Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.yu/filomat

Filomat **24:4** (2010), 9–16 DOI: 10.2298/FIL1004009J

SOME PROPERTIES OF K-CONTACT RIEMANNIAN MANIFOLDS ADMITTING A SEMI-SYMMETRIC NON-METRIC CONNECTION

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Abstract

In this paper we present an investigation of differential geometric structures arising on immersed manifolds in K-contact Riemannian manifolds admitting semi-symmetric non-metric connection. Some properties of semi-symmetric non-metric connection in K-Contact Riemannian manifolds are also obtained.

1 Introduction

In 1992, Agashe and Chafle [4] defined a semi-symmetric non-metric connection in a Riemannian manifold. Recently Chaubey [1] defined a new semi-symmetric non-metric connection in an almost contact metric manifold. The paper is organized as follows: In Section 2, we give brief introduction about K-Contact Riemannian manifolds. In Section 3, we first give some formulae for semi-symmetric non- metric connection which we use later. Then we have shown that a K-Contact Riemannian manifold admitting a semi-symmetric non-metric F-connection $\tilde{B}$ is completely integrable. In Section 4, we have studied the induced connection on the submanifold. At first it is shown that the induced connection on the almost contact metric submanifold of almost contact metric manifold with a semi-symmetric non-metric connection is also a semi-symmetric non-metric. Finally we have proved that the submanifold is totally geodesic (totally umbilical) with respect to the induced Riemannian connection D^* if and only if it is totally geodesic (totally umbilical) with respect to the induced semi-symmetric non-metric connection $\tilde{B}^*$.

2000 *Mathematics Subject Classifications.* 53B15.
Key words and Phrases. Semi-symmetric non-metric connection, K-Contact Riemannian manifold, induced connection, totally geodesic, totally umbilical.
Received: August 5, 2009
Communicated by Ljubiša Kočinac

2 Preliminaries

Let there exist an odd dimensional differentiable manifold $M_n, (n = 2m + 1)$, of differentiability class C^∞, a vector valued linear function F, a 1-form A and a vector field T, satisfying

$$\bar{\bar{X}} + X = A(X)T, \tag{2.1}$$

$$A(\bar{X}) = 0, \tag{2.2}$$

where

$$\bar{X} = F(X),$$

for arbitrary vector field X, then M_n is said to be an almost contact manifold and the system $\{F, A, T\}$ is said to give an almost contact structure to M_n. By virtue of (2.1) and (2.2), we find that

$$A(T) = 1, \tag{2.3}$$

$$\bar{T} = 0, \tag{2.4}$$

and

$$rank\{F\} = n - 1.$$

If the associated Riemannian metric g of type $(0, 2)$ in M_n satisfies the following condition

$$g(\bar{X}, \bar{Y}) = g(X, Y) - A(X)A(Y), \tag{2.5}$$

for arbitrary vector fields X, Y in M_n, then (M_n, g) is said to be an almost contact metric manifold and the structure $\{F, A, T, g\}$ is called an almost contact metric structure to M_n, [3].

Putting T for X in (2.5) and then using (2.3) and (2.4), we find that

$$A(X) = g(X, T). \tag{2.6}$$

If we define

$$'F(X, Y) = g(\bar{X}, Y), \tag{2.7}$$

then

$$'F(X, Y) +' F(Y, X) = 0. \tag{2.8}$$

For a K-Contact Riemannian manifold, we have

$$D_X T = \bar{X}, \tag{2.9}$$

where D is the Riemannian connection [3].

