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SOME PROPERTIES OF GREEN'S FUNCTIONS OF SHILOV-TYPE PARABOLIC SYSTEMS

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Abstract. For Shilov-type parabolic systems with nonnegative genus and coefficients of bounded smoothness, the properties of differentiability of Green's function with respect to spatial variables are studied.

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1. INTRODUCTION

Developing the idea advanced by Ya.I. Zhitomirskii in [10] concerning the description of the systems of partial differential equations parabolically steady to a change in coefficients, the following wide class of parabolic systems with variable coefficients was defined in [7]:

$$\partial_t u(t;x) = \{P_0(t;i\,\partial_x) + P_1(t,x;i\,\partial_x)\}u(t;x), \quad (t;x) \in \Pi_{(0;T]}. \tag{1.1}$$

Here, $u := \mathrm{col}(u_1,...,u_m)$, $\Pi_{(0;T]} := \{(t;x) : t \in (0;T],\ x \in \mathbb{R}^n\}$, and $P_0(t;i\,\partial_x)$ and $P_1(t,x;i\,\partial_x)$ are matrix differentiable expressions of the orders, respectively, p and p_1, $p > p_1 \geq 0$, with coefficients depending on the time variable t. In this case, the coefficients of the expression P_1 can depend also on the spatial variable x. It is also assumed that the corresponding system

$$\partial_t u(t;x) = P_0(t;i\,\partial_x)u(t;x), \quad (t;x) \in \Pi_{(0;T]}, \tag{1.2}$$

is uniformly parabolic by Shilov with parabolicity index h, $0 < h \leq p$, nonnegative genus μ, and reduced order p_0 [2]. In other words, there exist constants $\delta_0 > 0$ and $\delta_1 \geq 0$ such that, for all $x \in \mathbb{R}^n$ and $t \in [0;T]$, the following inequality holds:

$$\Lambda(t;x) := \max_{j \in \mathbb{N}_m} \mathrm{Re}\lambda_j(t;x) \leq -\delta_0 \|x\|^h + \delta_1.$$

Here, λ_j are roots of the equation $\det(P_0(t;\zeta) - \lambda E) = 0$, $\zeta \in \mathbb{C}^n$, E is the identity matrix of the order m, $\|\cdot\|$ is the Euclid norm in $\mathbb{R}^n$, and $\mathbb{N}_m := \{1,2,\ldots,m\}$.

We recall also that the reduced order p_0 of a system is called the exact power order of growth of a function $\Lambda(\cdot;\cdot)$ in the complex space $\mathbb{C}^n$ (for a parabolic system,

we have always $p_0 > 1$). In addition, the genus of a system parabolic by Shilov is called the largest index μ such that the function $\Lambda(\cdot;\cdot)$ in the domain $G^{\mu}(K) := [0;T]\times\{x+iy\in\mathbb{C}^n:\ \|y\|\le K(1+\|x\|)^{\mu}\}$ with some $K>0$ satisfies the estimate

$$\Lambda(t;x+iy)\le -\delta_*\|x\|^h+\delta_1,\delta_*>0,\delta_1\ge 0.$$

It is known that $1-(p_0-h)\le\mu\le 1$.

The following condition must be satisfied for (1.1):

$$(\mathrm{A}):\ 0\le p_1<h-n(1-h\mu/p_0)-(m-1)(p-h).$$

It is obvious that the class of systems (1.1) envelops completely the class of Shilov-type parabolic systems with nonnegative genus. In this connection, it was proposed in [10] to call such systems Shilov-type parabolic systems with nonnegative genus and variable coefficients. To a certain extent, these systems solve the problem of proper extension of the notion of parabolicity by Shilov to the linear systems of equations with variable coefficients, since the Shilov class of systems is not parabolically stable to a change in its coefficients [4].

Let us consider systems (1.1) under the condition that their coefficients are the complex-valued functions that are continuous in the variable t, infinitely differentiable with respect to the variable x, and bounded together with their derivatives in a ball $\Pi_{[0;T]}$. For these systems, Green's function $Z(t,x;\tau,\xi)$, $0\le\tau<t\le T$, $\{x,\xi\}\subset\mathbb{R}^n$, was constructed, and its main properties such as the smoothness and the behavior in vicinities of infinitely remote spatial points were studied [7]. On the basis of these results, the theory of the Cauchy problem was developed in [5, 8, 9] for such systems in spaces of the type S by Gelfand and Shilov [3]. In particular, the proper solvability of the Cauchy problem with generalized initial data like the Gevrey ultradistributions was established, the form of a representation of classical solutions with generalized limiting values on the input hyperplane was obtained, their qualitative properties were studied, and the sets of generalized input functions, for which the corresponding solutions are elements of the Schwartz space S or some space of the type S, were described.

In order to get similar results under weaker conditions imposed on the coefficients of system (1.1), it is necessary firstly to clarify the properties of corresponding Green's function Z. In the present work, we will study properties of the function $Z(t,x;\tau,\xi)$ with respect to spatial variables x and ξ under the minimum conditions of smoothness imposed on the coefficients of this system.

2. Auxiliary information

Let the differential expressions P_0 and P_1 of system (1.1) have the structure

$$P_0(t;i\partial_x)=\sum_{|k|_+\le p}A_{0,k}(t)\partial_x^k,\ P_1(t,x;i\partial_x)=\sum_{|k|_+\le p_1}A_{1,k}(t;x)\partial_x^k,$$

where $A_{0,k}(t) := i^{|k|_+}\left(a_{0,k}^{lj}(t)\right)_{l,j=1}^{m}$, $A_{1,k}(t;x) := i^{|k|_+}\left(a_{1,k}^{lj}(t;x)\right)_{l,j=1}^{m}$ are matrix coefficients, i is the imaginary unity, and $|k|_+ := k_1 + \ldots + k_n$, $k \in \mathbb{Z}_+^n$.

By $G(t,\tau;\cdot)$, $0 \le \tau < t \le T$, we denote Green's function of system (1.2). It is known that $G(t,\tau;\cdot) = F\left[\Theta_\tau^t(\xi)\right](t,\tau;\cdot)$, where $F[\cdot]$ is the Fourier transformation operator, and $\Theta_\tau^t(\cdot)$ is a matriciant of the corresponding Fourier duality of the system. The following proposition is proper [5, 6]:

$$\forall\ T>0\ \exists\ \delta>0\ \forall\ k\in\mathbb{Z}_+^n\ \exists\ c>0\ \forall\ t\in(\tau;T]\ \forall\ \tau\in[0;T)\ \forall\ \{x,\xi\}\subset\mathbb{R}^n:$$

$$|\partial_x^k G(t,\tau;x-\xi)| \le c(t-\tau)^{-\frac{n+|k|_++\gamma}{h}} e^{-\delta\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}, \tag{2.1}$$

where $\gamma := (m-1)(p-h)$ and $\alpha := \mu/p_0$, $|(a_{lj})_{l,j=1}^m| := \max\limits_{\{l,j\}\subset\mathbb{N}_m} |a_{lj}|$.

