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Some Properties of Graphs of Diameters*

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Abstract. The main result of this paper is as follows. Any two cycles of odd lengths of the graph of diameters G in three-dimensional Euclidean space have a common vertex. Some properties of graphs of diameters in two-dimensional Banach spaces with strictly convex metrics are also established. Applications are given.

Definition. Let V be a set in a metric space. We assign the following graph G to the set V. The vertices of $G = (V, E)$ are the points of V. Two vertices x_1, x_2 are adjacent ($x_1 x_2 \in E$) iff the distance between x_1, x_2 equals the diameter of the set V. This graph is called *the graph of diameters of* V (see [7], [8], and [11]). A segment with the ends $x_1, x_2 \in V$ in a Banach space B^n is called a diameter too if its length is equal to the diameter of V. In what follows, only sets of diameter one are considered.

Graphs of diameters were investigated in many papers in connection with the famous conjecture of Borsuk [2] (an excellent survey of the literature on Borsuk's conjecture is Grünbaum's paper [11]). In spite of the fact that at present this conjecture is disproved in large dimensions [15], it is probable that in small dimensions, for instance $n = 4$, this conjecture is true. The proof for $n = 3$ was given by Eggleston [6]. Other simple proofs for $n = 3$ were given by Grünbaum [10] and Heppes [12] (for $n = 2$ see [11]). However, the research of graphs of diameters represents an independent interest, but still there are no approaches to full description of these graph. Note that for two-dimensional Euclidean space this problem has a relatively easy solution (see [3]).

In this paper some properties of graphs of diameters in certain two-dimensional Banach spaces and in three-dimensional Euclidean space are presented.

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Let B^2 be a two-dimensional Banach space with strictly convex metric.

Theorem 1. *If G is the graph of diameters of a set V in B^2, then there exists $x \in V$ such that the graph of diameters G' of $V \backslash \{x\}$ is bipartite.*

To prove Theorem 1, we need a lemma.

Lemma. *Let V be a finite subset of B^2, then any two diameters of V have a common point.*

Proof. Denote the closed ball and the sphere with center x and of radius r by $B(x, r)$ and $S(x, r)$. First we show that if $V \subset B^n$ and x, y are the endpoints of a diameter of V, then there exist two parallel supporting hyperplanes of V passing through x and y, respectively.

Consider the ball $B(x, 1)$ and let π_1 be a supporting hyperplane of $B(x, 1)$ passing through y. Since V is the subset of $B(x, 1)$, we see that π_1 is a supporting hyperplane of V. Similarly, there exists a supporting hyperplane π_2 of $B(y, 1)$ passing through x such that π_1, π_2 are parallel.

Consider two diameters $[x, y]$ and $[a, b]$ of V. Assume $[x, y] \cap [a, b] = \emptyset$. Arguing as above, we see that there exist two parallel supporting lines π_1, π_2 of V passing through x and y, respectively, and two parallel supporting lines π_1', π_2' of V passing through a and b, respectively.

We consider two cases.

Case 1: *π_1 and π_1' are nonparallel*. Then the lines $\pi_1, \pi_2, \pi_1', \pi_2'$ form a parallelogram. The points x, y and a, b belong to the opposite sides of this parallelogram. Consequently, $[x, y]$ and $[a, b]$ have a common point.

Case 2: *if they are parallel, then $\pi_1 = \pi_1'$ and $\pi_2 = \pi_2'$*. We may assume that x, $a \in \pi_1$ and $y, b \in \pi_2$. Since π_1 is the supporting line of $B(y, 1)$ at the point x, we have $\|a - y\| \geq 1 \Rightarrow \|a - y\| = 1$. Hence, if $z \in [x, a]$, then $\|z - y\| = 1$. This contradicts the condition of the strict convexity for B^2. The lemma is proved. □

Proof of Theorem 1. In other words, we must prove that all cycles of odd lengths have a common vertex. Without loss of generality, it can be assumed that V is a finite set in B^2. It is easy to prove that there exists a direction e such that any line $\pi \parallel e$ intersects V in at most one point.

Evidently, there exists a line $\pi \parallel e$ such that π intersects all diameters of the set V. Denote by P_1 and P_2 two open half-planes, defined by the line π. Then we have

$$\operatorname{diam}(P_1 \cap V) < 1, \qquad \operatorname{diam}(P_2 \cap V) < 1, \qquad |\pi' \cap V| \leq 1.$$

The theorem is proved. □

Theorem 1 immediately implies the following two corollaries.

