) \leqslant a_h(x)(1 + |z|^p) & \forall (x, z) \in R^{2n} \\ f_h(x, \cdot) \quad \text{convex } \forall x \in R^n & \end{array} \right. \tag{2.1}$$

and assume that there exists a regular measure μ on Ap_n such that

$$\int a_h(x)\, dx \to \int d\mu \quad \text{weakly (as } h \to \infty) . \tag{2.2}$$

Set, for any $\Omega \in Ap_n$, $u \in \text{Lip}_{\text{loc}}$

$$F_h(\Omega, u) = \int_\Omega f_h(x, Du)\, dx . \tag{2.3}$$

To simplify notations, often we shall not write dx in (2.3).

We want to prove the following

THEOREM 2.1. – *For any $\Omega \in Ap_n$ such that $\int_{\partial\Omega} d\mu = 0$, if in $u \in \text{Lip}_{\text{loc}}$ there exists*

$$F(\Omega, u) = \Gamma(N, C^0(\Omega)^-) \lim_{\substack{h \to \infty \\ v \to u}} F_h(\Omega, v) ,$$

then there exists also

$$F_0(\Omega, u) = \Gamma(N, C_0^0(\Omega)^-) \lim_{\substack{h\to\infty \\ v\to u}} F_h(\Omega, v) ,$$

and we have

$$F_0(\Omega, u) = F(\Omega, u) .$$

We begin with the following

LEMMA 2.2. – *Let $a_h \in L^1_{\text{loc}}(R^n)$ verify* (2.2), *and $\Omega \in Ap_n$ be such that*

$$\int_{\partial\Omega} d\mu = 0 ,$$

and set $\forall \varepsilon > 0$

$$B_\varepsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\} .$$

Then there exists a countable set C such that $\forall \varepsilon \in R^+ - C$

$$\int_{\partial B_\varepsilon} d\mu = 0 .$$

Furthermore if $\varepsilon_n \in R^+ - C$ verify $\varepsilon_n > \varepsilon_{n+1}$, $\varepsilon_n \downarrow 0$, $\bigcup_n \bar{B}_{\varepsilon_n} = \bar{\Omega}$, then:

$$\lim_h \int_{\Omega - \bar{B}_{\varepsilon_n}} a_h(x)\, dx = \int_{\Omega - \bar{B}_{\varepsilon_n}} d\mu \qquad \forall n \in N$$

$$\lim_n \int_{\Omega - \bar{B}_{\varepsilon_n}} d\mu = 0 .$$

PROOF (of Lemma 2.2). – By the continuity of $x \to \operatorname{dist}(x, \partial\Omega)$ we deduce that B_ε is an open set and $\varepsilon_1 < \varepsilon_2 \Rightarrow B_{\varepsilon_1} \subset B_{\varepsilon_2}$; we have also

$$\bar{B}_\varepsilon = \{x \in \bar{\Omega} : \operatorname{dist}(x, \partial\Omega) \geq \varepsilon\} ,$$

$$\partial B_\varepsilon = \{x \in \bar{\Omega} : \operatorname{dist}(x, \partial\Omega) = \varepsilon\} .$$

For any $n \in N$ let us indicate by I_n the finite set of $\varepsilon > 0$ such that $\int_{\partial B_\varepsilon} d\mu > 1/n$ then one has

$$\int_{\partial B_\varepsilon} d\mu = 0 \qquad \forall \varepsilon \in]0, 1[- \bigcup_{n\in N} I_n .$$

And so there exists $(\varepsilon_n)_{n\in N}$ strictly decreasing to zero such that $\varepsilon_n \in]0,1[- \bigcup_{n\in N} I_n$ and

$$\bigcup_{n\in N} \bar{B}_{\varepsilon_n} = \bar{\Omega} .$$

Let us prove that $\forall \varepsilon > 0$

$$\overline{\Omega - \bar{B}_\varepsilon} \subseteq (\bar{\Omega} - \bar{B}_\varepsilon) \cup \partial B_\varepsilon. \tag{2.4}$$

In fact

$$\overline{\Omega - \bar{B}_\varepsilon} \subseteq \bar{\Omega} \cap \overline{\complement(\bar{B}_\varepsilon)} = \bar{\Omega} \cap \left(\complement(\bar{B}_\varepsilon) \cup \partial\complement(\bar{B}_\varepsilon)\right) = \bar{\Omega} \cap \left(\complement(\bar{B}_\varepsilon) \cup \partial\bar{B}_\varepsilon\right)$$
$$= \bar{\Omega} \cap \left(\complement(\bar{B}_\varepsilon) \cup \partial B_\varepsilon\right) = (\bar{\Omega} - \bar{B}_\varepsilon) \cup \partial B_\varepsilon .$$

Observe now that by (2.4) it follows

$$\mu(\Omega - \bar{B}_{\varepsilon_n}) \leqslant \mu\left(\overline{\Omega - \bar{B}_{\varepsilon_n}}\right) \leqslant \mu([\bar{\Omega} - \bar{B}_{\varepsilon_n}] \cup \partial B_{\varepsilon_n})$$
$$= \mu(\bar{\Omega}) - \mu(\bar{B}_{\varepsilon_n}) + \mu(\partial \bar{B}_{\varepsilon_n}) = \mu(\Omega) - \mu(\bar{B}_{\varepsilon_n}) = \mu(\Omega - \bar{B}_{\varepsilon_n})$$

and so

$$\lim_h \int_{\Omega - \bar{B}_{\varepsilon_n}} a_h(x)\, dx = \int_{\Omega - \bar{B}_{\varepsilon_n}} d\mu .$$

The last limiting relation follows by the regularity of μ.

We prove also the following technical lemma which will be used in the proof of Theorem 2.1.

LEMMA 2.3. – *Let $B_1 \in Ap_n$ and $B \subset\subset B_1$ and ψ such that*

$$\begin{cases} \psi \in C_0^1(B_1)\,, & 0 \leqslant \psi \leqslant 1 \\ \psi(x) = 1 & \forall x \in B \end{cases}$$

If $f = f(x,z) \geqslant 0$ is measurable in $x \in R^n$, convex in $z \in R^n$, set $\forall u, v \in C^1$; $t \in (0,1)$:

$$w^t = (1-t)[\psi v + (1-\psi)u] .$$

Then

$$\int_{B_1} f(x, Dw^t) \leqslant (1-t)\int_{B_1} f(x, Dv) + (1-t)\int_{B_1 - B} f(x, Du)$$
$$+ t\int_{B_1 - B} f\left(x, \frac{1-t}{t} D\psi(v-u)\right) + t\int_B f(x, 0) .$$

PROOF. – We have:

$$Dw^t = (1-t)[\psi\, Dv + (1-\psi)\, Du] + (1-t)\, D\psi(v-u)$$

and also:

$$f(x, Dw^t) \leqslant (1-t) f(x, \psi Dv + (1-\psi) Du) + t f\left(x, \frac{1-t}{t} D\psi(v-u)\right)$$
$$\leqslant (1-t)[f(x, Dv) + f(x, Du)] + t f\left(x, \frac{1-t}{t} D\psi(v-u)\right)$$

$$f(x, (1-t) Dv) \leqslant (1-t) f(x, Dv) + t f(x, 0) .$$

Then

$$\int_{B_1} f(x, Dw^t) = \int_{B_1 - B} f(x, Dw^t) + \int_B f(x, Dw^t) \leqslant (1-t) \int_{B_1-B} f(x, Dv) + (1-t) \int_{B_1-B} f(x, Dv)$$
$$+ t \int_{B_1-B} f\left(x, \frac{1-t}{t} D\psi(v-u)\right) + (1-t) \int_B f(x, Dv) + t \int_B f(x, 0) .$$

PROOF (of Theo. 2.1). – Clearly it suffices to prove that

$$F''_0(\Omega, u) = \Gamma(N^+, C^0_0(\Omega)^-) \lim_{\substack{h\to\infty \\ v \to u}} F_h(\Omega, v) \leqslant F(\Omega, u) . \tag{2.5}$$

To this aim let $u_h \xrightarrow{C^0(\Omega)} u$ such that $F(\Omega, u) = \lim_h F_h(\Omega, u_h)$ and let $B, B_1, B_2 \in Ap_n$ such that

$$B \subset\subset B_1 \subset\subset B_2 \subset\subset \Omega .$$

Denoting by ψ a function in the class $C^1_0(\Omega)$ such that

$$0 \leqslant \psi(x) \leqslant 1$$
$$\psi(x) = \begin{cases} 1 & \forall x \in B \\ 0 & \forall x \in \Omega - B_1 , \end{cases} \tag{2.6}$$

for any $t > 0$ let $\varphi^t \in C^1(\Omega)$ be such that

$$\varphi^t(x) = \begin{cases} 1 & \forall x \in B_1 \\ \dfrac{1}{1-t} & \forall x \in \Omega - B_2 \end{cases} \tag{2.7}$$

and set

$$\tilde{w}_h^t = \varphi^t(1-t)\big(u + \psi(u_h - u)\big)\,. \tag{2.8}$$

It is easy to check that $\forall t > 0$

$$\tilde{w}_h^t \xrightarrow[(h\to\infty)]{} \varphi^t u(1-t) \quad \text{in } C_0^0(\Omega) \tag{2.9}$$

and also

$$\varphi^t u(1-t) \xrightarrow[(t\to 0)]{} u \quad \text{in } C_0^0(\Omega)\,. \tag{2.10}$$

For any $\varepsilon > 0$ let $t(\varepsilon) < \frac{1}{2}$ be such that $\forall t \in]0, t(\varepsilon)[$

$$F_0''(\Omega, u) < F_0''\big(\Omega, \varphi^t u(1-t)\big) + \varepsilon\,; \tag{2.11}$$

(this follows by (2.10) and the lower semicontinuity of F_0'') then, using (1.3) we have by (2.5), (2.11) $\forall t \in]0, t(\varepsilon)[$

$$\begin{aligned} F_0''(\Omega, u) &\leqslant \limsup_h F_h(\Omega, \tilde{w}_h^t) + \varepsilon \leqslant \\ &\leqslant F(\Omega, u) + \limsup_h [F_h(\Omega, \tilde{w}_h^t) - F_h(\Omega, u_h)] + \varepsilon\,. \end{aligned} \tag{2.12}$$

We have

$$\begin{aligned} F_h(\Omega, \tilde{w}_h^t) - F_h(\Omega, u_h) &= \int_{\Omega - B_1} [f_h(x, D\tilde{w}_h^t) - f_h(x, Du_h)] + \\ &+ \int_{B_1} [f_h(x, Dw_h^t) - f_h(x, Du_h)] = a_h + b_h\,. \end{aligned} \tag{2.13}$$

By lemma 2.3 we deduce (with $f = f_h$, $v = u_h$, $w^t = \tilde{w}^t_{h|B_1}$)

$$b_h \leqslant -t\int_{B_1} f_h(x, Du_h) + (1-t)\int_{B_1 - B} f_h(x, Du) + t\int_{B_1 - B} f_h\left(x, \frac{1-t}{t}\, D\psi(u_h - u)\right) + t\int_B f_h(x, 0)$$

and also, by (2.1)

$$\begin{aligned} b_h \leqslant -t\int_{B_1} f_h(x, Du_h) &+ (1-t)\int_{B_1 - B} a_h(x)\,dx \sup_\Omega\, (1 + |Du|^p) + \\ &+ t\int_{B_1 - B} a_h(x)\,dx \sup_\Omega\, (1 + |D\psi(u_h - u)\,t^{-1}(1-t)|^p) + t\int_B a_h(x)\,dx\,. \end{aligned} \tag{2.14}$$

Furthermore

$$(2.15)\qquad a_h \leqslant \int_{\Omega - B_1} f_h(x, D\tilde{w}_h^t) \leqslant$$

$$\leqslant \int_{\Omega - B_1} a_h(x)\,dx \sup_{\Omega}\left[1 + 2^{p-1}(1-t)^p\left\{|\varphi^t Du|^p + \frac{t^p|u|^p}{\operatorname{dist}(\partial B_1, \partial B_2)^p}\right\}\right]$$

By (2.13), (2.14), (2.15) $\forall t \in]0, t(\varepsilon)[$ as $u_h \xrightarrow{C^0(\Omega)} u$

$$\lim_h{}'' \int_{\Omega} [f_h(x, D\tilde{w}_h^t) - f_h(x, Du_h)] \leqslant -t \lim_h{}'' \int_{B_1} f_h(x, Du_h)$$

$$(2.16)\qquad + (1-t)\sup_{\Omega}(1 + |Du|^p) \lim_h{}'' \int_{B_1 - B} a_h(x)\,dx + t \lim_h{}'' \int_{B_1} a_h(x)\,dx$$

$$+ \sup_{\Omega}\left[1 + 2^{p-1}(1-t)^p\left\{2^p|Du|^p + \frac{t^p|u|^p}{\operatorname{dist}(\partial B_1, \partial B_2)^p}\right\}\right] \lim_h{}'' \int_{\Omega - B_1} a_h(x)\,dx$$

By (2.12), (2.16) we deduce, passing to the limit as $t \to 0$,

$$(2.17)\qquad F_0''(\Omega, u) = F(\Omega, u) + \varepsilon + \lim_h{}'' \int_{B_1 - B} a_h(x)\,dx \sup_{\Omega}(1 + |Du|^p)$$

$$+ \sup_{\Omega}[1 + 2^{2p-1}|Du|^p] \lim_h{}'' \int_{\Omega - B_1} a_h(x)\,dx$$

$$\leqslant F(\Omega, u) + \varepsilon + 2\sup_{\Omega}[1 + 2^{2p-1}|Du|^p] \lim_h{}'' \int_{\Omega - B} a_h\,dx .$$

But, if as B we choose B_{ε_n} of lemma (2.2) we have $\forall n$

$$\lim_h \int_{\Omega - B_{\varepsilon_n}} a_h\,dx = \int_{\Omega - B_{\varepsilon_n}} d\mu$$

so that $\forall n$ by (2.17)

$$F_0''(\Omega, u) \leqslant F(\Omega, u) + \varepsilon + 2\sup_{\Omega}[1 + 2^{2p-1}|Du|^p] \int_{\Omega - B_{\varepsilon_n}} d\mu$$

and we deduce (2.5), passing to the limit as $n \to \infty$.

Let us now prove the following first consequence of Theo. 2.1

PROPOSITION 2.4. – *Let F_h be defined by* (2.1), (2.2), (2.3). *Then there exist a sequence (h_r) and F: $Ap_n \times \mathrm{Lip}_{\mathrm{loc}} \to [0, \infty]$ such that $\forall u$*

(2.18) *$F(\cdot, u)$ is a measure on Ap_n and is rich the set*

$$R(u) = \left\{U \in Ap_n : F(U, u) = \Gamma(N, C^0(U)^-) \lim_{\substack{r\to\infty \\ v\to u}} F_{h_r}(U, u)\right\}$$

PROOF. – First let us observe that there exists a countable family $\mathfrak{D}$ of open plurintervals which is dense in Ap_n and such that $\forall P \in \mathfrak{D} \int_{\partial P} d\mu = 0$.