3 Semi-Symmetric non-metric Connection

Here we consider a semi-symmetric non-metric connection $\tilde{B}$ on (M_n, g) given by [1]

$$\tilde{B}_X Y = D_X Y +' F(X,Y)T. \tag{3.10}$$

The torsion tensor S of the connection $\tilde{B}$ and the Riemannian metric g of the type $(0,2)$ satisfy the following conditions

$$S(X,Y) = 2'F(X,Y)T, \tag{3.11}$$
$$(\tilde{B}_X g)(Y,Z) = -A(Y)'F(X,Z) - A(Z)'F(X,Y). \tag{3.12}$$

Also curvature tensor with respect to the semi-symmetric non-metric connection in K-Contact Riemannian manifold is given by

$$\begin{aligned} R(X,Y,Z) &= K(X,Y,Z) + g(\bar{Y},Z)\bar{X} - g(\bar{X},Z)\bar{Y} \\ &\quad + g((D_X F)(Y) - (D_Y F)(X), Z)T. \end{aligned} \tag{3.13}$$

In an almost Grayan manifold following relations also hold [3]

$$(D_X{}'F)(\bar{\bar{Y}}, \bar{\bar{Z}}) + (D_X{}'F)(\bar{Y}, \bar{Z}) = 0 \tag{3.14}$$
$$(D_X{}'F)(\bar{Y}, \bar{\bar{Z}}) = (D_X{}'F)(\bar{\bar{Y}}, \bar{Z}). \tag{3.15}$$

Now we have

$$\begin{aligned} X('F(Y,Z)) &= (\tilde{B}_X{}'F)(Y,Z) +' F(\tilde{B}_X Y, Z) +' F(Y, \tilde{B}_X Z) \\ &= (D_X{}'F)(Y,Z) +' F(D_X Y, Z) +' F(Y, D_X Z). \end{aligned}$$

Using equations (3.10) and (2.4) in the above relation, we get

$$(\tilde{B}_X{}'F)(Y,Z) = (D_X{}'F)(Y,Z). \tag{3.16}$$

Theorem 3.1. *Let $\tilde{B}$ be a semi-symmetric non-metric connection, then*
a) in an almost Grayan manifold

$$'N(\bar{X}, \bar{Y}, \bar{Z}) +' N(\bar{\bar{X}}, \bar{Y}, \bar{\bar{Z}}) = 0, \tag{3.17}$$

b) in a K-contact Riemannian manifold

$$(\tilde{B}_X A)(Y) = 0. \tag{3.18}$$

Proof. a) We know that the Nijenhuis tensor N is defined by [3]

$$N(X,Y) = (D_{\bar{X}} F)Y - (D_{\bar{Y}} F)X - \overline{(D_X F)Y} + \overline{(D_Y F)X}.$$

Using (3.10), (2.4) and (2.6) in the above equation, we have

$$N(X,Y) = (\tilde{B}_{\bar{X}} F)Y - (\tilde{B}_{\bar{Y}} F)X - \overline{(\tilde{B}_X F)Y} + \overline{(\tilde{B}_Y F)X} + 2g(X,\bar{Y})T, \tag{3.19}$$
$$\begin{aligned} 'N(X,Y,Z) &= (\tilde{B}_{\bar{X}}{}'F)(Y,Z) - (\tilde{B}_{\bar{Y}}{}'F)(X,Z) + (\tilde{B}_X{}'F)(Y,\bar{Z}) \\ &\quad - (\tilde{B}_Y{}'F)(X,\bar{Z}) + 2g(X,\bar{Y})A(Z), \end{aligned} \tag{3.20}$$

where $'N(X,Y,Z) = g(N(X,Y)Z)$. Barring X , Y , Z in the equation (3.20) and then using (2.2), we get

$$'N(\bar{X},\bar{Y},\bar{Z}) = (\tilde{B}_{\bar{\bar{X}}}{}'F)(\bar{Y},\bar{Z}) - (\tilde{B}_{\bar{\bar{Y}}}{}'F)(\bar{X},\bar{Z}) + (\tilde{B}_{\bar{X}}{}'F)(\bar{Y},\bar{\bar{Z}}) - (\tilde{B}_{\bar{Y}}{}'F)(\bar{X},\bar{\bar{Z}}). \tag{3.21}$$