Corollary 1. *For any finite set $V \subset B^2$ there exists $x \in V$ such that the set $V \backslash \{x\}$ may be divided into two parts of smaller diameter.*

Corollary 2. *For any finite set $V \subset B^2$ there exists $U \subset V$ of smaller diameter such that $|U| \geq (|V| - 1)/2$.*

If V is the set of vertices of a regular $(2m+1)$-gon and $U \subseteq V$ such that $|U| \geq |V|/2$, then, evidently, $\operatorname{diam} U = \operatorname{diam} V = 1$. This means that the inequality in Corollary 2 is exact.

The following result is an analogue of Theorem 1 for three-dimensional Euclidean space E_3.

Theorem 2. *If G is a graph of diameters in E_3, then any two cycles of odd lengths have a common vertex.*

Proof. Consider a finite set $V \subset E_3$. There is a set W of constant width 1 such that $V \subset W$ (see [1]).

Let $C = \{x_1, x_2, \ldots, x_{2m+1}\}$, $x_i x_{i+1} \in E$, $1 \leq i \leq 2m+1$, be a cycle of length $2m+1$ in the graph of diameters G of the set V. Suppose x_{i-1}, x_i, x_{i+1} are three successive vertices on this cycle; then $\|x_{i-1} - x_i\| = \|x_{i+1} - x_i\| = 1$. Take the arc $\alpha_i(C)$ of the circle of center x_i and radius 1 between x_{i-1} and x_{i+1} of measure $< \pi$. Since W is the set of constant width, we see that $\alpha_i(C) \subset \operatorname{bd} W$ ($\operatorname{bd} W$ is the boundary of W).

It is easy to check that the union of all such arcs $\bigcup_{i=1}^{2m+1} \alpha_i(C)$ for the given odd cycle C is the closed curve $\gamma(C) \subset \operatorname{bd} W$. If $x \in \alpha_i(C)$, then the segment $[x_i, x]$ is a diameter. Therefore the vectors $\overline{x_i x}$ and $\overline{x x_i}$ are the unit normal vectors of the supporting hyperplanes of W at the points t and y.

Choose an origin O in the space E_3. Selecting O as the initial point of all vectors $\overline{x_i x}$ for $x \in \alpha_i(C)$, we denote by $\alpha_i^+(C)$ the set of endpoints of these vectors. Similarly, denote by $\alpha_i^-(C)$ the set of endpoints of $\overline{x x_i}$, $x \in \alpha_i(C)$.

Since the sets $\alpha_i^+(C)$ and $\alpha_i^-(C)$ consist of unit vectors, we see that $\alpha_i^+(C)$ and $\alpha_i^-(C)$ are two centrally symmetric arcs of the unit sphere $S(O, 1)$. For any cycle C of length $2m+1$ in the graph of diameters G, define

$$S(C) = \bigcup_{i=1}^{2m+1} \alpha_i^+(C) \cup \alpha_i^-(C).$$

It is easy to show that $S(C)$ is a closed, centrally symmetric curve without self-intersections consisting of $2(2m+1)$ arcs of a circle of radius 1.

Consider two cycles of odd lengths $C_1 = \{x_1, x_2, \ldots, x_{2m+1}\}$ and $C_2 = \{y_1, y_2, \ldots, y_{2k+1}\}$ of the graph G. The corresponding curves $S(C_1)$ and $S(C_2)$ are homeomorphic to the circle, have no self-intersections, and are centrally symmetric. Using Jordan's theorem, we get that there exists $x \in S(C_1) \cap S(C_2)$. We consider two cases.

Case 1. Suppose x is a common point for two arcs $\alpha_i^+(C_1)$ and $\alpha_j^+(C_2)$; then the vector Ox is the normal vector of the supporting hyperplane for a set W at the points x_i and y_j. Therefore x_i, y_j belong to the same supporting hyperplane for W. Since a set of constant width is strictly convex, we see that $x_i = y_j$, and the theorem is proved in this case.

