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Some Properties of Hypercentral Lie Algebras

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1.

In [1, Chap. 7] the following characterization of nilpotent Lie algebras has been shown: Let L be a Lie algebra with $L^2 \lneqq L$. Then L is nilpotent if and only if every proper subalgebra of L is a nilpotent subideal. S. Tôgô suggested to the authors that the similar characterization of hypercentral Lie algebras holds. The class of hypercentral Lie algebras is a natural transfinite extension of the class of nilpotent Lie algebras. Until now, however, very little is known about this class. Accordingly it seems to be desirable for us to know the properties of this class. In this paper, including the characterization stated above (Corollary 4), we investigate the class of hypercentral Lie algebras and present some of their properties.

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2.

Let L be a Lie algebra over an arbitrary field. The center of L is defined to be

$$\zeta_1(L) = \{x \in L: [x, L] = 0\}.$$

The transfinite upper central series $\{\zeta_\alpha(L)\}$ is defined as follows:

$$\zeta_{\alpha+1}(L)/\zeta_\alpha(L) = \zeta_1(L/\zeta_\alpha(L)) \quad \text{for any ordinal } \alpha,$$

$$\zeta_\alpha(L) = \bigcup_{\beta<\alpha} \zeta_\beta(L) \quad \text{for any limit ordinal } \alpha.$$

By set-theoretic consideration this series terminates for some ordinal, in the sense that from that ordinal onwards all terms are equal. This terminal $\zeta_\alpha(L)$ is called the hypercenter of L and denoted by $\zeta_*(L)$. If $L=\zeta_*(L)$ then L is called hypercentral, and the class of hypercentral Lie algebras is denoted by $\mathfrak{Z}$.

Let us recall several classes of Lie algebras according to [1]:

$L \in \mathfrak{F}$ iff L is finite-dimensional.

$L \in \mathfrak{F}_n$ iff L is of dimension $\leqq n$.

$L \in \mathfrak{N}$ iff L is nilpotent.

$L \in \mathrm{L}\mathfrak{N}$ iff every finitely generated subalgebra of L is nilpotent.

$L \in \mathfrak{E}$ iff for every $x, y \in L$ there exists a positive integer $m = m(x, y)$ such that $[x, {}_m y] = 0$.

The class $\mathfrak{E}$ is called the class of Engel algebras.

A subalgebra H of a Lie algebra L is called ascendant in L and denoted by H asc L if and only if there is an ascending series of subalgebras

$$H = H_0 \leqq H_1 \leqq \cdots H_\sigma = L$$

where $H_\alpha \lhd H_{\alpha+1}$ for all $0 \leqq \alpha < \sigma$ and $H_\lambda = \bigcup_{\mu < \lambda} H_\mu$ for all limit ordinals $\lambda \leqq \sigma$.

3.

We begin with the following

LEMMA 1. *Let L be an Engel algebra. If I is a hypercentral ideal of L and $x \in L$, then the subalgebra $I + \langle x \rangle$ is hypercentral.*

PROOF. Put $K = I + \langle x \rangle$. First we show that if $K \neq 0$ then $\zeta_1(K) \neq 0$. We may assume that $I \neq 0$. Let $Z = \zeta_1(I)$. Then $0 \neq Z \lhd K$. For any non-zero $y \in Z$ there exists a positive integer $n = n(x, y)$ such that $[y, {}_{n-1}x] \neq 0$ and $[y, {}_n x] = 0$. Since $[y, {}_{n-1}x] \in \zeta_1(I)$, we have $[[y, {}_{n-1}x], K] = 0$. Hence $0 \neq [y, {}_{n-1}x] \in \zeta_1(K)$.

Assume that $K \notin \mathfrak{Z}$. Then $K/\zeta_*(K) \neq 0$. Now obviously $K/\zeta_*(K) \in \mathfrak{E}$ and $(I + \zeta_*(K))/\zeta_*(K)$ is a $\mathfrak{Z}$-ideal of $K/\zeta_*(K)$. Furthermore

$$K/\zeta_*(K) = (I + \zeta_*(K))/\zeta_*(K) + (\langle x \rangle + \zeta_*(K))/\zeta_*(K).$$

Therefore from the first part of the proof we deduce that $\zeta_1(K/\zeta_*(K)) \neq 0$, which is contrary to the definition of $\zeta_*(K)$. Thus $K \in \mathfrak{Z}$.

PROPOSITION 2. *Let L be an Engel algebra and I be a hypercentral ideal of L. If L/I is finite-dimensional, then L is hypercentral.*

PROOF. Let $n = \dim L/I$. If $n \leqq 1$, then the assertion is trivial by Lemma 1. Let $n \geqq 2$ and assume inductively that the assertion is true for $n-1$. By using the well-known Engel's theorem on finite-dimensional Lie algebras $L/I \in \mathfrak{F} \cap \mathfrak{E} \leqq \mathfrak{N}$. Hence there exists J such that $I \leqq J \lneqq L$ and $L/J \in \mathfrak{F}_1$. It follows that $J/I \in \mathfrak{F}_{n-1}$. Since $J \in \mathfrak{E}$, by induction hypothesis $J \in \mathfrak{Z}$. Therefore by Lemma 1 we have $L \in \mathfrak{Z}$.