Again barring X , Z in the above equation and then using (2.1), we have

$$'N(\bar{\bar{X}},\bar{Y},\bar{\bar{Z}}) = -(\tilde{B}_{\bar{X}}{}'F)(\bar{Y},\bar{\bar{Z}}) - (\tilde{B}_{\bar{\bar{Y}}}{}'F)(\bar{\bar{X}},\bar{\bar{Z}}) - (\tilde{B}_{\bar{\bar{X}}}{}'F)(\bar{Y},\bar{Z}) + (\tilde{B}_{\bar{Y}}{}'F)(\bar{\bar{X}},\bar{Z}). \tag{3.22}$$

Adding equations (3.21) and (3.22) and then using (3.16), (3.14) and (3.15) we get relation (3.17).

b) Now taking co-variant derivative of (2.6) with respect to $\tilde{B}$, we get

$$\tilde{B}_X \{A(Y)\} = \tilde{B}_X \{g(T,Y)\}, \tag{3.23}$$

which implies

$$(\tilde{B}_X A)(Y) + A(\tilde{B}_X Y) = (\tilde{B}_X g)(T,Y) + g(\tilde{B}_X T, Y) + g(T, \tilde{B}_X Y). \tag{3.24}$$

Using (2.6) in the above equation, we get

$$(\tilde{B}_X A)(Y) = (\tilde{B}_X g)(T,Y) + g(\tilde{B}_X T, Y). \tag{3.25}$$

In consequence of equations (3.12), (2.3) and (2.4) the above equation assumes the form

$$(\tilde{B}_X A)(Y) = -'F(X,Y) + g(Y, D_X T). \tag{3.26}$$

Using the equation (2.9) in the above equation we get (3.18). □

Theorem 3.2. *A K-contact Riemannian manifold admitting a semi-symmetric non-metric F-connection $\tilde{B}$ is completely integrable.*

Proof. It is known that an almost contact metric manifold is completely integrable if it satisfies the following two conditions [3]

$$\overline{N(\bar{X},\bar{Y})} = 0 \Leftrightarrow{}' N(\bar{X},\bar{Y},\bar{Z}) = 0, \tag{3.27}$$

$$A(N(\bar{X},\bar{Y})) = A(N(X,Y)). \tag{3.28}$$

Since $\tilde{B}$ is F-connection, we have from (3.20)

$$'N(X,Y,Z) = 2g(X,\bar{Y})A(Z). \tag{3.29}$$

Barring X , Y , Z in (3.29) we at once get the condition (3.27). Now from equations (3.19) and (2.2), we have

$$A(N(X,Y)) = A(\tilde{B}_{\bar{X}}\bar{Y} - \tilde{B}_{\bar{Y}}\bar{X}) + 2g(X,\bar{Y}). \tag{3.30}$$

Using the fact $A(\tilde{B}_{\bar{X}}\bar{Y}) = -(\tilde{B}_{\bar{X}}A)(\bar{Y})$ in the above equation, we obtain

$$A(N(X,Y)) = -(\tilde{B}_{\bar{X}}A)(\bar{Y}) + (\tilde{B}_{\bar{Y}}A)(\bar{X}) + 2g(X,\bar{Y}). \tag{3.31}$$

By virtue of (3.18), equation (3.31) gives

$$A(N(X,Y)) = 2g(X,\bar{Y}). \tag{3.32}$$

Barring X , Y in the above equation and then using (2.5), we get

$$A(N(\bar{X},\bar{Y})) = 2g(X,\bar{Y}). \tag{3.33}$$

From the equations (3.32) and (3.33), we have the condition (3.28). □

4 Induced Connection on the Submanifold

Let $M_n, (n = 2m + 1)$ be an odd dimensional differentiable manifold of class C^∞ and M_{n-2} be a submanifold of M_n. Let $p : M_{n-2} \to M_n$ be an inclusion map such that $b \in M_{n-2} \Rightarrow p\,b \in M_n$. The map p induces a Jacobian map $P : T_{n-2} \to T_n$ where T_{n-2} is the tangent space to M_{n-2} at b and T_n is the tangent space to M_n at $p\,b$ such that λ in M_{n-2} at $b \Rightarrow P\lambda$ in M_n at $p\,b$. Let G be the metric tensor of M_n and g the induced metric tensor of M_{n-2} at the points bp and p respectively. Then

