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SOME PROPERTIES OF NEGATIVE PINCHED RIEMANNIAN MANIFOLDS OF DIMENSIONS 5 AND 7

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1. Let M be a compact orientable Riemannian manifold, and denote by $K^p(M, \boldsymbol{R})$ the vector space of Killing p-forms of the manifold M over the field $\boldsymbol{R}$ of real numbers. It has been shown [3] that if the manifold M is negative k-pinched and of even dimension $n = 2m$ (resp. odd dimension $n = 2m + 1$), and $k > 1/4$ (resp. $k > 2(m-1)/(8m-5)$), then $K^2(M, \boldsymbol{R}) = 0$. In this paper, we have improved the above result for negative pinched manifolds of dimensions 5 and 7.

2. We consider a compact orientable negative k-pinched Riemannian manifold M. If α, β are two exterior p-forms of the manifold, then the local product of the two forms α, β and the norm of α are defined by

$$(\alpha, \beta) = \frac{1}{p!}\alpha^{i_1\cdots i_p}\beta_{i_1\cdots i_p} = \frac{1}{p!}\alpha_{i_1\cdots i_p}\beta^{i_1\cdots i_p} ,$$

$$|\alpha|^2 = \frac{1}{p!}\alpha^{i_1\cdots i_p}\alpha_{i_1\cdots i_p} .$$

If η is the volume element of the manifold M, then the global product of the two exterior p-forms α, β and the global norm of α are defined by

$$\langle\alpha, \beta\rangle = \int_M (\alpha, \beta)\eta ,$$

$$\|\alpha\|^2 = \int_M |\alpha|^2\eta .$$

It is well known that the following relation holds [1, p. 187]:

$$\langle\alpha, \Delta\alpha\rangle = \|\delta\alpha\|^2 + \|d\alpha\|^2 . \tag{2.1}$$

We also have the formula [2, p. 3]:

$$\frac{1}{2}\Delta(|\alpha|^2) = (\alpha, \Delta\alpha) - |\nabla\alpha|^2 + \frac{1}{2(p-1)!}Q_p(\alpha) , \tag{2.2}$$

where

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$$(2.3)\qquad \begin{aligned} Q_p(\alpha) = (p-1)R_{klmn}\alpha^{kli_3\cdots i_p}\alpha^{mn}{}_{i_3\cdots i_p} \\ -2R_{kl}\alpha^{ki_2\cdots i_p}\alpha^{l}{}_{i_2\cdots i_p}\,, \end{aligned}$$

$$(2.4)\qquad |\nabla\alpha|^2 = \frac{1}{p!}\nabla^k\alpha^{i_1\cdots i_p}\nabla_k\alpha_{i_1\cdots i_p}\,.$$

If $\alpha \in K^p(M, R)$, then it is easy to prove, using the property of α [4, p. 66]:

$$\nabla_X\alpha(Y, X_2, \cdots, X_p) + \nabla_Y\alpha(X, X_2, \cdots, X_p) = 0\,, \quad \text{for } Y, X, X_l \in T(M)\,,$$

and the relation

$$(2.5)\qquad (\alpha, \Delta\alpha) = -(p+1)Q_p(\alpha)/p!\,,$$

where $l = 2, \cdots, p$.

Let P be a point of the manifold M, and consider a normal coodinate system on the manifold with origin at the point P. It is well known that there is an orthonormal basis $\{X_1, \cdots, X_n\}$ in the tangent space M_p such that its dual basis $\{X_1^*, \cdots, X_n^*\}$ has the property that the exterior 2-form α at the point P takes the form

$$(2.6)\qquad \alpha = \alpha_{12}X_1^* \wedge X_2^* + \alpha_{34}X_3^* \wedge X_4^* + \cdots + \alpha_{2m-1,2m}X_{2m-1}^* \wedge X_{2m}^*\,.$$

where $m = [n/2]$.

Since the manifold M is negative k-pinched, the components of the Riemannian curvature tensor at the point P satisfy the relations and the inequalities [3]:

$$\langle R(X_i, X_j)X_l, X_h\rangle = R_{ijhl}\,,$$
$$\sigma_{ij} = \sigma(X_i, X_j) = R_{ijij}\,,$$

$$(2.7)\qquad -1 \le \sigma_{ij} \le -k\,,\quad |R_{ijil}| \le \frac{1}{2}(1-k)\,,\quad |R_{ijhl}| \le \frac{2}{3}(1-k)\,,$$

where $i \ne j \ne h \ne l$.