In fact $\forall x_0 \in Q^n$ let $(P_\varepsilon(x_0))_{\varepsilon\in]0,1[^n}$ be the family of all open intervals with center in x_0; then with an argument similar to that of Lemma 2.2 it is easy to check that there exists a countable dense set $(\varepsilon_k)_k$ in $]0, 1[^n$ such that $\mu(\partial P_{\varepsilon_k}(x_0)) = 0$, $\forall k \in N$ furthermore we define $\mathfrak{D}$ the set of all finite unions of the elements of these countable families, when x_0 describes Q^n.

Using an abstract compactness result due to KURATOWSKI (Theorem 1.11) we can find a sequence $(h_p)\uparrow\infty$ such that $\forall B \in \mathfrak{D}$ there exists

$$F(B, u) = \Gamma(N, C^0(B)^-) \lim_{\substack{p\to\infty \\ v\to u}} F_{h_p}(B, v) .$$

Besides, by theorem 2.III of [15] it is enough to prove that F is sub-additive on $\mathfrak{D}$, i.e. $\forall C, A, B \in \mathfrak{D}$; $\forall u \in \mathrm{Lip}_{\mathrm{loc}}$

$$C \subseteq A \cup B \Rightarrow F(C, u) \leqslant F(A, u) + F(B, u) . \tag{2.19}$$

To this aim we pose $M = (1 + \sup_{A\cup B} |Du|^p)$ and observe that:

$$(A - B) = (A - \bar{B}) \cup (\partial B \cap A) ,$$
$$(A - \bar{B}) \cap (\partial B \cap A) = \emptyset .$$

We deduce:

$$\mu(A - B) = \mu(A - \bar{B}) ,$$
$$\mu(A - B) = \sup_{\substack{\Omega\in\mathfrak{D} \\ \Omega\subset\subset A-\bar{B}}} \mu(\Omega) .$$

Then there exist, $\forall \varepsilon > 0$, Ω_1^ε and $\Omega_2^\varepsilon \in \mathfrak{D}$ such that:

$$\bar{\Omega}_1^\varepsilon \cap \bar{\Omega}_2^\varepsilon = \emptyset ,$$
$$\Omega_1^\varepsilon \subseteq B , \qquad \Omega_2^\varepsilon \subseteq A - \bar{B} ,$$
$$\mu(A \cup B - (\Omega_1^\varepsilon \cup \Omega_2^\varepsilon)) < \varepsilon/M .$$

Then, as by Theorem 2.1, we have also $\forall B \in \mathfrak{D}$:

$$F(B, u) = \Gamma(N, C_0^0(B)^-) \lim_{\substack{p \to +\infty \\ v \to u}} F_{h_p}(B, v)\,.$$

Now, we have first:

$$\Gamma(N, C_0^0(\Omega_1^\varepsilon \cup \Omega_2^\varepsilon)^-) \lim_{\substack{p \to \infty \\ v \to u}} F_{h_p}(\Omega_1^\varepsilon \cup \Omega_2^\varepsilon, v) =$$
$$= \Gamma(N, C_0^0(\Omega_1^\varepsilon)^-) \lim_{\substack{p \to \infty \\ v \to u}} F_{h_p}(\Omega_1^\varepsilon, v) + \Gamma(N, C_0^0(\Omega_2^\varepsilon)^-) \lim_{\substack{p \to \infty \\ v \to u}} F_{h_p}(\Omega_2^\varepsilon, v)$$

We have also if $E \subseteq D$ and $E, D \in \mathfrak{D}$:

$$F(D, u) \leqslant F(E, u) + M \int_{D - \bar{E}} d\mu\,.$$

In fact, as:

$$\partial(D - \bar{E}) \subseteq \partial D \cup \partial E\,,$$

we have:

$$\mu(\partial(D - \bar{E})) = 0\,,$$

and if $u_p \to u$ in $C_0^0(E)$, $F_{h_p}(E, u_p) \to F(E, u)$, we deduce:

$$F(D, u) \leqslant \lim_{p \to \infty} \left(F_{h_p}(E, u_p) + \int_{D - \bar{E}} a_{h_p}(x)\, dx \right) = F(E, u) + M \int_{D - \bar{E}} d\mu\,.$$

Then by the monotonicity of:

$$B \in \mathfrak{D} \to F(B, u)\,,$$

we have that:

$$F(C, u) \leqslant F\big((\Omega_1^\varepsilon \cup \Omega_2^\varepsilon) \cup ((A \cup B) - (\Omega_1^\varepsilon \cup \Omega_2^\varepsilon))\big) \leqslant$$
$$\leqslant F(\Omega_1^\varepsilon \cup \Omega_2^\varepsilon) + M \int_{A \cup B - \overline{\Omega_1^\varepsilon \cup \Omega_2^\varepsilon}} d\mu \leqslant$$
$$\leqslant F(\Omega_1^\varepsilon) + F(\Omega_2^\varepsilon) + \varepsilon \leqslant F(A) + F(B) + \varepsilon\,. \qquad \square$$

In the following result we precise the set $\mathfrak{R}(u)$ of previous proposition and obtain the final result of this section.

THEOREM 2.5. – *Let* F_h *be defined by* (2.1), (2.2), (2.3); *then there exists a sequence* (h_r) *and* F: $Ap_n \times \mathrm{Lip}_{\mathrm{loc}} \to [0, \infty]$ *such that* $\forall u$ (2.18) *holds and*

$$(2.20) \qquad F(\Omega, u) = \Gamma(N, C^0(\Omega)^-) \lim_{\substack{r \to \infty \\ v \to u}} F_{h_r}(\Omega, v) = \Gamma(N, C_0^0(\Omega)^-) \lim_{\substack{r \to \infty \\ v \to u}} F_{h_r}(\Omega, v)$$

$\forall u \in \mathrm{Lip}_{\mathrm{loc}}$ *and* $\Omega \in Ap_n$ *such that*

$$\int_{\partial\Omega} d\mu = 0 .$$

PROOF. – Using Coroll. 2.I and the subsequent oss. 2.I in [15] we have only to prove that $\forall u \in \mathrm{Lip}_{\mathrm{loc}}$, $\forall \Omega \in Ap_n$ such that

$$\int_{\partial\Omega} d\mu = 0 ,$$

$F(\cdot, u)$ is sub-additive in the family of the open subsets of Ω. And this can be done similarly as done for (2.19).

REMARK. – The preceding theorem can be generalized with the same proof to the case that

$$F_h(\Omega, u) = \int_\Omega f_h(x, u, Du)$$

with

$$\begin{cases} f_h(x, y, z) = f'_h(x, z) + f''_h(x, y) , \\ 0 \leqslant f_h(x, y, z) \leqslant a_h(x)\left(1 + |y|^p + |z|^p\right) , \\ f'_h(x, \cdot) \text{ convex} . \end{cases}$$

Observe that we have never used the convexity of the limit functional $F(\Omega, \cdot)$.

3. – Integral representation theorems.

In this section, with the same assumptions of the previous one and moreover $\mu(\Omega) = \int_\Omega a(x)\, dx$, $a \in L^1_{\mathrm{loc}}$ we want to precise the nature of the Γ^- limit F given by Theorem 2.5. In particular we prove that F has an integral form similar to that of F_h, at least on a dense subset of Ap_n.

At the end, (3.25), we will assume

$$\int_\Omega a_h\, dx \to \int_\Omega a\, dx , \qquad \forall \Omega \in Ap_n$$

and obtain a representation formula for F of the kind

$$F(\Omega, u) = \int_\Omega f(x, Du) , \qquad \forall \Omega \in Ap_n$$

for a suitable convex function $f(x, \cdot)$.

All this will be a consequence of the following technical lemma.

LEMMA 3.1. – *If F is given by Theo. 2.5 and $A \in Ap_n$, then for any $u \in \mathrm{Lip}(A)$ there exists $w_h \in C^1(A)$ such that*

$$w_h \to u\,, \quad Dw_h \to Du\,, \quad \text{a.e. in } A$$

$$\|Dw_h\|_{C^0(A)} \leq \varrho \|Du\|_{C^0(A)}$$

$$F(A, u) = \lim_h F(A, w_h)$$

with ϱ depending only by the dimension of R^n.

PROOF. – Let $u \in \mathrm{Lip}(A)$ and fix $\sigma, t \in (0, 1)$. Then there exists a closed set C_t such that $C_t \subset A$ and

$$\int_{A-C_t} a(x)\,dx < (1-t)^p\,. \tag{3.1}$$

By Withney's lemma [58] there exists $w^t \in C^1(A)$ such that

$$u = w^t \quad \text{on} \quad C_t\,. \tag{3.2}$$

Let $I_h \in \mathfrak{D}$ be a decreasing sequence of subsets of A such that

$$\bigcap_{h \in N} I_h = C_t \tag{3.3}$$

and let $\varphi_h^t \in C^1$ be such that

$$\varphi_h^t(x) = \begin{cases} 1 & \forall x \in C_t \\ 0 & \forall x \in A - I_h \end{cases} \tag{3.4}$$

and set

$$\tilde{\varphi}_h^t(x) = \varphi_h(x)\,\mathrm{dist}\,(C_t, I_h)\,. \tag{3.5}$$

Let $u_h \in \mathrm{Lip}(A)$ be such that (for semplicity we don't indicate the subsequences)

$$u_h \xrightarrow{C^0(A)} u\,, \qquad F(A, u) = \lim_h F_h(A, u_h) \tag{3.6}$$

and set

$$w_h^t = u_h - u + w^t \tag{3.7}$$

so that

$$\left\{\begin{array}{ll} w_h^t = u_h\,, & \text{on} \quad C_t\,, \\ w_h^t \to w^t & \text{in } C^0(A), \text{ as } h \to \infty\,. \end{array}\right. \tag{3.8}$$

Furthermore set

$$\sigma t v_h^t = \sigma t[u_h \tilde{\varphi}_h^t + w_h^t(1 - \tilde{\varphi}_h^t)] \tag{3.9}$$

and observe that, by (3.5), (3.8), (3.9)

$$\sigma t v_h^t \to \sigma t w^t \quad \text{in } C^0(A) \text{ as } h \to \infty\,. \tag{3.10}$$

Let us estimate

$$\int_A f_h(x, t\sigma\, Dv_h^t)\, dx\,.$$

We have

$$Dv_h^t = \left(\tilde{\varphi}_h^t\, Du_h + (1 - \tilde{\varphi}_h^t)\, Dw_h^t\right) + (u_h - w_h^t)\, D\tilde{\varphi}_h^t$$

and so

$$\begin{aligned} \int_A f_h(x, t\sigma\, Dv_h^t) = \int_{A-I_h} f_h(x, \sigma t\, Dw_h^t) + \int_{I_h - C_t} f_h(x, t\{[\sigma\tilde{\varphi}_h^t\, Du_h + \sigma(1 - \tilde{\varphi}_h^t)\, Dw_h^t] + \\ + \sigma(u_h - w_h^t)\, D\tilde{\varphi}_h^t\} + \int_{C_t} f_h(x, \sigma t\, Du_h) = a_h + b_h + c_h\,. \end{aligned} \tag{3.11}$$

Using the convexity of $f_h(x, \cdot)$ we have

$$\begin{aligned} a_h &= \int_{A-I_h} f_h(x, t\sigma Du_h + t\sigma[Dw^t - Du]) \leqslant \\ &\leqslant t \int_{A-I_h} f_h(x, \sigma Du_h) + (1 - t) \int_{A-I_h} f_h\left(x, \frac{t\sigma}{1 - t}\,[Dw^t - Du]\right) \\ &\leqslant \sigma t \int_{A-I_h} f_h(x, Du_h) + t(1 - \sigma) \int_{A-I_h} f_h(x, 0) + (1 - t) \int_{A-I_h} f_h\left(x, \frac{t\sigma}{1 - t}\,[Dw^t - Du]\right), \end{aligned} \tag{3.12}$$

and also

$$c_h \leqslant \sigma t \int_{C_t} f_h(x, Du_h) + (1 - \sigma t) \int_{C_t} f_h(x, 0) \tag{3.13}$$

$$(3.14)\qquad b_h \leq t \int_{I_h - C_t} f_h(x, \sigma\tilde{\varphi}_h^t Du_h + \sigma(1-\tilde{\varphi}_h^t)Dw_h^t) +$$

$$+ (1-t)\int_{I_h - C_t} f_h\left(x, \frac{\sigma t}{1-t}(u_h - w_h^t)D\tilde{\varphi}_h^t\right) \leq$$

$$\leq t \int_{I_h - C_t} \tilde{\varphi}_h^t f_h(x, \sigma Du_h) + t\int_{I_h - C_t}(1-\tilde{\varphi}_h^t) f_h(x, \sigma Dw_h^t) +$$

$$+ (1-t)\int_{I_h - C_t} f_h\left(x, \frac{\sigma t}{1-t}(u_h - w_h^t)D\tilde{\varphi}_h^t\right) \leq$$

$$\leq t \int_{I_h - C_t} \tilde{\varphi}_h^t f_h(x, \sigma Du_h) + t\int_{I_h - C_t}(1-\tilde{\varphi}_h^t) f_h(x, \sigma Du_h + \sigma[Dw^t - Du]) +$$

$$+ (1-t)\int_{I_h - C_t} f_h\left(x, \frac{\sigma t}{1-t}(u_h - w_h^t)D\tilde{\varphi}_h^t\right) = b_h' + b_h'' + b_h''' .$$

We can estimate b_h' and b_h'':

$$(3.15)\qquad b_h'' \leq \sigma t\int_{I_h - C_t}(1-\tilde{\varphi}_h^t) f_h(x, Du_h) + (1-\sigma)t\int_{I_h - C_t} a_h(x)\left\{\sup_A\left(\frac{\sigma}{1-\sigma}|Dw^t - Du|\right)^p + 1\right\}$$

$$(3.16)\qquad b_h' \leq \sigma t\int_{I_h - C_t}\tilde{\varphi}_h^t f_h(x, Du_h) + t(1-\sigma)\int_{I_h - C_t}\tilde{\varphi}_h^t f_h(x, 0)$$

So by (3.11), ..., (3.16) we deduce, using (2.1)

$$(3.17)\qquad \int_A f_h(x, t\sigma Dv_h^t) \leq t\sigma\int_A f_h(x, Du_h) + t\int_{A - C_t} f_h(x, 0) - \sigma t\int_A f_h(x, 0) + \int_{C_t} f_h(x, 0)$$

$$+ (1-t)\int_{A - I_h} a_h\cdot\sup_A\left(\frac{t\sigma}{1-t}|Dw^t - Du|\right)^p + (1-\sigma)t\int_{I_h - C_t} a_h\cdot\sup_A\left\{\left(\frac{\sigma}{1-\sigma}|Dw^t - Dw|\right)^p + 1\right\}$$

$$+ (1-t)\int_{I_h - C_t} a_h\cdot\sup_A\left\{\left(\frac{t\sigma}{1-t}|u_h - w_h^t|\right)^p + 1\right\}.$$

Passing to the limit as $h\to\infty$ in (3.17), by (2.2), (3.6), we have:

$$\lim_h{}'' \int_A f_h(x, t\sigma Dv_h^t) \leq t\sigma F(A, u) + t\int_{A - C_t} a(x)\,dx + \int_{C_t} a(x)\,dx - \sigma t\int_A a(x)\,dx$$

$$+ (1-t)\sup_A\left\{\left(\frac{t\sigma}{1-t}|Dw^t - Du|\right)^p + 1\right\}\int_{A - C_t} a(x)\,dx .$$

And, also by (3.1), (3.10)

$$\Gamma(N, C^0(A)^-) \lim_{\substack{h\to\infty \\ v\to t\sigma w}} F_h(A, v) < t\sigma F(A, u) + (t-\sigma t)\int_A a(x)\,dx + (1-t)\int_{C_t} a(x)\,dx + (1-t)\,t\sigma \sup_A |Dw^t - Du|^p + (1-t)^{p+1}$$

by which as $t\to 1$ and $\sigma\to 1$ we have the result.