Here, we consider systems (1.1), which satisfy, in addition to condition (A), the following condition:

(B): the coefficients $a_{0,k}^{lj}(t)$, $a_{1,k}^{lj}(t;x)$ are continuous in the variable t uniformly with respect to x, differentiable with respect to the variable x up to the order α_* inclusively, and bounded together with their derivatives by complex-valued functions in a ball $\Pi_{[0;T]}$.

In [7], Green's function of system (1.1) was constructed in the form

$$Z(t,x;\tau,\xi) = G(t,\tau;x-\xi) + W(t,x;\tau,\xi),\ (t,x;\tau,\xi)\in\Pi_T^2, \tag{2.2}$$

where $\Pi_T^2 := \{(t,x;\tau,\xi)|\ 0\le\tau<t\le T, \{x,\xi\}\subset\mathbb{R}^n\}$,

$$W(t,x;\tau,\xi) := \int_\tau^t d\beta \int_{\mathbb{R}^n} G(t,\beta;x-y)\Phi(\beta,y;\tau,\xi)dy, \tag{2.3}$$

and

$$\Phi(t,x;\tau;\xi) = \sum_{l=1}^{\infty} K_l(t,x;\tau,\xi), \tag{2.4}$$

where

$$K_1(t,x;\tau,\xi) := P_1(t,x;i\,\partial_x)G(\tau,\xi;x-\xi),$$

$$K_l(t,x;\tau,\xi) := \int_\tau^t d\beta \int_{\mathbb{R}^n} K_1(t,x;\beta,y)K_{l-1}(\beta,y;\tau,\xi)dy,\ l>1. \tag{2.5}$$

In this case, it was established that condition (A) and the boundedness of the coefficients of system (1.1) ensure the absolute uniform convergence of the functional series (2.4) for all $\{x,\xi\}\subset\mathbb{R}^n$, $t\in(\tau;T]$, and $\tau\in[0,T)$. Moreover, its sum Φ and the iterated kernels K_l satisfy the estimates

$$|\Phi(t,x;\tau,\xi)| \le c_1(t-\tau)^{\alpha_0-(1+\alpha n)} e^{-\delta_1\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}, \tag{2.6}$$

$$|K_l(t,x;\tau,\xi)| \le c_0^l \left(\prod_{j=1}^{l-1} c_{(j\varepsilon)} B(\alpha_0, j\alpha_0) \right) \times (t-\tau)^{l\alpha_0-(1+\alpha n)} e^{-\delta(1-(l-1)\varepsilon)\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}, \ \varepsilon \in (0;1), \quad (2.7)$$

with the estimating constants independent of t, τ, x, and ξ. Here $\alpha_0 := 1+\alpha n-(n+p_1+\gamma)/h > 0$ and $B(\cdot,\cdot)$ is the Euler beta-function.

We note that estimates (2.1) and (2.6) for $\{x,\xi\} \subset \mathbb{R}^n$ and $0 \le \tau < t \le T$ guarantee the absolute convergence of the integral, by which the potential W is determined. Thus, the matrix function $Z(t,x;\tau,\xi)$ is properly determined by formula (2.2) on the whole set Π_T^2.

Completing this item, we present the following estimates from [1], which will be of importance in what follows:

$$e^{-\delta\left\{\left(\frac{\|x-y\|}{(t-\beta)^\alpha}\right)^{\frac{1}{1-\alpha}}+\left(\frac{\|y-\xi\|}{(\beta-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}\right\}} \le e^{-\delta\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}; \quad (2.8)$$

$$\int_{\mathbb{R}^n} e^{-\delta\left\{\left(\frac{\|x-y\|}{(t-\beta)^\alpha}\right)^{\frac{1}{1-\alpha}}+\left(\frac{\|y-\xi\|}{(\beta-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}\right\}} \frac{dy}{((t-\beta)(\beta-\tau))^{\alpha n}} \le \frac{c_\varepsilon e^{-\delta(1-\varepsilon)\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}}{(t-\tau)^{\alpha n}}, \ \delta > 0 \quad (2.9)$$

(here, $\{x,y,\xi\} \subset \mathbb{R}^n$, $\beta \in (\tau;t)$, $0 \le \tau < t \le T$, $\varepsilon \in (0;1)$, and $\delta > 0$, and the quantity c_ε depends only on ε).

3. Properties of Green's function

First, we estimate the derivatives of the iterated kernels K_l.

According to representation (2.5), the smoothness of the kernel $K_1(t,x;\tau,\xi)$ in the spatial variables x and ξ is determined, respectively, by the smoothness of the coefficients of system (1.1) and the function $G(t,\tau;x-\xi)$. Therefore, there exist the derivatives $\partial_\xi^r \partial_x^q K_1$ for $\{r,q\} \subset \mathbb{Z}_+^n$, $|q|_+ \le \alpha_*$, and the following equality holds:

$$\partial_\xi^r \partial_x^q K_1(t,x;\tau,\xi) = \sum_{|k|_+ \le p_1} \sum_{l=0}^{q} C_q^l \left(\partial_x^l A_{1,k}(t;x)\right) \left(\partial_{(x-\xi)}^{k+r+q-l} G(t,\tau;x-\xi)\right),$$

where C_q^l is a binomial coefficient. From whence, with regard for condition (B) and estimate (2.1) for $\{r,q\} \subset \mathbb{Z}_+^n$, $|q|_+ \le \alpha_*$, $(t,x;\tau,\xi) \in \Pi_T^2$, we get

$$|\partial_\xi^r \partial_x^q K_1(t,x;\tau,\xi)| \le c_{r,q}(t-\tau)^{-\frac{n+p_1+\gamma+|r+q|_+}{h}} e^{-\delta\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}} \quad (3.1)$$

(here, the estimating constants are independent of t,τ,x, and ξ).