Case 2. Now suppose that $x \in \alpha_i^-(C_1) \cap \alpha_j^+(C_2)$; then $x_i \in \alpha_j(C_2)$ and $y_j \in \alpha_i(C_1)$. Assume $x_i \neq y_{j-1}, y_{j+1}, y_j \neq x_{i-1}, x_{i+1}$. Denote the planes passing through the points x_{i-1}, x_i, x_{i+1} and y_{j-1}, y_j, y_{j+1} by π_1 and π_2, respectively. Now suppose π_1 and π_2 are not perpendicular. By p_{π_1} denote the orthogonal projection of E_3 onto π_1. Then $p_{\pi_1}(W)$ is the set of constant width one too. Since $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ are diameters of $p_{\pi_1}(W)$, we see that x_i is a corner point of $p_{\pi_1}(W)$.

On the other hand, x_i is contained in the interior of $p_{\pi_1}(\alpha_j(C_2)) \subset p_{\pi_1}(W)$. The set $p_{\pi_1}(\alpha_j(C_2))$ is an arc of an ellipse. Hence the point x_i is not a corner point. This contradiction proves the theorem for this case.

Now consider the case when π_1 and π_2 are perpendicular. Then the point y_j is contained in the interior of $\alpha_i(C_1)$. Let t be the midpoint of $\alpha_i(C_1)$, and let α, β be the measures of $\angle x_{i-1}x_i t$, $\angle x_{i-1}x_i y_j$, respectively. Then $\alpha > \beta$ and $\cos\alpha < \cos\beta$.

Consider the plane π passing through t and x_i and perpendicular to π_1. Since $W \subseteq B(x_{i-1}, 1) \bigcap B(x_{i+1}, 1)$, we have

$$p_\pi(W) \subseteq \pi B(x_{i-1}, 1) \cap B(x_{i+1}, 1) \cap \pi = B.$$

Clearly, B is a disk of radius $\cos\alpha$ with center at the point $z = (x_{i-1} + x_{i+1})/2 \in \pi$. Let S be its boundary circle. The set $\delta = p_\pi(\alpha_j(C_2))$ is an arc of an ellipse with semiaxes 1, $\cos\beta$, and center on a segment $[t, x_i]$. The circle S must touch δ at the point x_i.

Since $\cos\alpha > \cos\beta$, we obtain that the arc δ is situated outside B in some neighborhood of the point x_i. This contradiction proves the theorem. □

Remark. Combining the proof of Theorem 1 and the statement of Theorem 2, it is easy to obtain that for any finite set V in E_3 there exists $x \in V$ such that the set $V \backslash \{x\}$ may be divided into four parts of smaller diameters.

However, Theorem 2 leads to a stronger result. Namely, it implies an easy new solution of Borsuk's problem for finite subsets of E_3.

Corollary 3. *If V is a finite set in E_3, then it may be divided into four parts of smaller diameters.*

Proof. In other notation, we must prove that the graph of diameters G of a set V in E_3 has chromatic number $\chi(G) \leq 4$. If G contains no cycles of odd length, then, by König's theorem about bipartite graphs, we have $\chi(G) \leq 2$, and all is proved.

Therefore we assume that G contains cycles of odd length. Let $C = \{x_1, x_2, \ldots, x_{2m+1}\}$, $x_i x_{i+1} \in E$, $1 \leq i \leq 2m+1$, be a cycle of the least odd length $2m+1$ in G. It is easy to prove that if $|i - j| \geq 2$, then x_i and x_j are not adjacent in G. Therefore $\chi(C) = 3$. We color the vertices of the cycle C by the colors 1, 2, 3. Let C_1, C_2, C_3 be the color classes of C.

By Theorem 2, it follows that the graph of diameters G_1 of $V \backslash C$ contains no cycles of odd length. Consequently $\chi(G_1) \leq 2$. Let V_1 and V_2 be the color classes of G_1.

Now we prove that only one vertex $y \in G_1$ may be adjacent to three vertices of C and that any other vertex $x \in G_1$ is adjacent to at most two vertices of C.

Indeed, suppose that there exists a vertex $y \in G_1$ adjacent to three vertices x_i, x_j, x_k, $i < j < k$, of the cycle C. Trivially, $(j-i)+(k-j)+(2m+1+i-k) = 2m+1$ and therefore one of the numbers $j-i, k-j, 2m+1+i-k$ is odd. Let $l = j-i$ be the least odd number of the set $\{j-i, k-j, 2m+1+i-k\}$.

We consider two cases.