$$G(P\lambda, P\mu)ob = g(\lambda, \mu), \tag{4.34}$$

where λ and μ are arbitrary vector fields in the submanifold M_{n-2}. Let N_1 and N_2 be two mutually orthogonal unit normals to the submanifold M_{n-2} such that

$$\begin{aligned} &(a)\ G(P\lambda, N_1) = G(P\lambda, N_2) = G(N_1, N_2) = 0,\\ &(b)\ G(N_1, N_1) = G(N_2, N_2) = 1. \end{aligned} \tag{4.35}$$

Let the almost contact manifold M_n admit a semi-symmetric non-metric connection $\tilde{B}$ given by (3.10), then we have

$$\begin{aligned} &(a)\ \overline{P\lambda} = P\,f\,\lambda + \alpha(\lambda)N_1 + \gamma(\lambda)N_2,\\ &(b)\ T = P\,t + \rho N_1 + \sigma N_2, \end{aligned} \tag{4.36}$$

where t is a vector field in the submanifold M_{n-2}.

Let D^* be the induced connection on the submanifold M_{n-2}. Then we have

$$D_{P\lambda}P\mu = P(D^*_\lambda \mu) + h_1(\lambda,\mu)N_1 + h_2(\lambda,\mu)N_2, \tag{4.37}$$

where h_1 , h_2 are second fundamental tensors of the submanifold M_{n-2}. Let $\tilde{B}^*$ be the induced connection on the submanifold for semi-symmetric non-metric connection $\tilde{B}$ with respect to the unit normals N_1 , N_2. Then we have

$$\tilde{B}_{P\lambda}P\mu = P(\tilde{B}^*_\lambda \mu) + m_1(\lambda,\mu)N_1 + m_2(\lambda,\mu)N_2, \tag{4.38}$$

for arbitrary vector fields $\lambda\,,\mu$ and $m_1\,,m_2$ are tensor fields of type $(0,2)$ of the submanifold M_{n-2}.

It is well known that the necessary and sufficient conditions that the submanifold M_{n-2} be an almost contact metric submanifold with the structure $\{f,t,a\}$ in almost contact metric manifold M_n are [2]

$$(A(PX))ob = a(X), \tag{4.39}$$

$$F(PX) = pfX. \tag{4.40}$$

Theorem 4.1. *The induced connection on the almost contact metric submanifold of almost contact metric manifold with semi-symmetric non-metric connection is also a semi-symmetric non-metric connection.*

Proof. From equation (3.10), we have

$$\tilde{B}_{P\lambda}P\mu = D_{P\lambda}P\mu +' F(P\lambda, P\mu)T. \tag{4.41}$$

Using (4.37), (4.38) and (4.40) in the above equation, we obtain

$$\begin{aligned} P(\tilde{B}^*_\lambda\mu) + m_1(\lambda,\mu)N_1 + m_2(\lambda,\mu)N_2 \\ = P(D^*_\lambda\mu) + h_1(\lambda,\mu)N_1 + h_2(\lambda,\mu)N_2 + G(Pf\lambda, P\mu)T, \end{aligned} \tag{4.42}$$

from which we can find

$$\tilde{B}^*_\lambda\mu = D^*_\lambda\mu + g(f\lambda,\mu)t \tag{4.43}$$

$$(a)\, h_1(\lambda,\mu) = m_1(\lambda,\mu), \quad (b)\, h_2(\lambda,\mu) = m_2(\lambda,\mu). \tag{4.44}$$

Now

$$\begin{aligned} \lambda(g(\mu,\upsilon)) &= (\tilde{B}^*_\lambda g)(\mu,\upsilon) + g(\tilde{B}^*_\lambda\mu,\upsilon) + g(\mu,\tilde{B}^*_\lambda\upsilon) \\ &= g(D^*_\lambda\mu,\upsilon) + g(\mu, D^*_\lambda\upsilon) \\ \Rightarrow \quad (\tilde{B}^*_\lambda g)(\mu,\upsilon) &= -g(\bar{\lambda},\mu)g(t,\upsilon) - g(\bar{\lambda},\upsilon)g(\mu,t) \\ &= -g(\bar{\lambda},\mu)a(\upsilon) - g(\bar{\lambda},\upsilon)a(\mu), \end{aligned} \tag{4.45}$$

where $a(\lambda) = g(t,\lambda)$.