For $l>1$, we will use the representation

$$K_l(t,x;\tau,\xi)=\int\limits_{\tau}^{t_1}d\beta\int\limits_{\mathbb{R}^n}K_1(t,x;\beta,\eta+\xi)K_{l-1}(\beta,\eta+\xi;\tau,\xi)d\eta$$
$$+\int\limits_{t_1}^{t}d\beta\int\limits_{\mathbb{R}^n}K_1(t,x;\beta,x-z)K_{l-1}(\beta,x-z;\tau,\xi)dz,\ t_1:=\frac{t+\tau}{2}.$$

According to it,

$$\partial_\xi^r\partial_x^qK_l(t,x;\tau,\xi)=\sum_{|r_1|_+\le|r|_+}C_r^{r_1}\int\limits_{\tau}^{t_1}d\beta\int\limits_{\mathbb{R}^n}\left(\partial_\xi^{r_1}\partial_x^qK_1(t,x;\beta,\eta+\xi)\right)$$
$$\times\left(\partial_\xi^{r-r_1}K_{l-1}(\beta,\eta+\xi;\tau,\xi)\right)d\eta+\sum_{|q_1|_+\le|q|_+}C_q^{q_1}\int\limits_{t_1}^{t}d\beta\int\limits_{\mathbb{R}^n}\left(\partial_x^{q_1}K_1(t,x;\beta,x-z)\right)$$
$$\times\left(\partial_\xi^r\partial_x^{q-q_1}K_{l-1}(\beta,x-z;\tau,\xi)\right)dz,\ |q|_+\le\alpha_*,(t,x;\tau,\xi)\in\Pi_T^2. \tag{3.2}$$

Hence, the estimation of $|\partial_\xi^r\partial_x^qK_l(t,x;\tau,\xi)|$ is reduced to that of the expressions

$$|\partial_\xi^r\partial_x^qK_1(t,x;\tau,\eta+\xi)|,\quad|\partial_x^qK_1(t,x;\tau,x-z)|,$$

$$|\partial_\xi^rK_{l-1}(t,\eta+\xi;\tau,\xi)|,\quad|\partial_\xi^r\partial_x^qK_{l-1}(t,x-z;\tau,\xi)|.$$

In view of the boundedness of $\partial_x^qa_{1,k}^{lj}(t;x)$, $|q|_+\le\alpha_*$, and estimate (2.1), for all $\{q,r\}\in\mathbb{Z}_+^n$, $|q|_+\le\alpha_*$, $\{x,\eta,\xi\}\in\mathbb{R}^n$, $t\in(\tau;T]$, and $\tau\in[0;T)$, we have

$$\left|\partial_\xi^r\partial_x^qK_1(t,x;\tau,\eta+\xi)\right|\le m\sum_{|k|_+\le p_1}\sum_{|q_1|_+\le|q|_+}C_q^{q_1}\left|\partial_x^{q_1}A_{1,k}(t;x)\right|$$
$$\times\left|\partial_{(x-\eta-\xi)}^{k+r+q-q_1}G(t,\tau;x-\eta-\xi)\right|\le c_{r,q}(t-\tau)^{-\frac{n+p_1+\gamma+|r+q|_+}{h}}e^{-\delta\left(\frac{\|x-\eta-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}; \tag{3.3}$$

$$\left|\partial_x^qK_1(t,x;\tau,x-\xi)\right|=\left|\partial_x^q\Big(\sum_{|k|_+\le p_1}A_{1,k}(t;x)\partial_x^kG(t,\tau;\xi)\Big)\right|\le m\left|\partial_x^qA_{1,0}(t;x)\right|$$
$$\times\left|G(t,\tau;\xi)\right|\le\widehat{c}_q(t-\tau)^{-\frac{n+\gamma}{h}}e^{-\delta\left(\frac{\|\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}\le c_q(t-\tau)^{-\frac{n+p_1+\gamma}{h}}e^{-\delta\left(\frac{\|\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}. \tag{3.4}$$

We now estimate the expression $\left|\partial_\xi^r K_l(t,\eta+\xi;\tau,\xi)\right|$. Since

$$\partial_\xi^r K_1(t,\eta+\xi;\tau,\xi)=\sum_{|k|_+\le p_1}\partial_\xi^r A_{1,k}(t;\eta+\xi)\partial_\eta^k G(t,\tau;\eta),\ (t,x;\tau,\xi)\in \Pi_T^2, \quad (3.5)$$

we have, according to condition (A), that the iterated kernels $K_l(t,\eta+\xi;\tau,\xi)$ are differentiable with respect to the variable ξ only to the order α_*. This fact and (3.2) imply that $\partial_x^q K_l(t,x;\tau,\xi)$, $|q|_*\le\alpha_*$, is also a function differentiable with respect to ξ only to this order α_*.

Representation (3.5) and estimate (2.1) yield

$$\left|\partial_\xi^r K_1(t,\eta+\xi;\tau,\xi)\right|\le c_{1,r}(t-\tau)^{-\frac{n+p_1+\gamma}{h}}e^{-\delta\left(\frac{\|\eta\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}. \quad (3.6)$$

We note that

$$\partial_\xi^r K_2(t,\eta+\xi;\tau,\xi)=\partial_\xi^r\left(\int_\tau^t d\beta\int_{\mathbb{R}^n}K_1(t,\eta+\xi;\beta,y)K_1(\beta,y;\tau,\xi)dy\right).$$

Let us change the variable of integration in the last integral by the formula $y=z+\xi$. In view of estimates (3.6) and (2.9) and the equalities

$$\int_\tau^t((t-\beta)(\beta-\tau))^{\alpha_0-1}d\beta=(t-\tau)^{2\alpha_0-1}B(\alpha_0,\alpha_0) \quad (3.7)$$

and

$$\partial_\xi^r K_1(t,\eta+\xi;\tau,z+\xi)=\partial_\zeta^r K_1(t,(\eta-z)+\zeta;\tau,\zeta)\Big|_{\zeta=z+\xi},$$

we get

$$\begin{aligned}\left|\partial_\xi^r K_2(t,\eta+\xi;\tau,\xi)\right|&\le m\sum_{|r_1|_+\le|r|_+}C_r^{r_1}\int_\tau^t d\beta\int_{\mathbb{R}^n}\left|\partial_\xi^{r_1}K_1(t,\eta+\xi;\beta,z+\xi)\right|\\&\times\left|\partial_\xi^{r-r_1}K_1(\beta,z+\xi;\tau,\xi)\right|dz\le m\sum_{|r_1|_+\le|r|_+}C_r^{r_1}c_{1,r_1}c_{1,(r-r_1)}\\&\times\int_\tau^t\left((t-\beta)(\beta-\tau)\right)^{-\frac{n+p_1+\gamma}{h}}\int_{\mathbb{R}^n}e^{-\delta\left(\left(\frac{\|\eta-z\|}{(t-\beta)^\alpha}\right)^{\frac{1}{1-\alpha}}+\left(\frac{\|z\|}{(\beta-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}\right)}dz\,d\beta\\&\le c_{2,r}(\varepsilon)B(\alpha_0,\alpha_0)(t-\tau)^{\alpha_0-\frac{n+p_1+\gamma}{h}}e^{-\delta(1-\varepsilon)\left(\frac{\|\eta\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}},\ \varepsilon\in(0;1).\end{aligned}$$