THEOREM 3.2. – *Let F_h be defined by* (2.1), (2.3) *and assume that there exists $a \in L^1_{\mathrm{loc}}$ such that*

$$\int_\Omega a_h\,dx \to \int_\Omega a\,dx = \mu(\Omega)\,, \qquad \forall \Omega \in Ap_n \quad \text{such that} \quad \int_{\partial\Omega} a\,dx = 0\,. \tag{3.18}$$

Then, there exists $f = f(x, z)$ such that

$$\begin{cases} 0 \leqslant f(x, z) \leqslant a(x)(1 + |z|^p) \\ f(x, \cdot) \quad \text{convex} \end{cases} \tag{3.19}$$

and the functional F given by Theo. 2.5 *has the form*

$$F(\Omega, u) = \int_\Omega f(x, Du)\,d\mu \tag{3.20}$$

for any $u \in C^1(\mathbf{R}^n)$ and $\Omega \in Ap_n$ such that $\int_{\partial\Omega} a(x)\,dx = 0$.

PROOF. – Let us prove (3.20) for $u \in C^1(R^n)$. We know that $\forall u \in C^1$

(3.21) $F(\cdot, u)$ is a measure on Ap_n absolutely continuous with respect to

$$\mu(\Omega) = \int_\Omega a(x)\,dx\,.$$

And also that

$$u \to F(\Omega, u) \quad \text{is convex}\,. \tag{3.22}$$

$$0 \leqslant F(\Omega, u) \leqslant \int_\Omega a(x)\,dx\Big[\sup_\Omega (1 + |Du|^p)\Big] \qquad \forall\Omega \text{ s.t. } \int_{\partial\Omega} a\,dx = 0\,. \tag{3.23}$$

Fixed $r_0 > 0$ and $\Omega_0 \in Ap_n$ such that $\int_{\partial\Omega_0} a(x)\,dx = 0$, fixed $u \in C^1$, we have $\forall \Omega \subseteq \Omega_0$ such that $\int_{\partial\Omega} a\,dx = 0$ and $\forall v \in C^1$ such that $\|v - u\|_{C^1(\Omega)} \leqslant r_0$

$$0 \leqslant F(\Omega, v) \leqslant M_\Omega \int_\Omega a(x)\,dx$$

with $M_\Omega = 1 + (\sup_\Omega |Du| + r_0)^p$. So, by the convexity condition (3.22), we deduce

$$(3.24)\qquad \|v_1 - u\|_{C^1(\Omega)}, \quad \|v_2 - u\|_{C^1(\Omega)} \leqslant \frac{r_0}{2} \Rightarrow |F(\Omega, v_1) - F(\Omega, v_2)| \leqslant \frac{2\int_\Omega a\,dx \cdot M_\Omega}{r_0} \operatorname*{Max}_{x\in\Omega} |Dv_1(x) - Dv_2(x)|$$

For any $\lambda \in R^n$ there exists, by the Radon-Nykodim theorem, a measurable function $f_\lambda(x)$ such that

$$F(\Omega, \langle \lambda, x\rangle) = \int_\Omega a(x) f_\lambda(x)\,dx\,.$$

Given $x_0 \in R^n$ and $v \in C^1$, let $\lambda = Dv(x_0)$,

$$r_0 = \|v - \langle\lambda, x\rangle\|_{C^1(\Omega_0)}\,,$$

then

$$|F(\Omega, \langle\lambda, x\rangle) - F(\Omega, v)| \leqslant \frac{2\int_\Omega a\,dx \cdot M_\Omega}{r_0} \max_{x\in\Omega} |\lambda - Dv(x)|$$

moreover there exist f_v such that

$$F(\Omega, v) = \int_\Omega a(x) f_v(x)\,dx\,.$$

So that

$$\frac{1}{\int_\Omega a(x)\,dx} \int_\Omega a(x)\,[f_\lambda(x) - f_v(x)]\,dx \leqslant \frac{2M_{\Omega_0}}{r_0} \max_{x\in\Omega} |\lambda - Dv(x)|$$

If we set $f(x, \lambda) = f_\lambda(x)$, we can pass to the density points with respect to $\mu(\Omega) = \int_\Omega a\,dx$, and conclude that for μ — a.e. $x_0 \in R^n$

$$f(x_0, \lambda) = f_v(x_0) = f(x_0, Dv(x_0))$$

so that

$$\int_\Omega f(x, Dv)\,a(x)\,dx = F(\Omega, v)\,.$$

Let us now replace (3.18) with

$$(3.25)\qquad \int_\Omega a_h\,dx \to \int_\Omega a\,dx = \mu(\Omega)\,, \qquad \forall \Omega \in Ap_n$$

and prove that, in this case, the Γ limit with respect to $C^0(\Omega)$ is equal to the Γ limit with respect to the μ-measure convergence.

PROP. 3.3. – *If* $F(\Omega, u)$ *is given by Theo.* 2.5 *and* (3.25) *holds instead of* (2.2) *then* $\forall \Omega \in Ap_n$, $u \in \mathrm{Lip}_{\mathrm{loc}}$

$$F(\Omega, u) = \Gamma(N, C_0^0(\Omega)^-) \lim_{\substack{r\to\infty \\ v\to u}} F_{h_r}(\Omega, v) = \Gamma(N, \mathcal{M}(\Omega)^-) \lim_{\substack{r\to\infty \\ v\to u}} F_{h_r}(\Omega, v)\,.$$

PROOF. – Let $\psi_r \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ be such that $\psi_r \to u$ in μ-measure and

$$F_\mu(\Omega, u) = \Gamma(N, \mathcal{M}(\Omega)^-) \lim_{\substack{r\to\infty \\ v\to u}} F_{h_r}(\Omega, v) = \lim_r F_{h_r}(\Omega, \psi_r)\,,$$

and let $\varrho_r, \sigma_r \to 0$ such that

$$A_r = \{x\colon |\psi_r(x) - u(x)| > \varrho_r\}\,, \qquad \mu(A_r) < \sigma_r\,.$$

For any $\varepsilon_r \to 0$ if β_r is such that

$$\begin{cases} \beta_r \in C^1(R)\,, & 0 \leqslant \beta_r' \leqslant 1 \\ \beta_r(t) = t\,, \quad |t| \leqslant \varrho_r; & \beta_r'(t) = 0\,, \qquad \forall |t| \geqslant \varrho_r + \varepsilon_r \end{cases}$$

set

$$w_r = u + \beta_r(u_r - u)\,.$$

Then

$$\int_\Omega f_{h_r}(x, Dw_r) - \int_\Omega f_{h_r}(x, D\psi_r) = \int_\Omega f_{h_r}(x, (1-\beta_r')\,Du + \beta_r' D\psi_r) - \int_\Omega f_{h_r}(x, D\psi_r) \leqslant$$

$$\leqslant \int_\Omega (1-\beta_r') f_{h_r}(x, Du) + \int_\Omega (\beta_r' - 1) f_{h_r}(x, D\psi_r) \leqslant \int_{A_r} f_{h_r}(x, Du) \to 0\,.$$

And this proves the theorem.

We can now summarize the results of section 2 and the preceeding of this section giving the following

THEOREM 3.4. – *Let* F_h *be defined by* (2.1), (2.3) *and assume* (3.25). *Then there exists* F_{h_r} *and* $f = f(x, z)$ *satisfying* (3.19) *such that* $\forall \Omega \in Ap_n$, $u \in \mathrm{Lip}_{\mathrm{loc}}$

$$\int_\Omega f(x, Du)\,dx = \Gamma(N, C_0^0(\Omega)^-) \lim_{\substack{r\to\infty \\ v\to u}} F_{h_r}(\Omega, v) = \Gamma(N, \mathcal{M}(\Omega)^-) \lim_{\substack{r\to\infty \\ v\to u}} F_{h_r}(\Omega, v)\,.$$

PROOF. – We can prove, as in Theo. 2.5, that (2.20) holds $\forall \Omega \in Ap_n$. Since $F(\Omega, u) \leqslant \int_\Omega a(x)(1 + |Du|^p)\,dx$, $\forall \Omega \in Ap_n$, the result follows by Theo. 3.2, Lemma 3.1 and Prop. 3.3.

4. – Lower semicontinuity properties.

In this section we will consider a single functional

$$F(u) = \int_\Omega f(x, Du) , \qquad u \in \mathrm{Lip}\,(\Omega) \tag{4.1}$$

satisfying the conditions $(a \in L^1(\Omega))$

$$\begin{cases} 0 \leqslant f(x, z) \leqslant a(x)(1 + |z|^p) \\ f(x, \cdot) \quad \text{convex} \end{cases} \tag{4.2}$$

and analize its semicontinuity properties with respect to different topologies on the space $\mathrm{Lip}\,(\Omega)$.

It is well-known (e.g. see [36]) that if one adds to (4.2) the assumption

$$\lim_{|z| \to \infty} f(x, z) = +\infty \tag{4.3}$$

and f is continuous, then the functional (4.1) is lower semicontinuous in the $L^1(\Omega)$ topology on $H^{1,1}(\Omega)$.

But without the assumption (4.3) one can give examples in which (4.1) is l.s.c. in the weak topology of $H^{1,1}(\Omega)$ but not in the strong one of $L^1(\Omega)$ (see the final example).

In the following, using the result of previous sections, we will prove that to any function f verifying (4.2) another function $\bar{f}(x, z)$ convex in z can be associated such that the functional

$$u \in \mathrm{Lip}\,(\Omega) \to \bar{F}(u) = \int_\Omega \bar{f}(x, Du)$$

is the greatest L^1-lower semicontinuous functional less than F on $\mathrm{Lip}\,(\Omega)$.

For any f verifying (4.2) let us define (see SERRIN [50]) $\forall u \in L^1(\Omega)$

$$\bar{F}(u) = \operatorname{Inf}\left\{\liminf_h \int_\Omega f(x, Du_h) : u_h \in C^1(R^n),\ u_h \to u \ \text{in} \ L^1(\Omega)\right\}. \tag{4.4}$$

We want to give a representation of this functional for $u \in \mathrm{Lip}\,(\Omega)$.

REMARK. – If (4.2) is replaced by

$$\begin{cases} |\sqrt[p]{f(x, z)} - \sqrt[p]{f(x, z^1)}| \leqslant a(|z - z^1|) \\ |z|^p \leqslant f(x, z) \leqslant a(1 + |z|^p) \qquad p \geqslant 1 \ \ a \in R \\ f(x, \cdot) \quad \text{convex} \end{cases} \tag{4.4}$$

it is easy to check that if $p > 1$

$$\overline{F}(u) = \begin{cases} \int_\Omega f(x, Du) & \forall u \in H^{1,p}(\Omega) \\ +\infty & \forall u \in L^1(\Omega) - H^{1,p}(\Omega) \end{cases} \tag{4.5}$$

and if $p = 1$ that

$$\overline{F}(u) = \int_\Omega f(x, Du) \,, \qquad \forall u \in H^{1,1}(\Omega) \tag{4.6}$$

but $\overline{F}(u) < \infty$ also out of $H^{1,1}(\Omega)$. Precisely

$$\overline{F}(u) < \infty \qquad \text{iff} \qquad u \in BV(\Omega) \,. \tag{4.7}$$

In order to prove (4.5), (4.6) first observe that $\forall p \geq 1$

$$\overline{F}(u) = \int_\Omega f(x, Du) \,, \qquad \forall u \in C^1(R^n)$$

in fact $\overline{F}$ is the greatest $L^1(\Omega)$ lower semicontinuous functional on C^1 less than $u \to \int_\Omega f(x, Du)$ and, by (4.4), this functional is $L^1(\Omega)$-lower semicontinuous.

Then, using the relation

$$\left|\sqrt[p]{\overline{F}(u)} - \sqrt[p]{\overline{F}(v)}\right| \leq a \|u - v\|_{H^{1,p}(\Omega)}$$

which is proved in [48] and the first inequality in (4.4), we obtain

$$\overline{F}(u) = \int_\Omega f(x, Du) \,, \qquad \forall u \in H^{1,p}(\Omega) \,, \qquad p \geq 1 \,.$$

Furthermore, in the case $p > 1$ if $u \in L^1(\Omega) - H^{1,p}(\Omega)$ let $u_h \in C^1$ be such that $u_h \xrightarrow{L^1(\Omega)} u$ and

$$\overline{F}(u) = \lim_h \int_\Omega f(x, Du_h) \,.$$

If it was $\overline{F}(u) = \alpha < \infty$, then $(\|Du_h\|_{L^p(\Omega)})_h$ would be a bounded sequence and so

$$u_h \to u \qquad \text{in} \qquad H^{1,p}(\Omega) - \text{weak}$$

and also $u \in H^{1,p}(\Omega)$.

The proof of (4.7) is given in [50] theorem 3, p. 145. But clearly if there are no coercivity assumptions there are many difficulties for the identification of $\overline{F}$ also on C^1.

A first result in this direction is the following

PROPOSITION 4.1. – *Let f verify* (4.2), *and F be defined as in* (4.1); *then, setting* $\forall\sigma \in R$, $A \in Ap_n$, $A \subseteq \Omega$, $\lambda \in R^n$

$$\theta_\sigma(A, z) = \operatorname{Inf}\left\{\int_A f(x, Du) + \sigma\int_A |u - \langle z, x\rangle|^p\,dx\right\}, \qquad u \in \langle z, x\rangle + C_0^1(A) \tag{4.8}$$

and $\forall(x_0, z) \in R^{2n}$ *defining the z-convex function $\bar{f}$ by*

$$\bar{f}(x_0, z) = \lim_{\substack{|A|\to 0\\ x_0\in A}} \frac{1}{|A|} \lim_{\sigma\to\infty} \theta_\sigma(A, z) \tag{4.9}$$

we have

$$\overline{F}(u) = \int_\Omega \bar{f}(x, Du)\,, \qquad \forall u \in \operatorname{Lip}(\Omega)\,. \tag{4.10}$$

PROOF. – By Theo. 3.3 we have that there exists $\bar{f}(x, z)$ convex in z such that

$$\overline{F}(u) = \int_\Omega \bar{f}(x, Du)\,, \qquad \forall u \in \operatorname{Lip}(\Omega)\,.$$

Let us prove that $\forall z \in R^n$

$$\overline{F}(\langle z, x\rangle) = \lim_{\sigma\to\infty} \theta_\sigma(\Omega, z)$$

and so we will have (4.9) as a consequence of Lebesgue points theorem.