3. Suppose that the manifold M is of dimension 5, and let α be an element of the vector space $K^2(M, R)$. Then we form the following exterior 4-form

$$(3.1)\qquad \beta = \frac{1}{2}\alpha \wedge \alpha\,.$$

In this case, the formula (2.6) takes the form

$$(3.2)\qquad \alpha = \alpha_{12}X_1^* \wedge X_2^* + \alpha_{34}X_3^* \wedge X_4^*\,.$$

The relation (3.1) by virtue of (3.2) becomes

(3.3) $$\beta = \alpha_{12}\alpha_{34}X_1^* \wedge X_2^* \wedge X_3^* \wedge X_4^* .$$

From (3.2) and (3.3) we obtain

(3.4) $$|\alpha|^2 = \alpha_{12}^2 + \alpha_{34}^2 , \qquad |\beta| = \alpha_{12}\alpha_{34} .$$

For the exterior 4-form β the formula (2.4) becomes

(3.5) $$|\nabla\beta|^2 = \nabla^k\beta^{i_1i_2i_3i_4}\nabla_k\beta_{i_1i_2i_3i_4} , \qquad i_1 < i_2 < i_3 < i_4 .$$

In the general case, the coefficients $\beta_{i_1i_2i_3i_4}$ of the exterior 4-form β are given by

(3.6) $$\beta_{i_1i_2i_3i_4} = \alpha_{i_1i_2}\alpha_{i_3i_4} + \alpha_{i_1i_3}\alpha_{i_4i_2} + \alpha_{i_1i_4}\alpha_{i_2i_3} .$$

By means of (3.6) and from the fact that α is a Killing 2-form, the relation (3.5) becomes

(3.7) $$|\nabla\beta|^2 \leq \alpha_{12}^2 T_1 + \alpha_{34}^2 T_2 ,$$

where T_1 and T_2 are linear expressions of terms of the form $\underset{\lambda<\mu<\nu}{(\nabla_\lambda\alpha_{\mu\nu})^2}$ whose coefficients are 0, 1, 4. Since α is a Killing 2-form, we have

(3.8) $$|\nabla\alpha|^2 = 3\underset{\lambda<\mu<\nu}{(\nabla_\lambda\alpha_{\mu\nu})^2} .$$

From (3.7) and (3.8) and the property of T_1, T_2 we obtain the inequality

(3.9) $$|\nabla\beta|^2 \leq \frac{4}{3}|\nabla\alpha|^2|\alpha|^2 .$$

If we estimate $\frac{1}{2}Q_2(\alpha)$ from the formula (2.3), we have

$$\begin{aligned}\frac{1}{2}Q_2(\alpha) = &-(\sigma_{13} + \sigma_{14} + \sigma_{15} + \sigma_{23} + \sigma_{24} + \sigma_{25})\alpha_{12}^2 \\ &- (\sigma_{31} + \sigma_{32} + \sigma_{35} + \sigma_{41} + \sigma_{42} + \sigma_{45})\alpha_{34}^2 \\ &+ 4R_{1234}\alpha_{12}\alpha_{34} ,\end{aligned}$$

which gives the inequality, by means of (2.7) and (3.4),

(3.10) $$\frac{1}{2}Q_2(\alpha) \geq 6k|\alpha|^2 - \frac{8}{3}(1-k)|\beta| .$$

If we also estimate $\frac{1}{2}Q_4(\beta)$ from the same formula (2.3), we obtain

$$\frac{1}{2}Q_4(\beta) = -3!(\sigma_{15} + \sigma_{25} + \sigma_{35} + \sigma_{45})\alpha_{12}^2\alpha_{34}^2 ,$$

which implies the inequality, by means of the first of (2.7) and the second of (3.4),

$$\frac{1}{2}Q_4(\beta) \geq 4!k|\beta|^2 . \tag{3.11}$$

It is clear that the above calculations have been done at the point P with respect to the special orthonormal frame in the tangent space M_P.

4. If we integrate the formula (3.9), we obtain

$$\|\nabla\beta\|^2 \leq \frac{4}{3}\int_M |\alpha|^2|\nabla\alpha|^2\eta . \tag{4.1}$$

The relation (2.2) for the exterior 4-form β becomes

$$\frac{1}{2}\Delta(|\beta|^2) = (\beta, \Delta\beta) - |\nabla\beta|^2 + \frac{1}{6\cdot 2}Q_4(\beta) ,$$

from which we have

$$0 = \int_M (\beta, \Delta\beta)\eta - \|\nabla\beta\|^2 + \frac{1}{6}\int_M \frac{1}{2}Q_4(\beta)\eta . \tag{4.2}$$