We remark that the condition $L/I \in \mathfrak{F}$ cannot be replaced by $L/I \in \mathfrak{N}$. In fact, the Roseblade-Stonehewer algebra L[1, p. 120] lies in $\mathfrak{E} \backslash \mathfrak{Z}$ and has an abelian ideal V such that $L/V \in \mathfrak{N} \backslash \mathfrak{F}$.

PROPOSITION 3. *Let L be a Lie algebra. If I is a $\mathfrak{Z}$-ideal of L and H is an ascendant $\mathfrak{Z}$-subalgebra of L, then $I+H$ is an ascendant $\mathfrak{Z}$-subalgebra of L.*

PROOF. Put $J=I+H$. Obviously J asc L. We may assume that $J\neq 0$. First we assert that $\zeta_1(J)\neq 0$. If $I=0$ then $\zeta_1(J)=\zeta_1(H)\neq 0$. So we suppose that $I\neq 0$. Then $\zeta_1(I)\neq 0$. Let $(H_\alpha)_{\alpha\leqq\sigma}$ be an ascending series from H to L. Then there exists an ordinal β minimal with respect to $H_\beta\cap\zeta_1(I)\neq 0$. Clearly β is not a limit ordinal. If $\beta=0$, then $0\neq H\cap\zeta_1(I)\lhd H$ and hence we have easily $0\neq\zeta_1(H)\cap\zeta_1(I)\leqq\zeta_1(J)$. If $\beta\neq 0$, then $H_{\beta-1}\cap\zeta_1(I)=0$. It follows that $[H_\beta\cap\zeta_1(I), J]=0$. Hence $0\neq H_\beta\cap\zeta_1(I)\leqq\zeta_1(J)$, as asserted.

Now assume that $J\notin\mathfrak{Z}$. Then $J/\zeta_*(J)\neq 0$ and

$$J/\zeta_*(J) = (I+\zeta_*(J))/\zeta_*(J) + (H+\zeta_*(J))/\zeta_*(J).$$

Furthermore $(I+\zeta_*(J))/\zeta_*(J)$ is a $\mathfrak{Z}$-ideal of $J/\zeta_*(J)$ and $(H+\zeta_*(J))/\zeta_*(J)$ is an ascendant $\mathfrak{Z}$-subalgebra of $J/\zeta_*(J)$. By the preceding assertion we have $\zeta_1(J/\zeta_*(J))\neq 0$, which is a contradiction. Therefore $J\in\mathfrak{Z}$.

As a consequence of Proposition 3 we obtain the following

COROLLARY 4. *Let L be a Lie algebra with $L^2\lneqq L$. If every proper subalgebra of L is an ascendant $\mathfrak{Z}$-subalgebra of L, then L is hypercentral.*

PROOF. Since $L^2\lneqq L$, L has an ideal I of codimension 1. Hence $L=I+\langle x\rangle$ for some $x\in L$. Furthermore $I\in\mathfrak{Z}$ and $\langle x\rangle$ asc L. Therefore the statement follows from Proposition 3.

4.

In this section we shall present some more characterizations of hypercentral Lie algebras. The first one is a Lie analogue of [2, Theorem 2.19].

PROPOSITION 5. *Let L be a Lie algebra. Then $a\in\zeta_*(L)$ if and only if for any given sequence $x_1, x_2,\cdots$ of elements of L there exists a positive integer n such that $[a, x_1,\cdots, x_n]=0$.*

PROOF. Let $a\in\zeta_*(L)$ and $a_k=[a, x_1,\cdots, x_k]$ for $k=1, 2,\cdots$. Let α be an ordinal minimal with respect to $\zeta_\alpha(L)\ni a_k$ for some k. It is easy to see that α is not a limit ordinal. If $\alpha\neq 0$ then $a_{k+1}=[a_k, x_{k+1}]\in\zeta_{\alpha-1}(L)$, which contradicts the minimality of α. Hence $\alpha=0$. It follows that $a_k=0$ for some k.

Next let $a\notin\zeta_*(L)$. Then it is immediate that $[a, L]\nleqq\zeta_*(L)$ and therefore we can choose $x_1\in L$ such that $[a, x_1]\notin\zeta_*(L)$. In the same manner we obtain a sequence $x_1, x_2,\cdots$ of elements of L such that $[a, x_1,\cdots, x_k]\neq 0$ for any k. This completes the proof.