First observe that one can prove $\forall u \in C^1$ that

$$\overline{F}(u) = \operatorname{Inf}\left\{\liminf_h \int_\Omega f(x, Du_h) : u_h - u \in C_0^1(\Omega),\ u_h \xrightarrow{L^1(\Omega)} u\right\}.$$

Then for any $u_h \in \langle z, x\rangle + C_0^1(\Omega)$ such that $u_h \to \langle z, x\rangle$ in $L^1(\Omega)$ we have

$$\theta_\sigma(\Omega, z) \leqslant \liminf_h \int_\Omega f(x, Du_h)$$

then

$$\lim_{\sigma\to\infty} \theta_\sigma(\Omega, z) \leqslant \liminf_h \int_\Omega f(x, Du_h)$$

and clearly

$$\lim_{\sigma\to\infty} \theta_\sigma(\Omega, z) \leqslant \overline{F}(\langle z, x\rangle) .$$

Moreover for any $\varepsilon > 0$ let $\sigma \in]0, \infty[$ be such that

$$\varepsilon\sigma > 1 + \int_\Omega a(x)(1 + |Dw|^p)\, dx$$

and for any $K \in N$ let $\psi_k \in \langle z, x\rangle + C_0^1(\Omega)$ be such that

$$\int_\Omega f(x, D\psi_k) + \sigma \int_\Omega |\langle z, x\rangle - \psi_k| < \frac{1}{k} + \theta_\sigma(\Omega, z)$$

then it is clear that

$$\int_\Omega |\langle z, x\rangle - \psi_k| < \varepsilon .$$

And so

$$\lim_k{}' \int_\Omega f(x, D\psi_k) \leqslant \lim_{\sigma\to\infty} \theta_\sigma(\Omega, z) .$$

The result can be obtained with a diagonal process.

Let us now consider the case of an integrand $f(x, z)$ verifying only

(4.11) $$0 \leqslant f(x, z) \leqslant a(1 + |z|^p) , \quad p \geqslant 1 , \quad \forall (x, z) \in R^{2n} , \quad a \in R^+,$$

and observe that, without assuming any convexity of $f(x, \cdot)$, the functional F is not weakly lower semicontinuous in $H^{1,p}(\Omega)$. Nevertheless we have the following

THEOREM 4.2. – *Let $f(x, z)$ verify* (4.11), *then there exists $g(x, z)$ convex in z such that* $\forall u \in \mathrm{Lip}\,(\Omega)$

$$u \in \mathrm{Lip}\,(\Omega) \to \overline{F}(u) = \int_\Omega g(x, Du)$$

is the greatest $L^1(\Omega)$ lower semicontinuous functional on Lip (Ω) *less than*

$$u \in \mathrm{Lip}\,(\Omega) \to \int_\Omega f(x, Du) .$$

Furthermore $\overline{F}$ *is the weak–*$H^{1,p}(\Omega)$*–lower semicontinuous envelope of* F *if and only if*

$$g(x, z) = f^{**}(x, z) \ (*).$$

PROOF. – For any topological space (X, τ), and any functional

$$G: X \to R$$

let us denote by

$$sc^-(\tau)G: X \to R$$

the greatest τ-lower semicontinuous functional on X less than G.

We have then, by a result of EKELAND-TEMAM [21], that $\forall u \in H^{1,p}(\Omega)$

$$sc^-(\text{weak–}H^{1,p}(\Omega))\int_\Omega f(x, Du) = \int_\Omega f^{**}(x, Du)\,.$$

Then, clearly

$$sc^-(L^1(\Omega))\int_\Omega f(x, Du) \leqslant \int_\Omega f^{**}(x, Du) \leqslant \int_\Omega f(x, Du)$$

and also, by applying the $sc^-(L^1(\Omega))$ to the three terms and observing that

$$sc^-(L^1(\Omega))\,sc^-(L^1(\Omega))\int_\Omega f(x, Du) = sc^-(L^1(\Omega))\int_\Omega f(x, Du)\,,$$

we deduce

$$sc^-(L^1(\Omega))\int_\Omega f(x, Du) = sc^-(L^1(\Omega))\int_\Omega f^{**}(x, Du)\,.$$

But, as $f^{**}(x, \cdot)$ is convex, we can use Theo. 3.3 to deduce the existence of a convex function $g(x, z)$ such that

$$sc^-(L^1(\Omega))\int_\Omega f^{**}(x, Du) = \int_\Omega g(x, Du)$$

and also the result.

(*) Let us recall that ([21]) $f^*(x, z^*) = \sup\limits_{z \in R^n} (\langle z^*, z\rangle - f(x, z))$ and $f^{**}(x, z) = (f^*)^*(x, z)$.

COROLLARY 4.3. – *If f verifies*

$$|z|^p \leqslant f(x, z) \leqslant a(1 + |z|^p)$$

then

$$\bar{f}(x, z) = f^{**}(x, z)$$

and $\overline{F}(u)$ *is the convex envelope of F on whole* $H^{1,p}(\Omega)$.

EXAMPLE. – Let (x_k) be the sequence of rational numbers of $(0, 1)$ and set

$$S_k = \left]x_k - \frac{1}{2^k}, x_k + \frac{1}{2^k}\right[$$

$$C = \bigcup_{k \in N} S_k$$

Then the functional

$$F(u) = \int_0^1 (1 - \chi_C(x))\, u'(x)^2\, dx\,, \qquad u \in H^{1,2}(0, 1)$$

is not L^2-lower semicontinuous. In fact

$$\text{(4.12)} \qquad sc^-(L^2)F(u) = 0\,, \qquad \forall u \in H^{1,2}(0, 1)\,.$$

PROOF. – First let $u \in H^{1,2}(0, 1)$ be an increasing function. We want to show that there exists a sequence $(u_h) \subset H^{1,2}(0, 1)$ such that

$$\text{(4.13)} \qquad u_k \to u \quad \text{in } L^2\,, \qquad \lim_k F(u_k) = 0\,.$$

For any $k \in N$ consider the increasing rearrangement of $x_1, \ldots, x_k$

$$x'_1 < x'_2 < \ldots < x'_k\,.$$

Denote with $S(x'_i)$ the interval of the family (S_h) whose center is x'_i and with $2\delta_i$ its dimensions; and define $u_k(x)$ by the following conditions:

$$u_k(0) = u(x'_1)$$

$$u'_k(x) = \begin{cases} u'(x) & \text{if } x'_i \in S(x'_{i-1}),\quad x \in [x'_{i-1}, x'_i[\\ 0 & \quad \text{»} \qquad \text{»} \qquad x \notin [x'_{i-1}, x'_i[\\ u'(x)\dfrac{u(x'_i) - u(x'_{i-1})}{u(x'_{i-1} + \delta_i) - u(x'_{i-1})} & \text{if } x'_i \notin S(x'_{i-1}) \text{ and } x \in [x'_{i-1}, x'_{i-1} + \delta_{i-1}[\\ 0 & \text{if } x'_i \notin S(x'_{i-1}), \text{ and } x \in \,]x'_i - \delta_i, x'_i[\\ 0 & \text{if } x \notin \bigcup_{i=1}^{k} S(x'_i) \\ u'(x)\dfrac{u(1) - u(x^i_1)}{u(x'_k + \delta_k) - u(x'_k)} & \text{if } x \in \,]x'_k, x'_k + \delta_k[\end{cases}$$

Then we have

$$u(x) \leqslant u_{k+1}(x) \leqslant u_k(x) ,$$

and it is easy to check that

$$u_k \to u \quad \text{uniformly in } (0, 1)$$

and also that

$$\int_0^1 (1 - \chi_C(x)(u_k'(x))^2 \, dx \to 0 .$$

So we have (4.13).

The same result holds for u arbitrary, as it is difference of two increasing functions, and so we have also (4.12).

5. – Stability criteria.

In this section we consider some stability problems of Γ-convergence with respect to perturbations of obstacle type.

Let $\Omega \in Ap_n$ and $F_h(u)$ be a sequence of functionals verifying (2.1), (2.2), (2.3) and set, for any $\psi \in L^1(\Omega)$, $u \in \mathrm{Lip}(\Omega)$

$$F_h(u, \psi) = \begin{cases} F_h(u) & \text{if } u \geqslant \psi \text{ a.e.} \\ +\infty & \text{otherwise} \end{cases} \tag{5.1}$$

Let us suppose that $\forall u \in \mathrm{Lip}(\Omega)$

$$F(u) = \int_\Omega f(x, Du) = \Gamma(N, L^1(\Omega)^-) \lim_{\substack{h \to \infty \\ v \to u}} F_h(v) \tag{5.2}$$

The problem we want to consider now is that of deducing by (5.2) a relation of the kind

$$F(u, \psi) = \Gamma(N, L^1(\Omega)^-) \lim_{\substack{h \to \infty \\ v \to u}} F_h(v, \psi_h) \tag{5.3}$$

assuming that the sequence of obstacles ψ_h verify at least the condition

$$\psi_h \xrightarrow{L^1(\Omega)} \psi . \tag{5.4}$$

It is known that a sufficient condition for (5.3) is (5.2) together with $\psi_h \to \psi$, $L^\infty(\Omega)$ (see [31]). And also that the weak convergence in L^1 is not sufficient (see [40]).

Here we prove the following

THEOREM 5.1. – *If* (5.2) *and* (5.4) *hold and there exist* $\varphi_h \in \operatorname{Lip}(\Omega)$ *such that*

$$\varphi_h \geqslant \psi_h\,, \quad \varphi_h \to \psi \quad in\ L^1(\Omega) \tag{5.6}$$

$$f_h(x, D\varphi_h)\,, \quad weakly\ precompact\ in\ L^1(\Omega) \tag{5.7}$$

then we have (5.3).

PROOF. – Let $u - \psi \geqslant \varepsilon$ a.e. and $u_h \xrightarrow{L^1(\Omega)} u$ such that

$$F(u) = \lim_h F_h(u_h)\,, \qquad u_h \in \operatorname{Lip}(\Omega)\,.$$

Set

$$\tilde{u}_h = \max\{u_h, \varphi_h\}$$

so that

$$\tilde{u}_h \xrightarrow{L^1(\Omega)} u\,, \qquad \tilde{u}_h \geqslant \psi_h\,.$$

If we put

$$\Omega_h = \{x \in \Omega\colon \tilde{u}_h \neq u_h\}$$

we have

$$\lim_h |\Omega_h| = 0$$

and so using the inequalities

$$\int_\Omega f_h(x, D\tilde{u}_h) = \int_{\Omega - \Omega_h} f_h(x, Du_h) + \int_{\Omega_h} f_h(x, D\varphi_h) \leqslant \int_\Omega f_h(x, Du_h) + \int_{\Omega_h} f_h(x, D\varphi_h)$$

we have by (5.7)

$$\lim_h{}'' \int_\Omega f_h(x, D\tilde{u}_h) \leqslant F(u)\,.$$

So we have, if $u - \psi \geqslant \varepsilon$, the relation

$$F(u) = \Gamma(N, L^1(\Omega)^-) \lim_{\substack{h\to\infty\\ v\to u}} F_h(v, \psi_h)\,. \tag{5.8}$$

But if we recall that

$$F(u) = F(u + \varepsilon)\,, \qquad \forall \varepsilon > 0$$

we deduce that (5.8) holds $\forall u \geqslant \psi$.

Let us now assume that $u < \psi$ on a set of positive measure, and observe that if $u_h \to u$ in $L^1(\Omega)$ then $\forall h \geq h_0$ $u_h < \psi$ on a set of positive measure and, since $\psi_h \to \psi$ in $L^1(\Omega)$ we deduce $\forall h \geqslant h_0$ $u_h < \psi_h$ on a set of positive measure indipendent on h so that

$$\Gamma(N, L^1(\Omega)^-) \lim_{\substack{h\to\infty \\ v\to u}} F_h(v, \psi_h) = +\infty ,$$

and therefore (5.3) holds.

Let us now recall the definition of the capacity of a set A with respect to $\Omega \in Ap_n$

$$C^\Omega(A) = \inf\left\{\int_\Omega |Du|^2 dx \colon u \in H_0^{1,2}(\Omega), u \geq 1 \text{ on } A, u \geq 0 \text{ on } \Omega\right\}$$

and remind also that for a sequence $\psi_h \in L^\infty$ we put

(5.9) $$\psi_h \to \psi \qquad \text{in capacity}$$

if and only if

$$\forall \varepsilon > 0 \quad \lim_h C^\Omega(\{x \in \Omega \colon |\psi_h(x) - \psi(x)| \geq \varepsilon\}) = 0 .$$

We have the following corollary of Theo. 5.1

COROLLARY 5.2. – *Let us consider the sequence*

$$F_h(u) = \int_\Omega f_h(x, Du)$$

verifying

$$\lambda |z|^2 \leq f_h(x, z) \leq \Lambda |z|^2 , \qquad f_h(x, \cdot) \text{ convex} .$$

Then, if (5.2), (5.9) *hold,* $\psi \in Lip$ *and* ψ_h *equibounded, we have* (5.3).

PROOF. – Let us choose the functions φ_h. Evidently, for any $n \in N$ there exists $\tilde{\varphi}_{h_n} \in H_0^1(\Omega)$ such that

$$\tilde{\varphi}_{h_n} \geq 1 \qquad \text{on} \qquad \left\{x \colon |\psi_{h_n}(x) - \psi(x)| \geqslant \frac{1}{n}\right\}$$

$$\int_\Omega |D\tilde{\varphi}_{h_n}|^2 dx < \frac{1}{n}$$

and (h_n) is increasing.

Then, if we put $M = \sup|\psi_h|$ and

$$\tilde{\tilde{\varphi}}_{h_n} = 2M\tilde{\varphi}_{h_n},$$

we have

$$\tilde{\tilde{\varphi}}_{h_n} \to 0, \qquad \text{in } L^1(\Omega).$$

Setting

$$\varphi_{h_n} = \psi + \tilde{\tilde{\varphi}}_{h_n} + \frac{1}{n}$$

we have

$$\varphi_{h_n} \geqslant \psi_{h_n}, \qquad \varphi_{h_n} \to \psi, \qquad \text{in } L^1(\Omega)$$

and we have for F_{h_n}, φ_{h_n}, φ_{h_n} the assumptions of Theo. 5.1. Using the arbitrariety of the sub-sequence, we have the result.

We want to show now with an example that, when dealing with obstacle problems, the equality between $\Gamma(C^0)$ and $\Gamma(L^1)$ may fail.

EXAMPLE. – Let us consider the functional

$$F_h(u) = \int_\Omega |Du|^\alpha dx, \qquad 1 < \alpha < 2 \qquad u \in \mathrm{Lip}(\Omega)$$

where $\Omega = \{(x_1, x_2) \in R^2 \colon x_1^2 + x_2^2 < 1\}$.

Let us define ψ_h as the cone with height 1, center of his base in $(0, 0)$ and base equal to $C_h = \{(x_1, x_2) \colon x_1^2 + x_2^2 < 1/h\}$ and set $\psi_h = 0$ in $\Omega - \Omega_h$.

Then it is easy to check that

$$\psi_h \xrightarrow{L^1} 0$$

and (5.7) holds.

But if we set

$$F_0(u) = \Gamma(N, C^0(\Omega)^-) \lim_{\substack{h\to\infty \\ v\to u}} F_h(v)$$

we have that

$$F_0(u) = +\infty, \qquad \forall u \qquad \text{such that} \qquad u(0, 0) < 1.$$

Let us now consider the stability problem for $\Gamma(C^0)$ limits.