By reasoning analogously step by step, we arrive at the inequality

$$\left|\partial_\xi^r K_l(t,\eta+\xi;\tau,\xi)\right|\le c_{l,r}(\varepsilon)\Big(\prod_{j=1}^{l-1}B(\alpha_0,j\alpha_0)\Big)(t-\tau)^{(l-1)\alpha_0-\frac{n+p_1+\gamma}{h}}$$

$$\times e^{-\delta(1-(l-1)\varepsilon)\left(\frac{\|\eta\|}{(t-\tau)^{\alpha}}\right)^{\frac{1}{1-\alpha}}}, \tag{3.8}$$

which is satisfied for all $\{\eta,\xi\}\subset\mathbb{R}^n$, $|r|_+\le\alpha_*$, $0\le\tau<t\le T$, $\varepsilon\in(0;1)$, and $l\in\mathbb{N}\setminus\{1\}$ and, hence, until the existence of such number l_*, for which

$$\left|\partial_{\xi}^{r}K_{l_*}(t,\eta+\xi;\tau,\xi)\right|\le c_{l_*,r}(\varepsilon)\Big(\prod_{j=1}^{l_*-1}B(\alpha_0,j\alpha_0)\Big)e^{-\delta(1-(l_*-1)\varepsilon)\left(\frac{\|\eta\|}{(t-\tau)^{\alpha}}\right)^{\frac{1}{1-\alpha}}} \tag{3.9}$$

(here, the quantities $c_{l,r}(\varepsilon)>0$ do not depend on the variables t,τ,η, and ξ, which vary in the above-indicated way).

Since

$$\partial_{\xi}^{r}\partial_{x}^{q}K_l(t,x;\tau,\eta+\xi)=\partial_{\zeta}^{r}\partial_{x}^{q}K_l(t,x;\tau,\zeta)\Big|_{\zeta=\eta+\xi},$$

$$\partial_{\xi}^{r}\partial_{x}^{q}K_l(t,x-z;\tau,\xi)=\partial_{\xi}^{r}\partial_{y}^{q}K_l(t,y;\tau,\xi)\Big|_{y=x-z},$$

the expressions $\partial_{\xi}^{r}\partial_{x}^{q}K_l(t,x;\tau,\eta+\xi)$, $\partial_{\xi}^{r}\partial_{x}^{q}K_l(t,x-z;\tau,\xi)$ and $\partial_{\xi}^{r}\partial_{x}^{q}K_l(t,x;\tau,\xi)$ are of the same type. Therefore, with regard for representation (3.2) and estimates (3.3), (3.4), (3.8), and (2.9), we have

$$\left|\partial_{\xi}^{r}\partial_{x}^{q}K_2(t,x;\tau,\xi)\right|\le m2^{|r+q|_+}\Bigg(\sum_{|r_1|_+\le|r|_+}c_{r_1,q}c_{1,(r-r_1)}\int_{\tau}^{t_1}(t-\beta)^{-\frac{n+p_1+\gamma+|r_1+q|_+}{h}}$$

$$\times(\beta-\tau)^{-\frac{n+p_1+\gamma}{h}}\int_{\mathbb{R}^n}e^{-\delta\left(\left(\frac{\|x-\eta-\xi\|}{(t-\beta)^{\alpha}}\right)^{\frac{1}{1-\alpha}}+\left(\frac{\|\eta\|}{(\beta-\tau)^{\alpha}}\right)^{\frac{1}{1-\alpha}}\right)}d\eta d\beta$$

$$+\sum_{|q_1|_+\le|q|_+}c_{q_1}c_{r,(q-q_1)}\times\int_{t_1}^{t}(\beta-\tau)^{-\frac{n+p_1+\gamma+|r+q-q_1|_+}{h}}(t-\beta)^{-\frac{n+p_1+\gamma}{h}}$$

$$\int_{\mathbb{R}^n}e^{-\delta\left(\left(\frac{\|z\|}{(t-\beta)^{\alpha}}\right)^{\frac{1}{1-\alpha}}+\left(\frac{\|x-z-\xi\|}{(\beta-\tau)^{\alpha}}\right)^{\frac{1}{1-\alpha}}\right)}dzd\beta\Bigg)$$

$$\le m2^{|r+q|_+}c_{\varepsilon}e^{-\delta(1-\varepsilon)\left(\frac{\|x-\xi\|}{(t-\tau)^{\alpha}}\right)^{\frac{1}{1-\alpha}}}(t-\tau)^{-\alpha n}\Bigg(\sum_{|r_1|_+\le|r|_+}c_{r_1,q}c_{1,(r-r_1)}$$

$$\times\int_{\tau}^{t_1}(t-\beta)^{\alpha n-\frac{n+p_1+\gamma+|r_1+q|_+}{h}}(\beta-\tau)^{\alpha_0-1}d\beta+\sum_{|q_1|_+\le|q|_+}c_{q_1}c_{r,(q-q_1)}$$

$$\times\int_{t_1}^{t}(t-\beta)^{\alpha_0-1}(\beta-\tau)^{\alpha n-\frac{n+p_1+\gamma+|r+q-q_1|_+}{h}}d\beta\Bigg),\ |r|_+\le\alpha_*,|q|_+\le\alpha_*,\varepsilon\in(0;1).$$