Also from (4.43), we have

$$\begin{aligned} \tilde{B}^*_\lambda\mu - \tilde{B}^*_\mu\lambda - [\lambda,\mu] &= g(\bar{\lambda},\mu)t - g(\bar{\mu},\lambda)t \\ &= 2g(\bar{\lambda},\mu)t \\ &= 2\,{}'f(\lambda,\mu)t, \end{aligned} \tag{4.46}$$

where $g(f\lambda,\mu) = {}'f(\lambda,\mu)$.

The theorem follows from equations (4.45) and (4.46). □

Theorem 4.2. *(a) The mean curvature of the submanifold M_{n-2} with respect to the Riemannian connection D^* coincides with the mean curvature of the submanifold M_{n-2} with respect to the semi-symmetric non-metric connection $\tilde{B}^*$.*

(b) The submanifold M_{n-2} is totally geodesic with respect to the Riemannian connection D^* if and only if it is totally geodesic with respect to the semi-symmetric non-metric connection $\tilde{B}^*$.

(c) The submanifold M_{n-2} is totally umbilical with respect to the Riemannian connection D^* if and only if it is totally umbilical with respect to the semi-symmetric non-metric connection $\tilde{B}^*$.

Proof. Define D^*P and $\tilde{B}^*P$ respectively by

$$(D^*P)(\lambda,\mu) = (D^*_\lambda P)\mu = D_{P\lambda}P\mu - P(D^*_\lambda \mu), \tag{4.47}$$

$$(\tilde{B}^*P)(\lambda,\mu) = (\tilde{B}^*_\lambda P)\mu = \tilde{B}_{P\lambda}P\mu - P(\tilde{B}^*_\lambda \mu). \tag{4.48}$$

In view of (4.37) and (4.38) the equations (4.47) and (4.48) can be rewritten as

$$(D^*_\lambda P)\mu = h_1(\lambda,\mu)N_1 + h_2(\lambda,\mu)N_2, \tag{4.49}$$

$$(\tilde{B}^*_\lambda P)\mu = m_1(\lambda,\mu)N_1 + m_2(\lambda,\mu)N_2. \tag{4.50}$$

respectively.

Let λ_1, λ_2, ..., λ_{n-2} be $(n-2)$ orthonormal local vector fields in the submanifold M_{n-2}. Then the function $\frac{1}{n-2}\sum_{i=1}^{n-2} h(e_i,e_i)$ is called the mean curvature of the submanifold M_{n-2} with respect to the Riemannian connection D^* and $\frac{1}{n-2}\sum_{i=1}^{n-2} m(e_i,e_i)$ is called the mean curvature of the submanifold M_{n-2} with respect to the semi-symmetric non-metric connection $\tilde{B}^*$.

If h_1, h_2 vanish, then the submanifold M_{n-2} is said to be totally geodesic with respect to the Riemannian connection D^* and if h_1, h_2 is proportional to g, then the submanifold M_{n-2} is called totally umbilical with respect to the Riemannian connection D^*. Similarly if m_1, m_2 vanish, then the submanifold M_{n-2} is said to be totally geodesic with respect to the semi-symmetric non-metric connection $\tilde{B}^*$ and if m_1, m_2 is proportional to g, then the submanifold M_{n-2} is called totally umbilical with respect to the semi-symmetric non-metric connection $\tilde{B}^*$.

The proof at once follows from the equation (4.44). □

Acknowledgement. The authors are thankful to the referee for his/her valuable suggestions.

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