THEOREM 5.3. – *Let us assume that F_h and F are given as in* (5.1), (5.2) *ψ_h lower* *semicontinuous and*

$$\psi(x) = \Gamma(N, R^+) \lim_{\substack{h\to\infty \\ y\to x,\, y\in\Omega}} \psi_{h_k}(y) \qquad \forall x\in\bar{\Omega}\,, \tag{5.10}$$

$$F_0(u, \psi) = \Gamma(N, C^0(\Omega)^-) \lim_{k\to\infty} F_{h_k}(v, \psi_{h_k}) \qquad \forall u\in \mathrm{Lip}\,(\Omega). \tag{5.11}$$

Then

$$F_0(u, \psi) = \begin{cases} +\infty & \text{if } \exists x_0\in\bar{\Omega}: u(x_0) < \psi(x_0) \\ F(u) & \textit{otherwise} \end{cases}$$

PROOF. – *a*) Let $u(x_0) < \psi(x_0) - \varepsilon$; then $\forall x \in I(x_0, \delta)$

$$u(x) < \psi(x_0) - \frac{\varepsilon}{2}$$

and if $u_h \xrightarrow{C^0} u$ we have

$$u_h(x) < \psi(x_0) - \frac{\varepsilon}{4} \qquad \forall x\in I(x_0, \delta) \qquad h \geqslant h_0\,.$$

But using the properties of Γ^+ limits there exist (x_h) such that $x_h \to x_0$ and

$$\lim_k \psi_{h_k}(x_{h_k}) = \psi(x_0)$$

so that

$$\psi_{h_k}(x_{h_k}) > \psi(x_0) - \frac{\varepsilon}{4} > u_{h_k}(x_{h_k})\,.$$

But this implies that

$$\forall u_h \xrightarrow{C^0} u \Rightarrow F_0(u, \psi) = +\infty\,.$$

b) Let

$$u(x) > \psi(x) + \varepsilon\,, \qquad \forall x\,,$$

and $u_h \xrightarrow{C^0(\Omega)} u$ such that $F(u) = \lim_h F_h(u_h)$. Then $\forall h \geqslant h_0$

$$u_h(x) > \psi(x) + \frac{\varepsilon}{2}\,.$$

If there exist x_h such that

$$u_h(x_h) < \psi_h(x_h)$$

then as $x_h \in \Omega$ which is bounded, we have

$$x_{h_k} \to x_0$$

and so

$$u(x_0) = \lim_k{}'' u_{h_k}(x_{h_k}) \leqslant \lim_k{}'' \psi_{h_k}(x_h) \leqslant \Gamma(N^+, R^+) \lim_{\substack{h\to\infty \\ x\to x_0}} \psi_h(x) = \psi(x_0)$$

which is absurd.

REMARK. – Let us observe that *b*) is true also if we assume only

$$\psi = \Gamma(N^+, R^+) \lim_{h\to\infty} \psi_h$$

but *a*) can be false in this case.

In fact it is not necessary that $F_0(u, \psi) = +\infty$ if there exists x_0 such that

$$u(x_0) < \Gamma(N^+, R^+) \lim_{\substack{h\to\infty \\ x\to x_0}} \psi_h(x) ;$$

we can only say that it is $+\infty$ if it exists.

It suffices to consider the following example

$$F_h(u) = \int_0^1 u'^2\, dx$$

$$\psi_{2h}(x) = 0$$

$$\psi_{2h-1}(x) = \begin{cases} 1 & x \in \left[0, \dfrac{1}{2h-1}\right] \\ 0 & x \notin \left[0, \dfrac{1}{2h-1}\right] \end{cases}$$

Then

$$\Gamma(N^+, R^+) \lim_{\substack{h\to\infty \\ x\to x_0}} \psi_h(x) = \begin{cases} 0 & x_0 \in \,]0, 1] \\ 1 & x_0 = 0 \end{cases}$$

and there exist sequences (u_h) (e.g. $u_{2h} = 0$), $u_h \to u$ C^0, and:

$$\liminf_h F_h(u_h) = 0$$

$$\limsup_h F_h(u_h) = +\infty .$$

In the following we intend to consider the problem of the invariance of the Γ limit of a sequence of integrals of variational calculus under multiplication of the integrands by a continuous function.

Let us suppose (2.1), (2.3), (3.25) and that $\forall \Omega \in Ap_n$, $\forall u \in \mathrm{Lip}_{\mathrm{loc}}$

$$\int_\Omega f(x, Du) = \Gamma(N, C_0^0(\Omega)^-) \lim_{\substack{h\to\infty \\ v\to u}} \int_\Omega f_h(x, Dv) \tag{5.12}$$

we have the following

THEOREM 5.4. – *If* (5.12) *holds and* $m \in C^0(\Omega)$, $m(x) \geqslant 0$, *then* $\forall u \in \mathrm{Lip}\,(\Omega)$

$$\int_\Omega m(x) f(x, Du) = \Gamma(N, C_0^0(\Omega)^-) \lim_{\substack{h\to\infty \\ v\to u}} \int_\Omega m(x) f_h(x, Du) \tag{5.13}$$

PROOF. – We begin to prove (5.13) in the case that m is a simple function (Ω_i open)

$$m(x) = \sum_{i=1}^k \alpha_i \chi_{\Omega_i}(x)\,, \qquad |\Omega \cap \partial\Omega_i| = 0\,, \qquad \Big(\bigcup_{i=1}^k \bar\Omega_i\Big) \cap \Omega = \Omega\,.$$

Let $u_h \to u$ then

$$\begin{aligned}\int_\Omega m(x) f(x, Du) &= \sum_i \int_{\Omega_i} \alpha_i f(x, Du) \leqslant \\ &\leqslant \sum_i \liminf_h \int_{\Omega_i} \alpha_i f_h(x, Du_h) \leqslant \\ &\leqslant \liminf_h \sum_i \int_{\Omega_i} \alpha_i f_h(x, Du_h) = \\ &= \liminf_h \int_\Omega m(x) f_h(x, Du_h)\,.\end{aligned}$$

Furthermore we know by (5.12) that $\forall u \in \mathrm{Lip}\,(\Omega)$, $\forall i$ there exists a sequence $(w_{ih})_h$ in $\mathrm{Lip}\,(\Omega)$ such that

$$w_{ih} - u \in C_0^0(\Omega_i)\,, \qquad \|w_{ih} - u\| \xrightarrow[C_0^0(\Omega_i)]{(h\to\infty)} 0\,, \qquad \lim_h \int_{\Omega_i} f_h(x, Dw_{ih}) = \int_{\Omega_i} f(x, Du) \tag{5.14}$$

Let us set

$$w_h(x) = w_{ih}(x)\,, \qquad \forall x \in \Omega_i\,,$$

then clearly $w_h - u \in C_0^0(\Omega)$ and $\|w_h - u\|_{C_0^0(\Omega)} \xrightarrow{(h\to\infty)} 0$; moreover by (5.14) we deduce

$$\int_{\Omega_i} \sum_i \alpha_i f_h(x, Dw_{ih}) = \sum_i \int_\Omega \alpha_i \chi_{\Omega_i} f_h(x, Dw_h) = \int_\Omega m(x) f_h(x, Dw_h) \to \int_\Omega m(x) f(x, Du)\,.$$

Let us consider now the general case in which $m(x) \in C^0(\Omega)$, $m(x) \geqslant 0$ and let (m_n) be a sequence of simple functions such that

$$m_n \uparrow m\,, \qquad m_n(x) = \sum_{i=1}^{k_n} \alpha_i^n \chi_{\Omega_i^n}(x)\,, \qquad \Big(\bigcup_{i=1}^{k_n} \bar\Omega_i^n\Big) \cap \Omega = \Omega\,, \qquad |\Omega \cap \partial\Omega_i^n| = 0\,.$$

Fixed $u \in \mathrm{Lip}(\Omega)$, $n \in N$, let us define $\forall h \in N$, $i = 1, \ldots, k_n$

$$w^n_{i,h} \in \mathrm{Lip}(\Omega)$$
$$w^n_{i,h} - u \in C^0_0(\Omega^n_i)$$

such that

$$\lim_{h\to\infty} \|w^n_{i,h} - u\|_{C^0_0(\Omega^n_i)} = 0 \qquad \int_{\Omega^n_i} \alpha^i_n f(x, Du) = \lim_h \int_{\Omega^n_i} \alpha^i_n f_h(x, Dw^n_{i,h}) \,. \tag{5.15}$$

If we set $\forall x \in \Omega$

$$w^n_h(x) = w^n_{i,h}(x)\,, \qquad \forall x \in \Omega^n_i$$

we obtain

$$\lim_{h\to\infty} \|w^n_h - u\|_{C^0_0(\Omega)} = 0 \qquad \int_\Omega m_n(x) f(x, Du) = \lim_{h\to\infty} \int_\Omega m_n f_h(x, Dw^n_h)$$

Then $\forall n$

$$\begin{aligned}(5.16)\qquad & \Gamma(N, C^0(\Omega)^-) \lim_h \int_\Omega m f_h(x, Du) - \Gamma(N, C^0(\Omega)^-) \lim_h \int_\Omega m_n f_h(x, Du) \\ & \leq \lim_h \left[\int_\Omega m f_h(x, Dw^n_h) - \int_\Omega m_n f_h(x, Dw^n_h)\right] \leq \lim_h \int_\Omega (m - m_n) f_h(x, Dw^n_h) \\ & \leq \|m - m_n\|_{L^\infty(\Omega)} \lim_h \int_\Omega f_h(x, Dw^n_h)\, dx\end{aligned}$$

but by (5.15) we deduce

$$\int_\Omega f(x, Du) = \sum_{i=1}^{k_n} \int_{\Omega^n_i} f(x, Du) = \lim_h \sum_{i=1}^{k_n} \int_{\Omega^n_i} f_h(x, Dw^n_h) = \lim_h \int_\Omega f_h(x, Dw^n_h)$$

So by (5.17) and (5.16) passing to the limit as $n \to \infty$ we deduce, recalling that $m_n \leq m$, the result.

REMARK. - If $m \notin C^0$, the result of Theo. 5.4 can be not true: see the example of § 4.

6. - Some observations about the dependence on u.

In this section we consider a class of functionals of the form

$$F_h(\Omega, u) = \int_\Omega f_h(x, u, Du)\, dx \tag{6.1}$$

without any continuity assumption on $f_h(x, \cdot, z)$, for which we prove an equality of the kind

$$\Gamma(N, C^0(\Omega)^-) \lim_{\substack{h\to\infty \\ v\to u}} F_h(\Omega, v) = \Gamma(N, C_0^0(\Omega)^-) \lim_{\substack{h\to\infty \\ v\to u}} F_h(\Omega, v) .$$

Precisely let $f_h = f_h(x, y, z)$ be defined for $x, z \in R^n$, $y \in R$ and assume

$$\text{(6.2)} \qquad \begin{cases} 0 \leqslant f_h'(x, z) \leqslant f_h(x, y, z) \leqslant k f_h'(x, z) + f_h''(x, y) \\ \quad f_h(x, y, \cdot) \text{ convex} \\ 0 \leqslant f_h'(x, z) \leqslant a_h(x)(1 + |z|^p) \\ 0 \leqslant f_h''(x, y) = a_h(x)|y|^p \end{cases}$$

$$\text{(6.3)} \qquad \lim_h \int_\Omega f_h'(x, Du_h) < \infty \Rightarrow \exists \sigma > 1 \quad \text{such that} \quad \lim_{h\to\infty} \int_\Omega f_h'(x, \sigma D u_h) < +\infty ,$$

where $a_h \in L^1_{\text{loc}}$ verify

$$\text{(6.4)} \qquad \int_\Omega a_h \, dx \to \int_\Omega a(x) \, dx , \qquad \forall \Omega \in Ap_n .$$

We have the following

THEOREM 6.1. – *If in $\Omega \in Ap_n$, $u \in C^1$ there exist the following limits*

$$F_0(\Omega, u) = \Gamma(N, C_0^0(\Omega)^-) \lim_{\substack{h\to\infty \\ v\to u}} F_h(\Omega, v)$$

$$F(\Omega, u) = \Gamma(N, C^0(\Omega)^-) \lim_{\substack{h\to\infty \\ v\to u}} F_h(\Omega, v)$$

then

$$\text{(6.5)} \qquad F = F_0 .$$

PROOF. – Let $u_h \in C^1$ be such that $u_h \xrightarrow{C^0(\Omega)} u$ and

$$F(\Omega, u) = \lim_h F_h(\Omega, u_h) .$$

For any compact $K \subset\subset \Omega$ let $\delta = \text{dist}(K, R^n - \Omega)$ and set as in [16], [48]

$$\text{(6.6)} \qquad B_0 = K \qquad B_i = \left\{ x \in R^n : \text{dist}(x, K) < \frac{i\delta}{\nu} \right\} \qquad i = 1, \ldots, \nu ;$$

and let $\psi_1, \ldots, \psi_r$ satisfy the conditions

$$\text{(6.7)} \qquad \begin{cases} \psi_i \in C_0^1(B_i) \quad 0 \leqslant \psi_i(x) \leqslant 1 & \forall x \in B_i \\ \psi_i(x) = 1 & \forall x \in B_{i-1} \\ |D\psi_i(x)| \leqslant \dfrac{\nu+1}{\delta} & \forall x \in B_i \end{cases}$$

Set $\forall h \in N;\ i = 1, \ldots, \nu$

$$w_{ih} = u + (u_h - u)\psi_i$$

and observe that $\forall i = 1, \ldots, \nu$

$$w_{ih} \to u\,, \quad \text{in } C_0^0(\Omega)\,,$$

$$F_0(\Omega, u) \leqslant \liminf_h F_h(\Omega, w_{ih})\,.$$

For any $\varepsilon > 0$ there exists $\bar{h} \in N$ such that $\forall h \geqslant \bar{h}$

$$\text{(6.8)} \quad F_0(\Omega, u) - F(\Omega, u) - \varepsilon < \int_{\Omega - B_i} f_h(x, w_{ih}, Dw_{ih}) + \int_{B_i} f_h(x, w_{ih}, Dw_{ih}) - \int_{B_i} f_h(x, u_h, Du_h) \leqslant$$
$$\leqslant \int_{\Omega - K} f_h(x, u, Du) + \int_{B_i - B_{i-1}} [f_h(x, w_{ih}, Dw_{ih}) - f_h(x, u_h, Du_h)] \leqslant$$
$$\leqslant \int_{\Omega - K} a_h(x)\,dx \cdot \sup_\Omega\,(1 + |u|^p + |Du|^p) + \int_{B_i - B_{i-1}} f_h(x, w_{ih}, Dw_{ih})\,.$$