In view of the estimates

$$\int_{\tau}^{t_1}(t-\beta)^{\alpha n-\frac{n+p_1+\gamma+|r_1+q|_+}{h}}(\beta-\tau)^{\alpha_0-1}d\beta \le 2^{\frac{|r_1+q|_+}{h}}(t-\tau)^{2\alpha_0-\left(1+\frac{|r_1+q|_+}{h}\right)}B(\alpha_0,\alpha_0)$$

and

$$\int_{t_1}^{t}(t-\beta)^{\alpha_0-1}(\beta-\tau)^{\alpha n-\frac{n+p_1+\gamma+|r+q-q_1|_+}{h}}d\beta \le 2^{\frac{|r+q-q_1|_+}{h}}(t-\tau)^{2\alpha_0-\left(1+\frac{|r+q-q_1|_+}{h}\right)}B(\alpha_0,\alpha_0),$$

we get the inequality

$$\left|\partial_\xi^r\partial_x^q K_2(t,x;\tau,\xi)\right| \le c_{2,\xi}^{r,q}(t-\tau)^{2\alpha_0-\left(1+\alpha n+\frac{|r+q|_+}{h}\right)}B(\alpha_0,\alpha_0)e^{-\delta(1-\varepsilon)\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}.$$

By continuing stepwise the process of estimation, we obtain

$$\begin{aligned}\left|\partial_\xi^r\partial_x^q K_l(t,x;\tau,\xi)\right| &\le c_{l,\varepsilon}^{r,q}(t-\tau)^{l\alpha_0-\left(1+\alpha n+\frac{|r+q|_+}{h}\right)}\\ &\le e^{-\delta(1-(l-1)\varepsilon)\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}\Big(\prod_{j=1}^{l-1}B(\alpha_0,j\alpha_0)\Big),\end{aligned}\tag{3.10}$$

for all $|r|_+\le\alpha_*$, $|q|_+\le\alpha_*$, $\{x,\xi\}\subset\mathbb{R}^n$, $0\le\tau<t\le T$, $\varepsilon\in(0;1)$, and $l\in\mathbb{N}\backslash\{1\}$.

Let us pass to the estimation of the expression $\left|\partial_\xi^r\partial_x^q K_l(t,x;\tau,\xi)\right|$, which will be suitable for the establishment of the differentiability of the matrix function Φ with respect to the spatial variables. Directly from (3.10), we arrive at the existence of a number l^* such that

$$\left|\partial_\xi^r\partial_x^q K_{l^*}(t,x;\tau,\xi)\right| \le c_{l^*,\varepsilon}^{r,q}e^{-\delta(1-(l^*-1)\varepsilon)\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}\Big(\prod_{j=1}^{l^*-1}B(\alpha_0,j\alpha_0)\Big).$$

Let us set $l_+:=\max\{l_*,l^*\}$, $l_-:=\min\{l_*,l^*\}$, where l_* is the corresponding number from (3.9), $\varepsilon:=\frac{1}{r_*l_+}$, $\delta_*:=\delta\big(1-\frac{1}{r_*}\big)$, $r_*>2$, $T_0:=\max\{1,T\}$, and

$$c_*^0:=\max_{l\in\mathbb{N}_{l_+}\backslash\{1\}}\Big\{c_{1,r},c_{l,r}(\varepsilon)\Big(\prod_{j=1}^{l-1}B(\alpha_0,j\alpha_0)\Big),c_{r,q},c_{l,\varepsilon}^{r,q}\Big(\prod_{j=1}^{l-1}B(\alpha_0,j\alpha_0)\Big)\Big\},$$

$c_* := c_*^0 (T_0)^{l_+ - l_-}$. Then (3.8) and (3.10) imply that, for all $\{x, \xi, \eta\} \subset \mathbb{R}^n$, $0 \le \tau < t \le T$, $|r|_+ \le \alpha_*$, and $|q|_+ \le \alpha_*$,

$$\left|\partial_\xi^r \partial_x^q K_{l_+}(t, x; \tau, \xi)\right| \le c_* e^{-\delta_* \left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}},$$

$$\left|\partial_\xi^r K_{l_+}(t, \eta+\xi; \tau, \xi)\right| \le c_* e^{-\delta_* \left(\frac{\|\eta\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}.$$

In view of this result, estimate (2.8), the equality

$$\int\limits_{\mathbb{R}^n} e^{-\delta_0 \left(\frac{\|x-y\|}{(t-\beta)^\alpha}\right)^{\frac{1}{1-\alpha}}} \frac{dy}{(t-\beta)^{\alpha n}} = \int\limits_{\mathbb{R}^n} e^{-\delta_0 \|z\|^{\frac{1}{1-\alpha}}} dz =: \widehat{E} < +\infty,$$

representation (3.2), and inequalities (3.3) and (3.4), we obtain

$$\left|\partial_\xi^r K_{l_++1}(t, \eta+\xi; \tau, \xi)\right| \le \sum_{|r_1|_+ \le |r|_+} C_r^{r_1} \int\limits_\tau^t d\beta \int\limits_{\mathbb{R}^n} \left|\partial_\xi^{r_1} K_1(t, \eta+\xi; \beta, z+\xi) \partial_\xi^{r-r_1}\right.$$

$$\times \left. K_{l_+}(\beta, z+\xi; \tau, \xi)\right| dz \le m c_*^2 \Big(\sum_{|r_1|_+ \le |r|_+} C_r^{r_1} \Big) \int\limits_\tau^t (t-\beta)^{\alpha_0 - 1}$$

$$\times \int\limits_{\mathbb{R}^n} e^{-\delta_* \left(\left(\frac{\|\eta-z\|}{(t-\beta)^\alpha}\right)^{\frac{1}{1-\alpha}} + \left(\frac{\|z\|}{(\beta-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}} \right)} e^{-\frac{\delta}{r_*} \left(\frac{\|\eta-z\|}{(t-\beta)^\alpha}\right)^{\frac{1}{1-\alpha}}} \frac{dz}{(t-\beta)^{n\alpha}} d\beta$$

$$\le m c_r^0 \widehat{E} c_*^2 B(\alpha_0, 1) (t-\tau)^{\alpha_0} e^{-\delta_* \left(\frac{\|\eta\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}, \; c_r^0 := \sum_{|r_1|_+ \le |r|_+} C_r^{r_1};$$

$$\left|\partial_\xi^r \partial_x^q K_{l_++1}(t, x; \tau, \xi)\right| \le \sum_{|r_1|_+ \le |r|_+} C_r^{r_1} \int\limits_\tau^{t_1} d\beta \int\limits_{\mathbb{R}^n} \left|\partial_\xi^{r_1} \partial_x^q K_1(t, x; \beta, \eta+\xi)\right.$$

$$\times \left. \partial_\xi^{r-r_1} K_{l_+}(\beta, \eta+\xi; \tau, \xi)\right| d\eta + \sum_{|q_1|_+ \le |q|_+} C_q^{q_1} \int\limits_{t_1}^t d\beta \int\limits_{\mathbb{R}^n} \left|\partial_x^{q_1} K_1(t, x; \beta, x-z)\right.$$