By means of (2.1) and (3.11), the above equation (4.2) gives

$$\|d\beta\|^2 + \|\delta\beta\|^2 - \|\nabla\beta\|^2 + 4k\|\beta\|^2 \leq 0 ,$$

or finally

$$\|\nabla\beta\|^2 \geq 4k\|\beta\|^2 . \tag{4.3}$$

It is well known that the following formula holds

$$\frac{1}{2}\Delta(|\alpha|^4) = |\alpha|^2\Delta(|\alpha|^2) - (d(|\alpha|^2))^2 ,$$

from which we obtain

$$\int_M |\alpha|^2\Delta(|\alpha|^2)\eta = \int_M (d(|\alpha|^2))^2\eta \geq 0 . \tag{4.4}$$

Since α is a Killing 2-form, (2.2) takes the form, by means of (2.5),

$$\frac{1}{2}\Delta(|\alpha|^2) = -|\nabla\alpha|^2 - \frac{1}{4}Q_2(\alpha) ,$$

which, by integration of the manifold M and the inequalities (3.10) and (4.4), gives the inequality

$$3\int_M |\alpha|^2 |\nabla\alpha|^2\eta \leq \int_M [4(1-k)|\beta||\alpha|^2 - 9k|\alpha|^4]\eta \,. \tag{4.5}$$

The inequality (4.5) together with (4.1) and (4.3) implies

$$9\,\|\beta\|^2 k \leq \int_M [4(1-k)|\beta||\alpha|^2 - 9k|\alpha|^4]\eta \,,$$

or

$$\int_M [9k|\alpha|^4 - 4(1-k)|\beta||\alpha|^2 + 9|\beta|^2 k]\eta \leq 0 \,.$$

Let f be the function defined by

$$f = 9k|\alpha|^4 - 4(1-k)|\beta||\alpha|^2 + 9|\beta|^2 k \,,$$

which at the point P takes the form

$$f = 9k(\alpha_{12}^2 + \alpha_{34}^2)^2 - 4(1-k)\alpha_{12}\alpha_{34}(\alpha_{12}^2 + \alpha_{34}^2) + 9k\alpha_{12}^2\alpha_{34}^2 \,. \tag{4.6}$$

It is easy to show that if $k > 8/53$, then $f \geq 0$, where the equality holds if $\alpha_{12} = \alpha_{34} = 0$.

From the above we derive

Theorem I. *Let M be a compact orientable negative k-pinched manifold of dimension 5. If $k > 8/53$, then $K^2(M, \boldsymbol{R}) = 0$.*

5. We assume that the manifold M is of dimension 7. In this case, the relation (2.6) becomes

$$\alpha = \alpha_{12}X_1^* \wedge X_1^* + \alpha_{34}X_3^* \wedge X_4^* + \alpha_{56}X_5^* \wedge X_6^* \,. \tag{5.1}$$

Let γ be the exterior 6-form defined by

$$\gamma = \frac{1}{3!}\alpha \wedge \alpha \wedge \alpha \,,$$

which by means of (5.1) becomes

$$\gamma = \alpha_{12}\alpha_{34}\alpha_{56}X_1^* \wedge \cdots \wedge X_6^* \,. \tag{5.2}$$

From (5.1) and (5.2) we obtain

$$|\alpha|^2 = \alpha_{12}^2 + \alpha_{34}^2 + \alpha_{56}^2 \,, \qquad |\gamma| = \alpha_{12}\alpha_{34}\alpha_{56} \,. \tag{5.3}$$

If we apply the same technique as in § 3, we obtain, in this case, the inequalities

$$\frac{1}{2}Q_2(\alpha) \geq 10k\,|\alpha|^2 - \frac{8}{3}(1-k)\theta\,, \tag{5.4}$$

$$\frac{1}{2}Q_6(\gamma) \geq 6!k\,|\gamma|^2\,, \tag{5.5}$$

where

$$\theta = \alpha_{12}\alpha_{34} + \alpha_{34}\alpha_{56} + \alpha_{56}\alpha_{12}\,. \tag{5.6}$$

In the general case, the coefficients $\gamma_{\nu_1\cdots\nu_6}$ of the exterior 6-form γ are given by

$$\gamma_{\nu_1\cdots\nu_6} = \alpha_{\nu_1\nu_2}A + \alpha_{\nu_1\nu_3}B + \alpha_{\nu_1\nu_4}C + \alpha_{\nu_1\nu_5}D + \alpha_{\nu_1\nu_6}E\,, \tag{5.7}$$