But for any $t \in (0, 1)$, using the convexity of $f_h(x, y, \cdot)$:

$$\text{(6.9)} \qquad \int_{B_i - B_{i-1}} f_h(x, w_{ih}, Dw_{ih}) = \int_{B_i - B_{i-1}} f_h(x, w_{ih}, (1 - \psi_i)Du + \psi_i Du_h + (u_h - u)D\psi_i) \leqslant$$
$$\leqslant t \int_{B_i - B_{i-1}} f_h\left(x, w_{ih}, \frac{D\psi_i(u_h - u)}{t}\right) + (1 - t) \int_{B_i - B_{i-1}} f_h\left(x, w_{ih}, \frac{(1 - \psi_i)Du + Du_h\psi_i}{1 - t}\right) \leqslant$$
$$\leqslant t \int_{B_i - B_{i-1}} a_h(x)\,dx \sup_\Omega \left[1 + (|u_h - u| + |u|)^p + \left(\frac{\nu+1}{\delta}\right)^p \frac{|u_h - u|^p}{t^p}\right] +$$
$$+ (1 - t) \int_{B_i - B_{i-1}} f_h\left(x, w_{ih}, \frac{Du}{1 - t}\right) + (1 - t) \int_{B_i - B_{i-1}} f_h\left(x, w_{ih}, \frac{Du_h}{1 - t}\right) \leqslant$$
$$\leqslant t \int_{B_i - B_{i-1}} a_h\,dx \sup_\Omega \left[1 + (|u_h - u| + |u|)^p + \left(\frac{\nu+1}{\delta}\right)^p \frac{|u_h - u|^p}{t^p}\right] +$$

$$+ (1-t) \int\limits_{B_i - B_{i-1}} a_h\, dx \sup_\Omega \left[1 + (|u_h - u| + |u|)^p + \frac{|Du|^p}{(1-t)^p}\right] +$$

$$+ (1-t) \int\limits_{B_i - B_{i-1}} a_h(x)\, dx \sup_\Omega (|u| + |u_h - u|)^p + (1-t)\, k \int\limits_{B_i - B_{i-1}} f_h'\left(x, \frac{Du_h}{1-t}\right) dx\,.$$

Then we have by (6.8), (6.9) $\forall h \geqslant \bar{h}$

$$(6.10) \qquad F_0(\Omega, u) - F(\Omega, u) - \varepsilon < \int\limits_{\Omega - K} a_h\, dx \sup_\Omega (1 + |u|^p + |Du|)^p +$$

$$+ \frac{t}{\nu} \int\limits_{\Omega - K} a_h(x)\, dx \sup_\Omega \left[1 + (|u_h - u| + |u|)^p + \left(\frac{\nu + 1}{\delta}\right)^p \frac{|u_h - u|^p}{t^p}\right] +$$

$$+ \frac{1-t}{\nu} \int\limits_{\Omega - K} a_h(x)\, dx \cdot \sup_\Omega \left[1 + (|u_h - u| + |u|)^p + \frac{|Du|^p}{(1-t)^p}\right] +$$

$$+ \frac{1-t}{\nu} \int\limits_{\Omega - K} a_h\, dx \sup_\Omega (|u_h - u| + |u|^p + \frac{1-t}{\nu}\, k \int\limits_{\Omega - K} f_h'\left(x, \frac{Du_h}{1-t}\right).$$

We pass to the limit as $h \to \infty$ and obtain, thanks to (6.3)

$$F_0(\Omega, u) - F(\Omega, u) - \varepsilon < \int\limits_{\Omega - K} a\, dx \sup_\Omega (1 + |u|^p + |Du|^p) +$$

$$+ \frac{t}{\nu} \int\limits_{\Omega - K} a\, dx \sup_\Omega [1 + |u|^p] + \frac{1-t}{\nu} \int\limits_{\Omega - K} a\, dx \sup_\Omega \left[1 + |u|^p + \frac{|Du|^p}{(1-t)^p}\right] +$$

$$+ \frac{1-t}{\nu} \int\limits_{\Omega - K} a\, dx \cdot \sup_\Omega |u|^p + \frac{1-t}{\nu}\, k \lim_h{}'' \int\limits_{\Omega - K} f_h'\left(x, \frac{Du_h}{1-t}\right) dx\,.$$

To the limit as $\nu \to \infty$ and $K \uparrow \Omega$ we have

$$0 \leqslant F_0(\Omega, u) - F(\Omega, u) < \varepsilon$$

and also the result.

REMARK. – A sufficient condition to have (6.3) is the following:

(6.11) $\forall C \subset\subset R^n$ there exist $\zeta_C > 1$, $\sigma_C > 1$, such that

$$\lim_{|z| \to \infty} [f_h(x, z\sigma_C) - \zeta_C f_h(x, z)] = 0 \qquad \text{uniformly for } h \in N,\ x \in C.$$

Observe also that quadratic forms for $p = 2$, « area-like » integrands for $p = 1$ and functionals (for every $p \geq 1$) such that

$$|z|^p \leq f_h(x, z) \leq a\big(|z|^p + 1\big)$$

satisfy (6.11).

7. – Homogenization formulas.

In this section we want to indicate that in some cases it is possible to have more informations on the Γ-limit of a sequence f_h, and in particular, for the homogenization problem, an explicit formula.

Let us consider a sequence $f_h(x, z)$ verifying

$$0 \leq m_h(x)|z|^p \leq f_h(x, z) \leq M_h(x)(1 + |z|^p)\,, \qquad p \geq 2\,,$$

and assume that m_h^{-1}, $M_h \in L^1_{\text{loc}}$ and $\forall \Omega \in Ap_n$

$$\Big(\int_\Omega m_h(x)^{1-p}\,dx\Big)^{1/(1-p)} \to \Big(\int_\Omega m_{(p)}(x)^{1-p}\,dx\Big)^{1/(1-p)}$$

$$\int_\Omega M_h(x)\,dx \to \int_\Omega M(x)\,dx$$

$$\int_\Omega f_0(x, Du)\,dx = \Gamma(N, L^1(\Omega)^-) \lim_{\substack{h\to\infty\\ v\to u}} \int_\Omega f_h(x, Dv)\,.$$

LEMMA 7.1. – *In the previous assumptions we have*

$$m_{(p)}(x)|z|^p \leq f_0(x, z) \leq M(x)|z|^p\,.$$

PROOF. – Using Hölder inequality we obtain easily for $\Omega \in Ap_n$

$$\Big(\int_\Omega m_h(x)^{1-p}\,dx\Big)^{1/(1-p)} \Big(\int_\Omega |Du|^{p-1}\,dx\Big)^{p/(p-1)} \leq \int_\Omega f_h(x, Du)\,dx \leq \int_\Omega M_h(x)(1 + |Du|^p)\,dx\,.$$

Passing to the Γ-limit and then choosing $u(x) = \langle z, x\rangle$, $z \in R^n$, we have:

$$\Big(\frac{1}{|\Omega|}\int_\Omega m_{(p)}(x)^{1-p}\,dx\Big)^{1/(1-p)} |\Omega||z|^p \leq \int_\Omega f_0(x, z)\,dx \leq (1 + |z|^p)\int_\Omega M(x)\,dx\,,$$

and so, using Lebesgue points, we obtain the assertion.

We can pass now to the homogenization. Set $Y =]0, 1[^n$ and consider a function $f = f(x, z)$ verifying

$$\begin{cases} m(x)|z|^p \leqslant f(x, z) \leqslant M(x)(1 + |z|^p) & p \geqslant 2 \\ f(x, \cdot) \quad \text{convex} \end{cases} \tag{7.1}$$

and assume that $m^{-1}, M \in L^1_{\text{loc}}$ and $\forall z$

$$m, M, f(\cdot, z), \quad \text{are } Y\text{-periodic } ^{(6)} . \tag{7.2}$$

Set, $\forall u \in \text{Lip}^1_{\text{loc}}$ and $\Omega \in Ap_n$

$$F_h(\Omega, u) = \int_\Omega f(hx, Du)\, dx . \tag{7.3}$$

For any $g \in L^1_{\text{loc}}$ Y-periodic, we put

$$(g)_Y = \int_Y g(x)\, dx .$$

Define $m_{(p)}$ as

$$m_{(p)} = (m^{-(p-1)})_Y$$

and observe that for any $\Omega \in Ap_n$

$$m_{(p)}|\Omega| = \lim_h \int_\Omega m(hx)^{1-p}\, dx . \tag{7.4}$$

Moreover, using Hölder inequality, we have

$$\frac{1}{\left(\int_\Omega m(hx)^{1-p}\, dx\right)^{1/(1-p)}} \left(\int_\Omega |Du|^{p-1} dx\right)^{p/(p-1)} \leqslant F_h(\Omega, u) . \tag{7.5}$$

If we consider an integer multiple Y' of Y, then for any $\Omega \subset Y'$ we have

$$\left(\int_\Omega m(hx)^{1-p}\, dx\right)^{1/(p-1)} \leqslant c(Y')$$

and so, $\forall \Omega \subset Y'$

$$\frac{1}{c(Y')} \left(\int_\Omega |Du|^{p-1} dx\right)^{1/(1-p)} \leqslant F_h(\Omega, u) . \tag{7.6}$$

---

(6) A function $\varphi: R^n \to R^m$ is Y-periodic iff for any i the function $x_i \to \varphi(x_1, \ldots, x_i, \ldots, x_n)$ is 1-periodic.

We know that there exist a subsequence of F_h (which we continue to denote with F_h) and $f_0(x, z)$ such that for any $\Omega \in Ap_n$ and $\forall u \in \mathrm{Lip}_{\mathrm{loc}}$

$$F(\Omega, u) = \int_\Omega f_0(x, Du)\,dx = \Gamma(N, L^1(\Omega)^-) \lim_{\substack{h\to\infty \\ v\to u}} \int_\Omega f(hx, Dv)\,dx\,. \tag{7.7}$$

We have now the following result

LEMMA 7.1. – *In the hypothesis* (7.1), (7.2), *the function* $f_0(x, z)$ *given by* (7.7) *is independent on* x, *and satisfies the following inequalities*

$$m_{(p)}^{1/(1-p)}|z|^p \leqslant f_0(z) \leqslant (M)_Y(1 + |z|^p)\,. \tag{7.8}$$

PROOF. – The first assertion can be proved for example as in [30]. The inequality (7.8) follows from Lemma 7.1.

In the following we want to extend the limit relation (7.7) from the space $\mathrm{Lip}_{\mathrm{loc}}$ to BV_{loc}, in order to obtain the convergence of the solutions of some minimum problems relative to F_h as a consequence.

To this aim let us consider the L^1-semicontinuous extension to BV_{loc} of the functional F_h:

$$\phi_h(\Omega, u) = sc^-(L^1(\Omega)) \begin{cases} F_h(\Omega, u) & u \in \mathrm{Lip}_{\mathrm{loc}} \\ +\infty & u \in BV_{\mathrm{loc}} - \mathrm{Lip}_{\mathrm{loc}} \end{cases} \tag{7.9}$$

and prove the following

LEMMA 7.3. – *If* ϕ_h *are given by* (7.9) *and* F_h *verify* (7.3), (7.7), *then there exists* (h_r) *such that, for any* $k \in N$ (*if* $\Omega_k = \{x \in R^n\colon |x| < k\}$) *there exist the limits*

$$\phi(\Omega_k, u) = \Gamma(N, L^1(\Omega_k)^-) \lim_{\substack{r\to\infty \\ v\to u}} \phi_{h_r}(\Omega_k, v) \qquad \forall u \in BV_{\mathrm{loc}} \tag{7.10}$$

and one has

$$\phi(\Omega_k, u) = F(\Omega_k, u)\,, \qquad \forall u \in \mathrm{Lip}_{\mathrm{loc}},\ k \in N\,. \tag{7.11}$$

PROOF. – Using the KURATOWSKY compactness Theorem 1.11 and the separability of L^1 we have the existence of the Γ-limits in (7.10) with a diagonal process. As it is clearly $\phi \leqslant F$, then to prove (7.11) let $u_r \in BV_{\mathrm{loc}}$, $u_r \to u \in \mathrm{Lip}_{\mathrm{loc}}$ in $L^1(\Omega_k)$ and

$$\phi(\Omega_k, u) = \lim_r \phi_{h_r}(\Omega_k, u_r)\,.$$

Then we can find $v_r \in \mathrm{Lip}_{\mathrm{loc}}$ such that

$$|\phi_{h_r}(\Omega_k, v_r) - \phi_{h_r}(\Omega_k, u_r)| < 1/r$$

$$\|v_r - u_r\|_{L^1(\Omega_k)} < 1/r ,$$

so that

$$F(\Omega_k, u) \leqslant \liminf_r F_{h_r}(\Omega_k, v_r) = \lim_r \phi_{h_r}(\Omega_k, u_r) .$$

Useful to the sequel will be the following

PROPOSITION 7.4. – *If $\phi(\Omega_k, u)$ is defined by* (7.10), *then the functional*

$$\phi(u) = \sup_k \phi(\Omega_k, u) \qquad \forall u \in BV_{\mathrm{loc}} \tag{7.12}$$

is convex, L^1_{loc}-semicontinuous, satisfies the relations

$$\phi\big(u(x-y)\big) = \phi\big(u(x)\big) , \qquad \forall y \in R^n,\ u \in BV_{\mathrm{loc}} \tag{7.13}$$

$$\phi(u) = \int_{R^n} f_0(Du)\, dx , \qquad \forall u \in BV_{\mathrm{loc}} . \tag{7.14}$$

PROOF. – This result can be obtained similarly as it is done in [30] in the quadratic case, and so we don't enter into all the details.

After the proof of (7.13) which is similar to that of Lemma 4.4 in [30], for $u \in BV_{\mathrm{loc}}$, if $\alpha \in C_0(R^n)$ satisfies

$$\int_{R^n} \alpha(x)\, dx = 1 , \qquad \alpha \geqslant 0,\ \text{spt. } \alpha \subset \{|x| < 1\} \tag{7.15}$$

set:

$$\alpha_k * u(x) = k^n \int_{R^n} \alpha(ky)\, u(x-y)\, dy$$

and observe that $\forall \Omega \in Ap_n$ (see [0])

$$\lim_k \int_\Omega |\alpha_k * u - u|\, dx = 0 ,$$

while if $\int_{\partial\Omega} |Du| = 0$

$$\lim_k \int_\Omega |D(\alpha_k * u)|\, dx = \int_\Omega |Du| .$$

Then by JENSEN's inequality and (7.13), (7.15) we deduce

$$\phi(\alpha_k * u) \leqslant \phi(u)$$

and also, by the semicontinuity of ϕ,

$$\phi(u) = \lim_k \phi(\alpha_k * u) ,$$

which clearly implies (7.14).

In the following we want to give an « homogenization » result by which it is possible to describe the limit $f_0(z)$ solving a differential problem connected with $f(x, z)$.

The first results of this kind were obtained by SANCHEZ-PALENCIA [41], [42], [43], DE GIORGI-SPAGNOLO [20], BABUŠKA [2] for a uniformly coercive quadratic functional of the kind

$$F(u) = \int_\Omega a_{ij}(x)\, D_i u\, D_j u\, dx$$

$$a_{ij} = a_{ji} \in L^\infty(\Omega)$$

$$\lambda_0 |z|^2 \leqslant a_{ij}(x) z_i z_j ,$$

and then where considered in more general situations by several authors: [5], [6], [7], [8], [9], [12], [13], [24], [25], [28], [30], [31], [40], [56].

For the homogenization of non quadratic functionals which have properties of coercivity on Sobolev spaces see [2], [28], [56].

Here we generalize most of this results in the sense that we have some weak coercivity properties and we don't assume any regularity of $f(x, z)$.

To this aim let us denote with C^1_{per} the set of all the functions $u \in C^1$ which are Y-periodic, and let W_Y be the space of all $u \in BV_{\text{loc}}$ such that there exists $(u_h) \subset C^1_{\text{per}}$ such that

$$u_h \to u \quad \text{in } L^1(Y) , \qquad \int_Y |Du_h| \to \int_Y |Du| ,$$

where $\int_Y |Du|$ indicates the total variation of Du (see § 1).