Case 1: $l = 1$. Then $m = 1$ and G contains three mutually adjacent vertices $C = \{x_1, x_2, x_3\} \subseteq V$. Obviously, only one vertex $y \in G_1$ may be joined by an edge in G with all vertices of C.

Case 2: $l \geq 3$. Then $l \leq 2m+1-2-2 = 2m-3$. Take the vertices $y, x_i, x_{i+1}, \ldots, x_k$. We get a cycle of odd length $\leq 2m-1$. This contradiction proves the statement in this case.

Assume, without loss of generality, that V_2 contains no vertex y adjacent to three vertices of C. We color all vertices of V_1 by the color 4. Since any vertex $x \in V_2$ is adjacent to at most two vertices of C, we see that x is adjacent to no vertex of a certain color class C_i of C. We color x by the color i. Finally, we obtain the 4-coloring of G. The corollary is proved. □

Remark 1. Let C be an odd cycle $C = \{x_1, x_2, \ldots, x_{2m+1}\}$ of minimal length > 3 in an arbitrary graph G. Suppose that a vertex x is adjacent to two vertices x_i, x_j, $i < j$, of a cycle C; then it is easy to see that either $j-i = 2$ or $2m+1+i-j$.

Remark 2. A sketch of the proofs of Theorem 2 and Corollary 3 was given in [4]. Other proofs of Borsuk's conjecture for finite sets in E_3, based on the properties of graphs of diameters, were given by Grünbaum [9], Heppes and Révész [13], [14], and Straszewicz [16]. The proofs are based on the inequality $|E| \leq 2|V| - 2$, which holds for any finite graph of diameters $G = (V, E)$ in E_3.

Remark 3. Arguing as above, it is not hard to prove that if $|T_1 \cap T_2| \leq 2$ for every two complete subgraphs on four vertices T_1 and T_2 of a simple graph G and if for any k cycles of odd lengths in G there exist two cycles having a common vertex, then $\chi(G) \leq 2k+2$.

Remark 4. Reasoning as in the proof of Theorem 2, it is easy to conclude that any cycle of odd length in a graph of diameters in B^2 (where B^2 is a two-dimensional Banach space with strictly convex metric) and any edge of the graph have a common vertex.

Corollary 4. *If a set $V \subseteq E_3$ (finite or infinite) has diameter d, then there exists such an integer m that in any finite subset $W \subseteq V$ there exists such a subset $U \subseteq W$ that $|U| \geq (|W| - 2m - 1)/2$ and* $\operatorname{diam} U < d$.

Proof. If G contains no cycles of odd length, then, by König's theorem, we have $\chi(G) \leq 2$. Then we may assume that $m = 0$, and all is proved.

Therefore we assume that G contains cycles of odd length. Let C be a cycle of the least odd length $2m+1$ in G. Take a finite set W such that $|W| \geq 2m+1$. By Theorem 2, $W \backslash C$ contains a stable subset on $\geq (|W|-2m-1)/2$ vertices. This concludes the proof. □

In conclusion we formulate two problems:

(1) Does Theorem 2 hold for three-dimensional Banach spaces with a strictly convex metric?
(2) Can we choose such universal $m \in \mathbb{N}$, that for an arbitrary finite set V in E_3 there exists U such that $U \subseteq V$, $|U| \geq (|V|-m)/2$, and $\operatorname{diam} U < \operatorname{diam} V$?

The next definition was given in [5]. Consider a simple graph $G=(V,E)$. Let $q \in \mathbb{N}$ and let $p(q,G)$ be the least of the numbers p such that, for any $W \subseteq V$, $|W| \geq p$, there exists a stable subset consisting of $\geq q$ vertices. This function is called the stable function of G.

Corollary 3 is equivalent to the following statement: $p(q,G) \leq 2q$ for any graph of diameters in B_2 with a strictly convex metric. Using estimates of the number of parts of smaller diameter, in which it is possible to divide a bounded set of n-dimensional Euclidean space E_n (see [3]), it is easy to see that, for any graph of diameters G in E_n, we have

$$p(q,G) \leq c^n(q-1)+1 \qquad \text{for some} \quad c>1.$$

Also it easily follows from [15] that there exist $c_1>1$ and a graph of diameters in E_n such that

$$p(q,G) \geq c_1^n(q-1)+1.$$

It would be interesting to find exact estimates of $p(q,G)$ for dimensions $n=3,4$.

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