$$\times \left. \partial_\xi^r \partial_x^{q-q_1} K_{l_+}(\beta, x-z; \tau, \xi)\right| dz \Big) \le m c_*^2 \Bigg(\sum_{|r_1|_+ \le |r|_+} C_r^{r_1} \int\limits_\tau^{t_1} (t-\beta)^{\alpha_0 - \left(1 + \frac{|r_1+q|_+}{h}\right)}$$

$$\times \int\limits_{\mathbb{R}^n} e^{-\delta_* \left(\left(\frac{\|x-\eta-\xi\|}{(t-\beta)^\alpha}\right)^{\frac{1}{1-\alpha}} + \left(\frac{\|\eta\|}{(\beta-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}} \right)} e^{-\frac{\delta}{r_*} \left(\frac{\|x-\eta-\xi\|}{(t-\beta)^\alpha}\right)^{\frac{1}{1-\alpha}}} \frac{d\eta}{(t-\beta)^{\alpha n}} d\beta$$

$$+\sum_{|q_1|_+\le|q|_+} C_q^{q_1}\int_{t_1}^{t}(t-\beta)^{\alpha_0-1}\int_{\mathbb{R}^n} e^{-\delta_*\left(\left(\frac{\|z\|}{(t-\beta)^\alpha}\right)^{\frac{1}{1-\alpha}}+\left(\frac{\|x-z-\xi\|}{(\beta-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}\right)}$$

$$\times e^{-\frac{\delta}{r_*}\left(\frac{\|z\|}{(t-\beta)^\alpha}\right)^{\frac{1}{1-\alpha}}}\frac{dz}{(t-\beta)^{\alpha n}}d\beta\Bigg)\le mc_*^2\widehat{E}e^{-\delta_*\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}\int_{\tau}^{t}(t-\beta)^{\alpha_0-1}d\beta$$

$$\times\left(\left(\sum_{|r_1|_+\le|r|_+} C_r^{r_1}(t-t_1)^{-\frac{|r_1+q|_+}{h}}\right)+c_q\right)\le mc_*^2\widehat{E}e^{-\delta_*\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}(t-\tau)^{\alpha_0}$$

$$\times B(\alpha_0,1)\left(\left(2^{\frac{|r+q|_+}{h}}\sum_{|r_1|_+\le|r|_+} C_r^{r_1}(t-\tau)^{-\frac{|r_1+q|_+}{h}}\right)+c_q\right)$$

$$\le mc_{r,q}^0c_*^2\widehat{E}(2T_0)^{\frac{|r+q|_+}{h}}B(\alpha_0,1)(t-\tau)^{\alpha_0-\frac{|r+q|_+}{h}}e^{-\delta_*\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}},\ c_{r,q}^0:=c_r+c_q.$$

Applying the method of induction, we can verify firstly the validity of the estimate

$$\left|\partial_\xi^r K_{l_++l}(t,\eta+\xi;\tau,\xi)\right|$$

$$\le c_*(mc_r^0c_*\widehat{E}(t-\tau)^{\alpha_0})^l e^{-\delta_*\left(\frac{\|\eta\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}\left(\prod_{j=1}^{l-1}B(\alpha_0,1+j\alpha_0)\right),\quad(3.11)$$

and, hence, the estimate

$$\left|\partial_\xi^r\partial_x^q K_{l_++l}(t,x;\tau,\xi)\right|\le c_*\left(mc_{r,q}^0c_*\widehat{E}(2T_0)^{\frac{|r+q|_+}{h}}\right)^l(t-\tau)^{l\alpha_0-\frac{|r+q|_+}{h}}$$

$$\times e^{-\delta_*\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}\left(\prod_{j=1}^{l-1}B(\alpha_0,1+j\alpha_0)\right),\quad(3.12)$$

for $|r|_+\le\alpha_*$, $|q|_+\le\alpha_*$, $(t,x;\tau,\xi)\in\Pi_T^2$ and $l\in\mathbb{N}\backslash\{1\}$.

The following propositions hold true.

Lemma 1. *The matrix function $\Phi(t,x;\tau,\xi)$ on the set Π_T^2 is a function differentiable with respect to each of the spatial variables x and ξ to the order α_* inclusively, and their derivatives satisfy the following estimates:*

$$\left|\partial_\xi^r\partial_x^q\Phi(t,x;\tau,\xi)\right|\le c_1(t-\tau)^{\alpha_0-(1+\alpha n+\frac{|r+q|_+}{h})}e^{-\delta_*\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}},\quad(3.13)$$

$$\left|\partial_\xi^r\Phi(t,\eta+\xi;\tau,\xi)\right|\le c_2(t-\tau)^{\alpha_0-(1+\alpha n)}e^{-\delta_*\left(\frac{\|\eta\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}},\{\eta,\xi\}\subset\mathbb{R}^n\quad(3.14)$$

(here, the estimating constants c_1,c_2, and δ_* are independent of t,τ,x,ξ,η).

Proof. In any way, let us fix a point $(x_0;\xi_0)$ from $\mathbb{R}^{2n}$, and let us consider a ball $\mathbb{K}^{\delta}_{(x_0;\xi_0)}$ with radius $\delta>0$, which is centered at the point $(x_0;\xi_0)$, in this space. Then, in view of structure (2.4) of the function Φ and the differentiability of the iterated kernels K_l with respect to spatial variables on $\mathbb{R}^{2n}$ to the order α_* inclusively, we can conclude that, in order to prove the differentiability of the matrix function Φ at the point $(x_0;\xi_0)$ to the indicated order, it is necessary only to prove the uniform convergence of the formally differentiated series (2.4) in the variables x and ξ on the set $\mathbb{K}^{\delta}_{(x_0;\xi_0)}$, $\delta>0$ (at every fixed t and τ, $0\le\tau<t\le T$):

$$\sum_{l=1}^{\infty}\partial_{\xi}^{r}\partial_{x}^{q}K_l(t,x;\tau,\xi),\quad |r|_+\le\alpha_*,\ |q|_+\le\alpha_*. \tag{3.15}$$