We want to prove the following

LEMMA 7.5. – *If F_h are defined by* (7.3) *and f_0 by* (7.7), (7.8) *for any $z \in R^n$ set* $\forall u \in W_Y$

$$\Psi_h(u) = sc^-(L^1(Y)) \begin{cases} F_h(Y, u + z, x) & u \in W_Y \cap \text{Lip}(Y) \\ + \infty & u \in W_Y - \text{Lip}(Y) \end{cases} \tag{7.15'}$$

and

$$\Psi(u) = sc^{-}(L^1(Y)) \begin{cases} \int_Y f_0(Du + z)\,dx & u \in W_Y \cap H^{1,p}(Y) \\ +\infty & u \in W_Y - H^{1,p}(Y) \end{cases} \tag{7.16}$$

then

$$\Psi(u) = \Gamma(N, L^1(Y)^-) \lim_{\substack{h\to\infty \\ v\to u}} \Psi_h(v) \qquad \forall u \in W_Y . \tag{7.17}$$

PROOF. – By Kuratowsky theorem we know that there exists (h_r) and a convex function $\theta(u)$ such that

$$\theta(u) = \Gamma(N, L^1(\Omega)^-) \lim_{\substack{r\to\infty \\ v\to u}} \Psi_{h_r}(v) \qquad \forall u \in W_Y .$$

The proof since now is similar to that of Lemma 5.1 in [30].

We can give the principal result of this section.

THEOREM 7.6. – *Let f satisfy* (7.1), (7.2) *and f_0 be defined by* (7.7) *then*: (7.8) *holds and*

$$f_0(z) = \mathrm{Min}\left\{ \int_Y f(x, Du + z)\,dx \colon u \in W_Y \right\} . \tag{7.18}$$

PROOF. – Using (7.17) and the fact that the solutions of the problems

$$\mathrm{Min}\left\{ \int_Y f(hx, Du + z)\,dx \colon u \in W_Y \right\} = M_h(z) \tag{7.19}$$

are in a bounded set of BV and therefore in a compact of BV with respect to the topology of L^1, we deduce from Theo. 1.12 that

$$M_h(z) \to \mathrm{Min}\left\{ \int_Y f_0(Du + z) \colon u \in W_Y \right\} = M_0(z) .$$

So we obtain (7.18) if we first prove that

$$M_0(z) = f_0(z) , \qquad \forall z \tag{7.20}$$

and then that

$$\lim_h M_h(z) = \mathrm{Min}\left\{ \int_Y f(x, Du + z) \colon u \in W_Y \right\} . \tag{7.21}$$

The relation (7.20) is obvious. In order to verify (7.21), observe that the condition

$$\int_Y Du\,dx = z\,, \qquad Du \;\; Y\text{-periodic}$$

is equivalent to

$$u(x) - \langle z, x\rangle\,, \qquad Y\text{-periodic}\,.$$

We want to check that $\forall h \in N$

$$M_h(z) = M_1(z)\,.$$

Let u be a solution of $M_h(z)$, $u_1 = u + \langle z, x\rangle$ and observe that the function

$$\tilde{u}_1(x_1, \ldots, x_n) = h^{-n} \sum_{i_1, \ldots, i_n = 0}^{h-1} u_1\left(x_1 + \frac{i_1}{h}, \ldots, x_n + \frac{i_n}{h}\right)$$

is such that $D\tilde{u}_1$ is $(1/h)\,Y$-periodic and

$$\int_Y D\tilde{u}_1\,dx = h^n \int_{(1/h)Y} D\tilde{u}_1\,dx = z\,.$$

Let us prove that $\tilde{u}_1$ is also a solution of $M_h(z)$. In fact, using the convexity assumption on $f(x, \cdot)$,

$$\int_Y f(hx, D\tilde{u}_1) \leqslant h^{-n} \sum_{i_1, \ldots, i_n = 0}^{h-1} \int_Y f\left(hx, Du_1\left(x_1 + \frac{i_1}{h}, \ldots, \frac{i_n}{h}\right)\right) dx \tag{7.22}$$

and, $\forall (i_1, \ldots, i_n)$ it is possible to prove

$$\int_Y f\left(hx, Du_1\left(x_1 + \frac{i_1}{h}, \ldots, x_n + \frac{i_n}{h}\right)\right) = \int_Y f(hx, Du_1(x))\,. \tag{7.23}$$

To prove (7.23) we make a change of variables in the first integral of (7.23)

$$x_k \to x_k - i_k/h\,,$$

put $i = (i_1, \ldots, i_n)$ and observe that

$$\int_Y f\left(hx, Du_1\left(x_1 + \frac{i_1}{h}, \ldots, x_n + \frac{i_n}{h}\right)\right) = \int_{Y+i/h} f(hx - i, Du_1(x)) =$$

$$= \int_{Y+i/h} f(hx, Du_1(x)) = \int_Y f(hx, Du_1(x))\,.$$

Furthermore, as $D\tilde{u}_1$ is $(1/h)\,Y$-periodic:

$$M_h(z) = \int_Y f(hx, D\tilde{u}_1(x))\,dx = h^n \int_{(1/h)Y} f(hx, D\tilde{u}_1(x))\,dx = \tag{7.24}$$

$$= \int_Y f\left(y, D_x\tilde{u}_1\left(\frac{y}{h}\right)\right) dy = \int_Y f(y, D_y\tilde{u}(y))\,dy$$

where we have set

$$\tilde{u}(y) = h\tilde{u}_1\left(\frac{y}{h}\right) \qquad \forall y \in Y.$$

By (7.24) we obtain

$$M_h(z) \geqslant M_1(z)$$

if we check that

$$\int_Y D_y\tilde{u}(y)\,dy = z\,.$$

To this aim observe that $D\tilde{u}$ is Y-periodic, as $D\tilde{u}_1$ is $(1/h)\,Y$-periodic, and

$$\int_Y D_y\,\tilde{u}(y)\,dy = \int_Y D_y h\tilde{u}_1\left(\frac{y}{h}\right)dy = \int_Y D_x u_1\left(\frac{y}{h}\right)dy = h^n \int_{(1/h)Y} D_x\,\tilde{u}_1(x)\,dx = z\,.$$

Let us now show that

$$M_1(z) \geqslant M_h(z)\,.$$

Let $\tilde{u}_2$ be a solution of $M_1(z)$ and set

$$u_2 = \tilde{u}_2 + \langle z, x\rangle\,, \qquad \tilde{u}(y) = \frac{1}{h}\,u_2(hy) \qquad y \in \left]0, \frac{1}{h}\right[^n$$

then extend $\tilde{u}$ on Y by periodicity [7] so that

$$\int_Y D_y\tilde{u}(y)\,dy = h^n \int_{(1/h)Y} D_y\frac{1}{h}\,(u_2(hy))\,dy = \int_Y D_x u_2(x)\,dx\,.$$

---

[7] As Du_2 is Y-periodic, then $u_2(x) - \langle z, x\rangle$ is Y-periodic, and so $\tilde{u}(x) - \langle z, x\rangle$ is $1/h$ Y-periodic. Let us extend this function by periodicity to Y and continue to denote it by $\tilde{u} - \langle z, x\rangle$.

Moreover

$$\int_Y f(x, D_x u_2(x))\, dx = h^n \int_{(1/h)Y} f(hy, D_x u_2(hy)\, dy =$$

$$= h^n \int_{(1/h)Y} f\left(hy, D_y \frac{1}{h} u_2(hy)\right) dy = \int_Y f(hy, D_y u(y))\, dy$$

and so we have the result.

COROLLARY 7.7. – *If $f(x, \cdot)$ is strictly convex, then the solution u_1^h of $M_h(z)$ is $(1/h)$ Y-periodic and $h \cdot u_1^h(y/h)$ is the solution of $M_1(z)$. Moreover $\forall h, k \in N$:*

$$u_1^k\left(\frac{y}{k}\right) = \frac{h}{k} u_1^h\left(\frac{y}{h}\right).$$

PROOF. – It suffices to make in the proof of theorem 7.6 the assumption of the uniqueness of the solution of $M_h(z)$ for any h.

8. – Limit cases.

In this section we want to study the case of the « explosions » of the coefficients of Dirichlet type functionals around a one dimensional or a bidimensional surface in R^3.

We show (Theo. 8.3) that without the assumption of uniform integrability of the functions a_h one can have

$$\Gamma(N, C^0(\Omega)) \lim_{\substack{h\to\infty \\ v\to u}} \int_\Omega a_h Dv^2\, dx \neq \Gamma(N, L^1(\Omega)) \lim_{\substack{h\to\infty \\ v\to u}} \int_\Omega a_h Dv^2\, dx$$

and also that the first limit is not the trace on Ap_n of a measure, as a function of Ω (Prop. 8.4 and Prop. 8.5).

To this aim set (similarly as in [29])

$$\Omega_0 = \{(x_1, x_2, y) \in R^3 \colon |y| < 1,\ x_1^2 + x_2^2 < 1,\ x_2 > 0\}$$
$$B_h = \{(x_1, x_2, y) \in \Omega_0 \colon x_1^2 + x_2^2 < 1/h\}$$
$$\omega = \{(0, 0)\} \times]-1, 1[\,, \qquad \text{[see fig. 1]}$$

and define, with $x = (x_1, x_2)$, $dx = dx_1\, dx_2$

$$a_h(x) = \begin{cases} h^2 & \text{if } (x, y) \in B_h \\ 1 & \text{if } (x, y) \notin B_h \end{cases}$$

$$F_h(\Omega, u) = \int_\Omega a_h |Du|^2\, dx\, dy\,, \qquad u \in \mathrm{Lip}\,(\Omega)\,.$$

We begin with the following

LEMMA 8.1. - *For any* $u \in C^1$

$$\int_{\Omega_0} |Du|^2 dx\, dy + \frac{\pi}{2} \int_{-1}^{1} |Du(0, 0, y)|^2 dy = \lim_h F_h(\Omega_0, u)$$

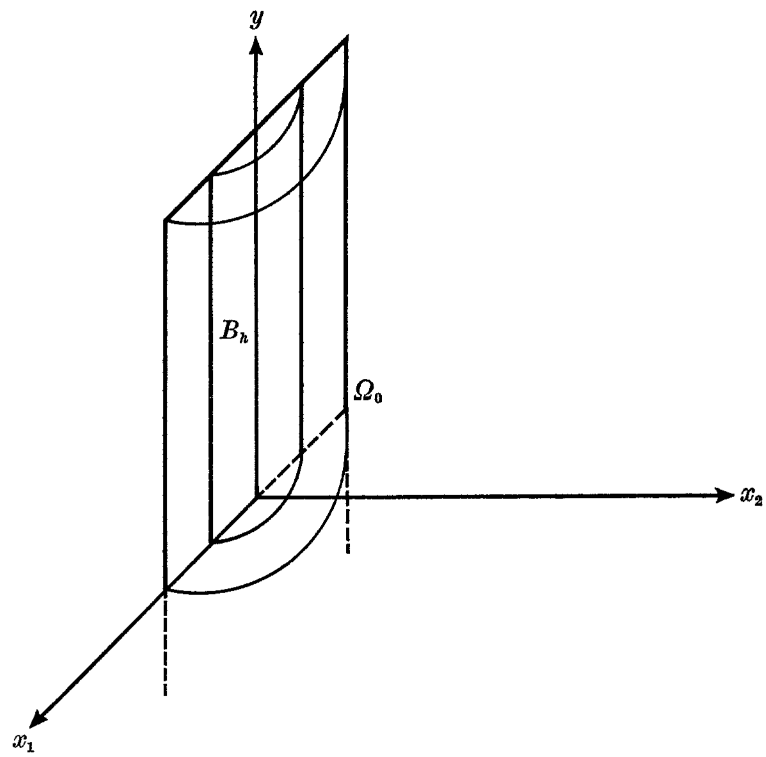

Figure 1

PROOF. - It suffices to observe that

$$F_h(\Omega_0, u) = h^2 \int_{B_h} |Du|^2 dx\, dy + \int_{\Omega_0 - B_h} |Du|^2 dx\, dy =$$

$$= \frac{1}{\pi/2h^2} \frac{\pi}{2} \int_{B_h \cap \{y=0\}} \left(\int_{-1}^{1} |Du(x, y)|^2 dy \right) dx + \int_{\Omega - B_h} |Du|^2 dx\, dy,$$

and so, as $h \to \infty$ we have the assertion.

Let us now prove the

LEMMA 8.2. – *For any* $u \in C^1$ *there exists* $u_h \in \mathrm{Lip}(\Omega_0)$ *such that:*

$$u_h \to u\,, \quad \text{in } C^0(\Omega_0) \tag{8.1}$$

and

$$\int_{\Omega_0} |Du|^2\,dx\,dy + \frac{\pi}{2}\int_{\omega}\left|\frac{\partial u}{\partial y}(0,0,y)\right|^2 dy = \lim_h F_h(\Omega_0, u)\,. \tag{8.2}$$

PROOF. – Let us define, in polar coordinates

$$u_h(\varrho,\theta,y) = \begin{cases} u(\varrho,\theta,y) & 2/h < \varrho < 1 \\ u(0,0,y) & 0 < \varrho < 1/h \\ \left[u\left(\dfrac{2}{h},\theta,y\right) - u(0,0,y)\right]h\varrho + 2u(0,0,y) - u\left(\dfrac{2}{h},\theta,y\right) & 1/h < \varrho < 2/h. \end{cases}$$

We have clearly (8.1). Moreover,

$$\begin{aligned} F_h(\Omega_0, u_h) &= \int_{\varrho \geq 2/h} |Du|^2\,dx\,dy + h^2 \int_{-1}^{1}\int_{0<\varrho<1/h}\left|\frac{\partial u}{\partial y}(0,0,y)\right|^2 dx\,dy + \\ &+ \int_{-1}^{1}\int_{1/h<\varrho<2/h} |Du_h|^2\,dx\,dy = \int_{\varrho\geq 2/h}|Du|^2\,dx\,dy + \frac{\pi}{2}\int_{-1}^{1}\left|\frac{\partial u}{\partial y}(0,0,y)\right|^2 dy + \\ &+ \int_{-1}^{1} dy \int_{1/h}^{2/h} d\varrho \int_0^{\pi}\left[\left|\frac{\partial u_h}{\partial \varrho}\right|^2 + \frac{1}{\varrho^2}\left|\frac{\partial u_h}{\partial \theta}\right|^2 + \left|\frac{\partial u_h}{\partial y}\right|^2\right]\varrho\,d\varrho\,d\theta = a_h + b + c_h\,. \end{aligned} \tag{8.3}$$

Let us indicate by k the Lipschitz constant of u on Ω_0. We have $\forall \varrho \in]1/h, 2/h[$

$$\left|\frac{\partial u_h}{\partial y}(\varrho,\theta,y)\right|^2 \leq 8h^2\varrho^2 k^2 + 18k^2$$

$$\left|\frac{\partial u_h}{\partial \varrho}(\varrho,\theta,y)\right|^2 \leq 4k^2$$

$$\left|\frac{\partial u_h}{\partial \theta}(\varrho,\theta,y)\right|^2 \leq k^2\varrho^2 \; (^8)$$

so that

$$c_h = o\left(\frac{1}{h}\right) \quad \text{as } h \to \infty\,.$$

(8) We use the fact that if $u(\theta, \varrho)$ has a Lipschitz constant equal to k, then $\left|\dfrac{\partial u}{\partial \theta}(\varrho,\theta)\right| \leq k\varrho$.