PROPOSITION 6. *Let L be an Engel algebra. Then the following statements are equivalent*:

(1) *L is hypercentral.*

(2) *L has an ascending series of ideals whose factors are* 1-*dimensional.*

(3) *L has an ascending series of ideals whose factors are finite-dimensional.*

PROOF. To show that (1) implies (2), let $(\zeta_\alpha(L))_{\alpha\leqq\sigma}$ be the upper central series of L. For each α, any subspace X with $\zeta_\alpha(L)\subseteqq X\subseteqq\zeta_{\alpha+1}(L)$ is an ideal of L. Therefore we obtain a required series by adding some members between $\zeta_\alpha(L)$ and $\zeta_{\alpha+1}(L)$ for each α.

It is obvious that (2) implies (3).

Finally we show that (3) implies (1). Let $(L_\alpha)_{\alpha\leqq\sigma}$ be an ascending series of ideals of L whose factors are finite-dimensional. Since $L\in\mathfrak{E}$ we have $L_{\alpha+1}/L_\alpha \in\mathfrak{E}\cap\mathfrak{F}\leqq\mathfrak{N}$ for each α. Hence L has an ascending $\mathfrak{F}_1$-series $(H_\alpha)_{\alpha\leqq\tau}$. Now suppose that $L\notin \mathrm{L}\mathfrak{N}$. Then there exists an ordinal α minimal with respect to $H_\alpha\notin\mathrm{L}\mathfrak{N}$. Here α is not a limit ordinal and $H_\alpha=H_{\alpha-1}+<x>$ for some $x\in H_\alpha$. For any finite subset F of H_α there is an $\mathfrak{F}$-subalgebra H of $H_{\alpha-1}$ such that $F\subseteqq H+<x>$. Hence $F\subseteqq <H^x>+<x>$. Since $L\in\mathfrak{E}$ and $H\in\mathfrak{F}$ we have $[H,{}_kx]=0$ for some positive integer k. Therefore $<H^x>\in\mathrm{L}\mathfrak{N}\cap\mathfrak{G}\leqq\mathfrak{F}$, whence $<H^x>+<x>\in\mathfrak{F}\cap\mathfrak{E}\leqq\mathfrak{N}$. Hence $H_\alpha\in\mathrm{L}\mathfrak{N}$, which contradicts the choice of α. Thus $L\in\mathrm{L}\mathfrak{N}$.

Now if $L\notin\mathfrak{Z}$, then $L\lneqq\zeta_*(L)$. Therefore there exists an ordinal β minimal with respect to $L_\beta\nleqq\zeta_*(L)$. β is then neither 0 nor a limit ordinal, and therefore $\beta-1$ exists. Since $L_{\beta-1}\leqq L_\beta\cap\zeta_*(L)\lneqq L_\beta$, we have

$$(L_\beta+\zeta_*(L))/\zeta_*(L)\cong L_\beta/(L_\beta\cap\zeta_*(L))\in\mathfrak{F}\backslash(0),$$

$$(L_\beta+\zeta_*(L))/\zeta_*(L)\lhd L/\zeta_*(L)\in\mathrm{L}\mathfrak{N}.$$

Then by [1, Lemma 7.1.6] there exists a positive integer r such that

$$(L_\beta+\zeta_*(L))/\zeta_*(L)\leqq\zeta_r(L/\zeta_*(L)),$$

which is in contradiction to $\zeta_r(L/\zeta_*(L))=0$. Therefore $L\in\mathfrak{Z}$.

In Proposition 2 we have proved that in an Engel algebra L with an ideal I, $I\in\mathfrak{Z}$ and $L/I\in\mathfrak{F}$ imply $L\in\mathfrak{Z}$. In this context it is worth while noting the following

COROLLARY 7. *Let L be an Engel algebra with an ideal I. If* $I\in\mathfrak{F}$ *and* $L/I\in\mathfrak{Z}$ *then* $L\in\mathfrak{Z}$.

5.

Finally we summarize the results in Sections 3 and 4 in the following theorem.

THEOREM 8. *Let L be a non-trivial Lie algebra. Then the following statements are equivalent*:

(1) *L is hypercentral.*

(2) $L^2 \lneqq L$ *and every proper subalgebra of L is an ascendant hypercentral subalgebra of L.*

(3) *L is the sum of a hypercentral ideal of L and an ascendant hypercentral subalgebra of L.*

(4) *For any given sequence* $x_0, x_1, \cdots$ *of elements of L there exists a non-negative integer n such that* $[x_0, x_1, \cdots, x_n] = 0$.

(5) *L is an Engel algebra and has an ascending series of ideals whose factors are* 1*-dimensional.*

(6) *L is an Engel algebra and has an ascending series of ideals whose factors are finite-dimensional.*

(7) *L is an Engel algebra and has a hypercentral ideal I such that L/I is finite-dimensional.*

(8) *L is an Engel algebra and has a finite-dimensional ideal I such that L/I is hypercentral.*

References

[1] R. K. Amayo and I. Stewart, Infinite-dimensional Lie Algebras, Noordhoff, Leyden, 1974.

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