Directly from estimates (3.10) and (3.12) and the equality

$$\prod_{j=0}^{l-1}B(\alpha_0,1+j\alpha_0)=\frac{\bigl(\Gamma(\alpha_0)\bigr)^l}{\Gamma(1+l\alpha_0)},$$

where $\Gamma(\cdot)$ is the Euler gamma-function, for $\{r,q\}\subset\mathbb{Z}^n_+$, $|r|_+\le\alpha_*$, $|q|_+\le\alpha_*$, and $(t,x;\tau,\xi)\in\Pi^2_T$, we have

$$\begin{aligned}
&\Bigl|\sum_{l=1}^{\infty}\partial_{\xi}^{r}\partial_{x}^{q}K_l(t,x;\tau,\xi)\Bigr|\le\sum_{l=1}^{l_+}\Bigl|\partial_{\xi}^{r}\partial_{x}^{q}K_l(t,x;\tau,\xi)\Bigr|+\sum_{l=l_++1}^{\infty}\Bigl|\partial_{\xi}^{r}\partial_{x}^{q}K_l(t,x;\tau,\xi)\Bigr|\\
&\le c_*\Biggl(\sum_{l=1}^{l_+}(t-\tau)^{l\alpha_0-(1+\alpha n+\frac{|r+q|_+}{h})}\\
&\quad+\sum_{l=1}^{\infty}\bigl(mc^0_{r,q}c_*\widehat{E}(2T_0)^{\frac{|r+q|_+}{h}}\bigr)^l(t-\tau)^{l\alpha_0-\frac{|r+q|_+}{h}}\\
&\quad\times\Bigl(\prod_{j=1}^{l-1}B(\alpha_0,1+j\alpha_0)\Bigr)\Biggr)e^{-\delta_*\left(\frac{\|x-\xi\|}{(t-\tau)^{\alpha}}\right)^{\frac{1}{1-\alpha}}}\le c_1(t-\tau)^{\alpha_0-\left(1+\alpha n+\frac{|r+q|_+}{h}\right)}\\
&\quad\times e^{-\delta_*\left(\frac{\|x-\xi\|}{(t-\tau)^{\alpha}}\right)^{\frac{1}{1-\alpha}}}.
\end{aligned}$$

From whence, we get the uniform convergence of series (3.15) in x and ξ and, hence, the validity of estimates (3.13).

Due to the corresponding estimates (3.8) and (3.11), we can verify analogously the validity of estimate (3.14).

The lemma is proved. □

Lemma 2. *The volumetric potential $W(t,x;\tau,\xi)$ on the set Π^2_T is a function differentiable with respect to each of the spatial variables x and ξ to, respectively, the*

orders $\alpha_* + p_1$ and α_* inclusively. In this case,

$$\partial_\xi^r \partial_x^q W(t,x;\tau,\xi) = \sum_{l=0}^{r} C_r^l \int_\tau^{t_1} d\beta \int_{\mathbb{R}^n} \partial_\xi^l \partial_x^q G(t,\beta;x-y-\xi)\partial_\xi^{r-l}\Phi(\beta,y+\xi;\tau,\xi)dy$$

$$+ \int_{t_1}^{t} d\beta \int_{\mathbb{R}^n} \partial_x^q G(t,\beta;x-y)\partial_\xi^r \Phi(\beta,y;\tau,\xi)dy,\ |q|_+ \le p_1, |r|_+ \le \alpha_*, \quad (3.16)$$

$$\partial_\xi^r \partial_x^q W(t,x;\tau,\xi) = \sum_{l=0}^{r} C_r^l \int_\tau^{t_1} d\beta \int_{\mathbb{R}^n} \partial_\xi^l \partial_x^q G(t,\beta;x-y-\xi)\partial_\xi^{r-l}\Phi(\beta,y+\xi;\tau,\xi)dy$$

$$+ \int_{t_1}^{t} d\beta \int_{\mathbb{R}^n} \partial_\eta^k G(t,\beta;\eta)\partial_\xi^r \partial_x^{q-k} \Phi(\beta,x-\eta;\tau,\xi)d\eta,\ |r|_+ \le \alpha_*, \quad (3.17)$$

$$|k|_+ = p_1,\ p_1 < |q|_+ \le \alpha_* + p_1.$$

Proof. For $|q|_+ \le p_1$ and $|r|_+ \le \alpha_*$, we use the representation

$$W(t,x;\tau,\xi) = \int_\tau^{t_1} d\beta \int_{\mathbb{R}^n} G(t,\beta;x-y-\xi)\Phi(\beta,y+\xi;\tau,\xi)dy$$

$$+ \int_{t_1}^{t} d\beta \int_{\mathbb{R}^n} G(t,\beta;x-y)\Phi(\beta,y;\tau,\xi)dy.$$

From whence, by the formal differentiation under the sign of integral, we obtain equality (3.16). Hence, in order to substantiate the validity of equality (3.16), it is sufficient to prove the uniform convergence of the following integrals in the variables x and ξ on $\mathbb{R}^{2n}$:

$$\mathcal{J}_1^{r,l,q}(t_1,x;\tau,\xi) := \int_\tau^{t_1} d\beta \int_{\mathbb{R}^n} |\partial_\xi^l \partial_x^q G(t,\beta;x-y-\xi)||\partial_\xi^{r-l}\Phi(\beta,y+\xi;\tau,\xi)|dy,$$

$$|l|_+ \le |r|_+;\ \mathcal{J}_2^{r,q}(t,x;t_1,\xi) := \int_{t_1}^{t} d\beta \int_{\mathbb{R}^n} |\partial_x^q G(t,\beta;x-y)||\partial_\xi^r \Phi(\beta,y;\tau,\xi)|dy. \quad (3.18)$$

This convergence becomes obvious, if we take condition (A) and the following estimates into account for $\{x,\xi\}\subset\mathbb{R}^n$ and $0\le\tau<t\le T$:

$$\mathcal{J}_1^{r,l,q}(t_1,x;\tau,\xi)\le cc_2\widehat{E}e^{-\delta_*\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}(t-t_1)^{-\frac{n+\gamma+|l+q|_+}{h}}$$
$$\times\int_\tau^{t_1}(\beta-\tau)^{\alpha_0-1}d\beta,\ |l|_+\le|r|_+;\tag{3.19}$$

$$\mathcal{J}_2^{r,q}(t,x;t_1,\xi)$$
$$\le cc_1\widehat{E}e^{-\delta_*\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}(t_1-\tau)^{-\frac{n+p_1+\gamma+|r|_+}{h}}\int_{t_1}^{t}(t-\beta)^{\alpha_0-1+\frac{p_1-|q|_+}{h}}d\beta.\tag{3.20}$$

These estimates follow directly from (2.1), (3.13), and (3.14).