We have as a consequence the

THEOREM 8.3. – *For any* $u \in C^1$

$$F(\Omega_0, u) = \int_{\Omega_0} |Du|^2 dx\, dy + \frac{\pi}{2}\int_{\omega}\left|\frac{\partial u}{\partial y}(0, 0, y)\right|^2 dy = \Gamma\,(N, C_0(\Omega_0)^-)\lim_{\substack{h\to\infty\\ v\to u}} F_h(\Omega_0, v)\,. \tag{8.4}$$

PROOF. – It suffices to prove that

$$u_h \to u \quad \text{in} \quad C^0(\Omega_0) \Rightarrow F(\Omega_0, u) \leqslant \liminf_h F_h(\Omega_0, u_h)\,. \tag{8.5}$$

Now, for any k

$$\liminf_h \int_{\Omega_0 - B_k} a_h |Du_h|^2 dx\, dy \geqslant \liminf_h \int_{\Omega_0 - B_k} a_h |Du_h|^2 dx\, dy \geqslant \int_{\Omega_0 - B_k} |Du|^2 dx\, dy\,,$$

and so, as $k \to \infty$

$$\liminf_h \int_{\Omega_0 - B_h} a_h |Du_h|^2 dx\, dy \geqslant \int_{\Omega_0} |Du|^2 dx\, dy\,. \tag{8.6}$$

Furthermore

$$\int_{B_h} a_h |Du_h|^2 dx\, dy = \int_{-1}^{1} dy \int_{\varrho<1/h} h^2 |Du_h|^2 dx \geqslant h^2 \int_{\varrho<1/h} dx \int_{-1}^{1} |D_y u_h|^2 dy\,. \tag{8.7}$$

Let us now divide $(-1, 1)$ with the points $a_0 = -1 < a_1 < \ldots < a_k = 1$ such that $a_i - a_{i-1} = \text{const.}$ and let $\sigma_h \to 0$ be such that

$$|u(x, a_i) - u_h(x, a_i)| < \sigma_h\,, \qquad \forall x, \forall i\,.$$

Then

$$\left|\frac{u_h(x, a_i) - u_h(x, a_{i-1})}{a_i - a_{i-1}}\right| = m_i(x) + \frac{\eta(\sigma_h, x)}{a_i - a_{i-1}}$$

where

$$m_i(x) = \left|\frac{u(x, a_i) - u(x, a_{i-1})}{a_i - a_{i-1}}\right|$$

and uniformly in x

$$\lim_{t\to 0} |\eta(t, x)| = 0\,.$$

We have

$$\sum_{i=1}^{k}\int_{a_{i-1}}^{a_i}|Du_h|^2\,dy \geqslant \sum_{i=1}^{k}\left(m_i(x)+\frac{\eta(\sigma_h,x)}{a_i-a_{i-1}}\right)^2(a_i-a_{i-1})$$

and, after an integration,

$$\int_{\varrho<1/h} h^2\,dx\int_{-1}^{1}|D_y u_h|^2\,dy \geqslant h^2\sum_{i=1}^{k}(a_i-a_{i-1})\int_{\varrho<1/h}\left[m_i^2(x)+\frac{2m_i(x)\,\eta(\sigma_h,x)}{a_i-a_{i-1}}+\frac{\eta^2(\sigma_h,x)}{(a_i-a_{i-1})^2}\right]dx\,,$$

and then as $h\to\infty$, since $\eta(\sigma_h,x)\to 0$

$$\liminf_h \int_{\varrho<1/h} h^2\,dx\int_{-1}^{1}|D_y u_h|^2\,dy \geqslant \liminf_h h^2\sum_{i=1}^{k}(a_i-a_{i-1})\int_{\varrho<1/h} m_i^2(x)\,dx=\sum_{i=1}^{k}(a_i-a_{i-1})\frac{\pi}{2}\,m_i^2(0)\,.$$

Passing to the limit as $a_i-a_{i-1}\to 0$

$$\liminf_h h^2\int_{\varrho<1/h} dx\int_{-1}^{1}|D_y u_h|^2\,dy \geqslant \frac{\pi}{2}\int_{-1}^{1}|D_y u(0,y)|^2\,dy$$

using the continuity of $D_y u$. And the proof is complete.

The difference between $\Gamma(C^0)$ and $\Gamma(L^1)$ is clarified by the

PROPOSITION 8.4. – *For any* $u\in C^1$

$$\int_{\Omega_0}|Du|^2\,dx\,dy = \Gamma(N, L^1(\Omega_0))\lim_{\substack{h\to\infty\\ v\to u}} F_h(\Omega_0, v)\,.$$

PROOF. – See [29].

REMARK. – Let us observe that in this example $F(\Omega, u)$ of (8.4) is not the trace on Ap_n of a measure. In fact if we choose the sequence

$$I_k=\Omega_0-\bar{B}_k\,,$$

we have that it is increasing to Ω_0, satisfies

$$F(I_k, u)=\int_{I_k}|Du|^2\,dx\,dy$$

while

$$F(\Omega_0, u) = \int_{\Omega_0} |Du|^2 dx\,dy + \frac{\pi}{2}\int_{\omega} \left|\frac{\partial u}{\partial y}(0, 0, y)\right|^2 dy\,.$$

Let us now consider the sequence of functions defined on $C =]0, 1[^3$ by

$$a_h(x, y, z) = \begin{cases} h & (x, y, z) \in C_h \\ 1 & \text{otherwise} \end{cases}$$

where (see fig. 2)

$$C_h = \{(x, y, z) \in C : z < 1/h\}$$

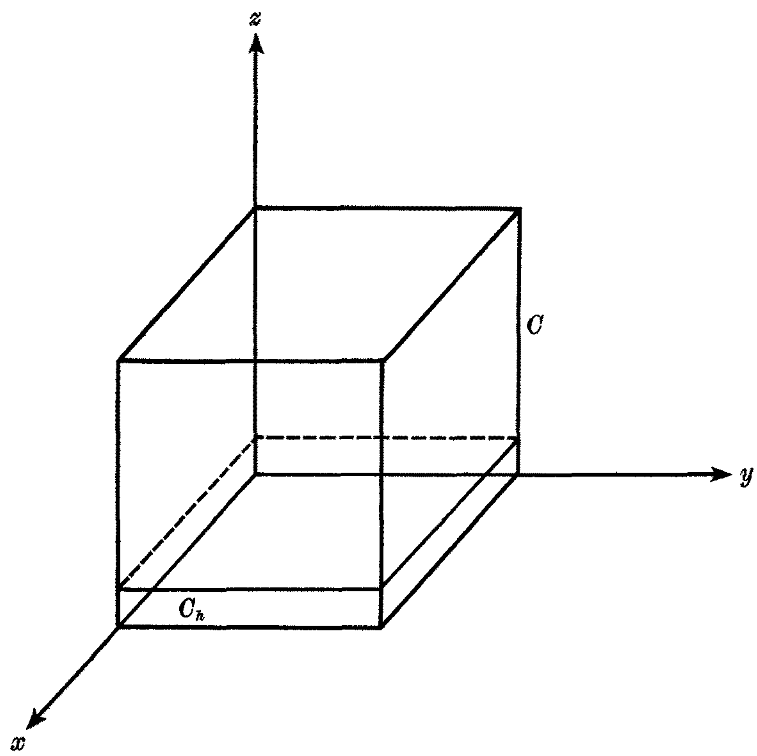

Figure 2

and the functionals

$$F_h(u) = \int_C a_h |Du|^2 dx\,dy\,dz\,, \qquad u \in \mathrm{Lip}\,(C)\,.$$

It is easy to check, similarly as in Lemma 8.1, that

$$\int_C |Du|^2\,dx\,dy\,dz + \int_S |Du(x, y, 0)|^2\,dx\,dy = \lim_h F_h(u) \qquad \forall u \in C^1$$

where $S =]0, 1[^2 \times \{0\}$.

Let us remind that the trace operator on $H^1(C)$ is completely continuous, so that:

$$\sup_z \|u(x, y, z)\|_{L^2(S)} \leqslant k \|u\|_{H^1(C)}\,; \tag{8.8}$$

(8.9) if L is bounded in H^1, then for any $u \in L$:

$$\|u(x, y, z_1) - u(x, y, z_2)\|_{L^2(S)} \leqslant \omega_L(|z_1 - z_2|)\,,$$

with $\lim_{\varrho \to 0} \omega_L(\varrho) = 0$;

$$u_h \to u \quad \text{weakly in} \quad H' \Rightarrow \sup_z \|u_h(x, y, z)\|_{L^2(S)} \to 0\,. \tag{8.10}$$

Let us define for any $u \in H^1(C)$:

$$\tilde{u}(x, y) = h \int_0^{1/h} u(x, y, z)\,dz\,.$$

LEMMA 8.5. – *If $u_h \to u$ in $H^{1,2}(C)$ then $\tilde{u}_h \to u(x, y, 0)$ weakly in $L^2(S)$.*

PROOF. – Let us prove first that:

$$\|\tilde{u}_h\|_{L^2(S)} \leqslant M\,. \tag{8.11}$$

We have:

$$\int_S |\tilde{u}_h|^2\,dx\,dy = \int_S h^2 \Big(\int_0^{1/h} u_h(x, y, z)\,dz\Big)^2 dx\,dy \leqslant h \int_0^{1/h} dz \int_S |u_h(x, y, z)|^2\,dx\,dy\,.$$

From (8.8) we deduce (8.11). Let u^* be a weak limit of $\tilde{u}_h$ with respect to $L^2(S)$. We have:

$$u(x, y, 0) - u^*(x, y) = u(x, y, 0) - u_h(x, y, 0) + u_h(x, y, 0) - \\ - \tilde{u}_h(x, y) + \tilde{u}_h(x, y) - u^*(x, y)\,. \tag{8.12}$$

By (8.10) the first term goes to zero, the third one tends to zero weakly in L^2:

let us estimate the second one using (8.9):

$$\int_S |u_h(x,y,0)-\tilde u_h(x,y)|^2\,dx\,dy=\int_S |u_h(x,y,0)-h\int_0^{1/h} u_h(x,y,z)\,dz|^2\,dx\,dy=$$
$$=\int_S h^2\left|\int_0^{1/h}[u_h(x,y,0)-u_h(x,y,z)]\,dz\right|^2 dx\,dy\leqslant h\int_S dx\,dy\int_0^{1/h}|u_h(x,y,0)-u_h(x,y,z)|^2\,dz\leqslant$$
$$\leqslant \sup_{z\in]0,1/h[}\sqrt{\int_S |u_h(x,y,0)-u_h(x,y,z)|^2\,dx\,dy}$$

and so also the second term tends to zero.

LEMMA 8.6. – *If* $u_h\to u$ *weakly in* $H^1(C)$ *and*

$$\int_C a_h|Du_h|^2\,dx\,dy\,dz\leqslant M\,,$$

then

$$\tilde u_h\to u(x,y,0)\qquad \textit{weakly in } H^1(S)\,.$$

PROOF. – We have:

$$\int_S |D\tilde u_h|^2\,dx\,dy=\int_S\left[\left\{h\int_0^{1/h}\frac{\partial u_h}{\partial x}\,dz\right\}^2+\left\{\int_0^{1/h}\frac{\partial u_h}{\partial y}\,dz\right\}^2\right]dx\,dy\leqslant \tag{8.13}$$
$$\leqslant\int_S\left(\left\{h\left[\int_0^{1/h}\left|\frac{\partial u_h}{\partial x}\right|^2dx\right]^{\frac12}\frac{1}{\sqrt h}\right\}^2+\left\{h\left[\int_0^{1/h}\left|\frac{\partial u_h}{\partial y}\right|^2dz\right]^{\frac12}\frac{1}{\sqrt h}\right\}^2\right)dx\,dy\leqslant$$
$$\leqslant\int_S h\int_0^{1/h}|Du_h|^2\,dx\,dy\,dz\leqslant\int_{C-C_h}a_h|Du_h|^2\,dx\,dy\,dz\leqslant M\,.$$

By Lemma 8.5 we have the result.

LEMMA 8.7. – *If* $u_h\to u$ *in* $L^2(C)$ *and*

$$\liminf_h\int_C a_h|Du_h|^2\,dx\,dy\,dz<+\infty\,,$$

then

$$\int_S\left[\left|\frac{\partial u}{\partial x}(x,y,0)\right|^2+\left|\frac{\partial u}{\partial y}(x,y,0)\right|^2\right]dx\,dy+\int_S|Du|^2\,dx\,dy\,dz\leqslant\liminf_h\int_C a_h|Du_h|^2\,dx\,dy\,dz\,.$$

PROOF. – Using (8.13) and lemma 8.6 we have:

$$\liminf_h \int_C a_h |Du_h|^2\, dx\, dy\, dz \geqslant \liminf_h \int_C |Du_h|^2\, dx\, dy\, dz + \liminf_h \int_{C-C_h} h\left[\left|\frac{\partial u_h}{\partial x}\right|^2 + \left|\frac{\partial u_h}{\partial y}\right|^2\right] dx\, dy\, dz \geqslant$$

$$\geqslant \int_C |Du|^2\, dx\, dy\, dz + \liminf_h \int_S \left[\left|\frac{\partial \tilde{u}_h}{\partial x}\right|^2 + \left|\frac{\partial \tilde{u}_h}{\partial y}\right|^2\right] dx\, dy \geqslant$$

$$\geqslant \int_C |Du|^2\, dx\, dy\, dz + \int_S \left[\left|\frac{\partial u}{\partial x}(x, y, 0)\right|^2 + \left|\frac{\partial u}{\partial y}(x, y, 0)\right|^2\right] dx\, dy\,.$$

We have the following:

THEOREM 8.8. – *For any* $u \in C^1$:

$$F(u) = \int_C |Du|^2\, dx\, dy\, dz + \int_S \left[\left|\frac{\partial u}{\partial x}(x, y, 0)\right|^2 + \left|\frac{\partial u}{\partial y}(x, y, 0)\right|^2\right] dx\, dy =$$

$$= \Gamma(N, C^0(C)^-) \lim_{\substack{h\to\infty \\ v\to u}} F_h(v) = \Gamma(N, L^1(\Omega)^-) \lim_{\substack{h\to\infty \\ v\to u}} F_h(v)\,.$$

PROOF. – For any $u \in C^1$ set

$$u_h(x, y, z) = \begin{cases} u(x, y, 0) & 0 < z < 1/h \\ u(x, y, z) & 2/h < z < 1 \\ \left[u\left(x, y, \dfrac{2}{h}\right) - u(x, y, 0)\right] hz + 2u(x, y, 0) - u\left(x, y, \dfrac{2}{h}\right) & 1/h < z < 2/h\,. \end{cases}$$

It is easy to show that:

$$u_h \to u \quad \text{in } C_0(C)\,,$$

and:

$$F(u) = \lim F_h(u_h)\,.$$

By Lemma (8.7) we have the result.

Compare this kind of result with those of [41]. □

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