We now prove the validity of formula (3.17). For this purpose, we fix any $k\in\mathbb{Z}_+^n$ such that $|k|_+=p_1$. Then, according to (3.16) for $p_1<|q|_+\le\alpha_*+p_1$ and $|r|_+\le\alpha_*$, we have

$$\partial_\xi^r\partial_x^q W(t,x;\tau,\xi)=\sum_{l=0}^{r}C_r^l\partial_x^{q-k}\int_\tau^{t_1}d\beta\int_{\mathbb{R}^n}\partial_\xi^l\partial_x^k G(t,\beta;x-y-\xi)\partial_\xi^{r-l}\Phi(\beta,y+\xi;\tau,\xi)$$

$$\times dy+\partial_x^{q-k}\int_{t_1}^{t}d\beta\int_{\mathbb{R}^n}\partial_\eta^k G(t,\beta;\eta)\partial_\xi^r\Phi(\beta,x-\eta;\tau,\xi)d\eta,\ (t,x;\tau,\xi)\in\Pi_T^2.$$

Hence, it remains to substantiate the possibility to introduce the operation ∂_x^{q-k} under the signs of the corresponding integrals. In other words, we should prove the uniform convergence in x and ξ of the following integrals on $\mathbb{R}^{2n}$ for $0\le\tau<t\le T$:

$$\int_\tau^{t_1}d\beta\int_{\mathbb{R}^n}\partial_\xi^l\partial_x^q G(t,\beta;x-y-\xi)\Phi(\beta,y+\xi;\tau,\xi)dy,$$

$$\int_{t_1}^{t}d\beta\int_{\mathbb{R}^n}\partial_\eta^k G(t,\beta;\eta)\partial_\xi^r\partial_x^{q-k}\Phi(\beta,x-\eta;\tau,\xi)d\eta.$$

By reasoning similarly to the case of integrals (3.18) and using estimates (2.1), (3.13), and (3.14), we get the necessary convergence of the indicated integrals.

The lemma is proved. □

The main result can be formulated as the following proposition.

Theorem 1. *Let system (1.1) satisfy conditions (A) and (B). Then the corresponding function $Z(t,x;\tau,\xi)$ defined by equality (2.2) is a function differentiable with respect to each of the spatial variables x and ξ on the set Π_T^2 to, respectively, the orders $\alpha_* + p_1$ and α_* inclusively, and*

$$\exists \delta > 0\, \forall \{r,q\} \subset \mathbb{Z}_+^n, |q|_+ \le \alpha_* + p_1, |r|_+ \le \alpha_*, \exists c > 0\, \forall (t,x;\tau,\xi) \in \Pi_T^2 :$$

$$|\partial_\xi^r \partial_x^q Z(t,x;\tau,\xi)| \le c(t-\tau)^{-\frac{n+|r+q|_+ +\gamma}{h}} e^{-\delta\left(\frac{\|x-\xi\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}; \tag{3.21}$$

$$|\partial_\xi^k Z(t,x+\xi;\tau,\xi)| \le c_k (t-\tau)^{\beta_k - \frac{n+\gamma}{h}} e^{-\delta_1\left(\frac{\|x\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}, \tag{3.22}$$

where $|k|_+ \le \alpha_*$, $0 \le \tau < t \le T$, $\{x,\xi\} \subset \mathbb{R}^n$, $\beta_k := \begin{cases} 0, & k = 0, \\ \alpha_0, & k \ne 0 \end{cases}$ (here, the estimating constants are independent of t, τ, x, and ξ).

Proof. With regard for structure (2.2) and the infinite differentiability of the function $G(t,\tau;\xi)$ with respect to the variable ξ, the smoothness of the function $Z(t,x;\tau,\xi)$ in the variables x and ξ becomes obvious directly from the assertion of Lemma 2.

Let $|q|_+ \le p_1$ and $|r|_+ \le \alpha_*$. Then, according to (3.16), we get

$$|\partial_\xi^r \partial_x^q Z(t,x;\tau,\xi)| \le |\partial_{x-\xi}^{r+q} G(t,\tau;x-\xi)| + \sum_{l=0}^{r} C_r^l \mathcal{J}_1^{r,l,q}(t_1,x;\tau,\xi) + \mathcal{J}_2^{r,q}(t,x;t_1,\xi).$$

From whence, by using estimates (2.1), (3.19), and (3.20), we obtain assertion (3.21).

In a similar way, by using formula (3.17), we verify the validity of assertion (3.21) also for $p_1 < |q|_+ \le \alpha_*$ and $|r|_+ \le \alpha_*$.

Then, according to estimates (2.1) and (3.14), we have

$$Y_k(t,x;\tau,\xi) := \Big| \int_\tau^t d\beta \int_{\mathbb{R}^n} G(t,\beta;x-\zeta)\partial_\xi^k \Phi(\beta,\zeta+\xi;\tau,\xi)d\zeta \Big|$$

$$\le cc_2 \int_\tau^t (t-\beta)^{\alpha_0 + \frac{p_1}{h} - 1}(\beta-\tau)^{\alpha_0 - 1} \int_{\mathbb{R}^n} \exp\Big\{-\delta_0\Big\{\Big(\frac{\|x-\zeta\|}{(t-\beta)^\alpha}\Big)^{\frac{1}{1-\alpha}}$$

$$+ \Big(\frac{\|\zeta\|}{(\beta-\tau)^\alpha}\Big)^{\frac{1}{1-\alpha}}\Big\}\Big\} \frac{dy\,d\beta}{((t-\beta)(\beta-\tau)^{\alpha n}}, \delta_0 := \min\{\delta, \delta_*\}, |k|_+ \le \alpha_*.$$

Using estimate (2.9) and equality (3.7), we get

$$Y_k(t,x;\tau,\xi) \le c_\varepsilon (t-\tau)^{\alpha_0 - \frac{n+\gamma}{h}} e^{-\delta_0(1-\varepsilon)\left(\frac{\|x\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}}, \quad \varepsilon \in (0;1),$$

where $|k|_+ \le \alpha_*$, $0 \le \tau < t \le T$ and $\{x,\xi\} \subset \mathbb{R}^n$. From whence, with regard for inequality (2.1) and the representation

$$Z(t,x+\xi;\tau,\xi) = G(t,\tau;x) + \int_\tau^t d\beta \int_{\mathbb{R}^n} G(t,\beta;x-\zeta)\Phi(\beta,\zeta+\xi;\beta,\xi)d\zeta,$$

we arrive at estimate (3.22).

The theorem is proved. □

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