where

$$\begin{aligned} A &= \alpha_{\nu_3\nu_4}\alpha_{\nu_5\nu_6} + \alpha_{\nu_3\nu_5}\alpha_{\nu_6\nu_4} + \alpha_{\nu_3\nu_6}\alpha_{\nu_4\nu_5}\,, \\ B &= \alpha_{\nu_2\nu_4}\alpha_{\nu_6\nu_5} + \alpha_{\nu_2\nu_5}\alpha_{\nu_4\nu_6} + \alpha_{\nu_2\nu_6}\alpha_{\nu_5\nu_4}\,, \\ C &= \alpha_{\nu_2\nu_3}\alpha_{\nu_5\nu_6} + \alpha_{\nu_2\nu_5}\alpha_{\nu_6\nu_3} + \alpha_{\nu_2\nu_6}\alpha_{\nu_3\nu_5}\,, \\ D &= \alpha_{\nu_2\nu_3}\alpha_{\nu_6\nu_4} + \alpha_{\nu_2\nu_4}\alpha_{\nu_3\nu_6} + \alpha_{\nu_2\nu_6}\alpha_{\nu_4\nu_3}\,, \\ E &= \alpha_{\nu_2\nu_3}\alpha_{\nu_4\nu_5} + \alpha_{\nu_2\nu_4}\alpha_{\nu_5\nu_3} + \alpha_{\nu_2\nu_5}\alpha_{\nu_3\nu_4}\,, \end{aligned}$$

The formula (2.4) for the exterior 6-form γ becomes

$$|\nabla\gamma|^2 = \nabla^k\gamma^{i_1\cdots i_6}\nabla_k\gamma_{i_1\cdots i_6}\,, \qquad i_1 < i_2 < \cdots < i_6\,,$$

which, by means of (5.7) and from the fact that α is a Killing 2-form, is reduced to

$$|\nabla\gamma|^2 \leq \alpha_{12}^2\alpha_{34}^2\sum\nolimits_1 + \alpha_{34}^2\alpha_{56}^2\sum\nolimits_2 + \alpha_{56}^2\alpha_{12}^2\sum\nolimits_3\,, \tag{5.8}$$

where $\sum_1, \sum_2, \sum_3$ are linear expressions of the terms of the form $\underset{\lambda<\mu<\nu}{(\nabla_\lambda\alpha_{\mu\nu})^2}$ whose coefficients are 0, 1, 2, 5.

From (3.8), (5.8) and the property of $\sum_1, \sum_2, \sum_3$ we derive the inequality

$$|\nabla\gamma|^2 \leq \frac{5}{3}\,|\nabla\alpha|^2\,(\alpha_{12}^2\alpha_{34}^2 + \alpha_{34}^2\alpha_{56}^2 + \alpha_{56}^2\alpha_{12}^2)\,,$$

which, by means of the inequality

$$(\alpha_{12}^2 + \alpha_{34}^2 + \alpha_{56}^2)^2 \geq 3(\alpha_{12}^2\alpha_{34}^2 + \alpha_{34}^2\alpha_{56}^2 + \alpha_{56}^2\alpha_{12}^2)\,,$$

takes the form

$$(5.9)\qquad |\nabla\gamma|^2 \leq \frac{5}{9}|\nabla\alpha|^2|\alpha|^4 .$$

6. From (5.9) we obtain

$$(6.1)\qquad \frac{9}{5}\|\nabla\gamma\|^2 \leq \int_M |\alpha|^4|\nabla\alpha|^2\eta .$$

It is well known that the following relation holds

$$\frac{1}{3}\Delta(|\alpha|^6) = |\alpha|^4\Delta(|\alpha|^2) - 2|\alpha|^2(d(|\alpha|^2))^2 ,$$

which implies

$$(6.2)\qquad \int_M |\alpha|^4\Delta(|\alpha|^2)\eta \geq 0 .$$

Since α is a Killing 2-form, then the relation (2.2), by virtue of (2.5), becomes

$$\frac{1}{2}\Delta(|\alpha|^2) = -|\Delta\alpha|^2 - \frac{1}{4}Q_2(\alpha) ,$$

or

$$\frac{1}{2}|\alpha|^4\Delta(|\alpha|^2) = -|\alpha|^4|\nabla\alpha|^2 - |\alpha|^4\frac{1}{4}Q_2(\alpha) ,$$

from which by integration on the manifold M and by the inequalities (5.4) and (6.2) we obtain

$$(6.3)\qquad 3\int_M |\alpha|^4|\nabla\alpha|^2\eta \leq \int_M [4(1-k)\theta|\alpha|^4 - 15k|\alpha|^6]\eta .$$

The formula (2.2) for the 6-form γ becomes

$$\frac{1}{2}\Delta(|\gamma|^2) = (\gamma, \Delta\gamma) - |\nabla\gamma|^2 + \frac{1}{5!\cdot 2}Q_6(\gamma) ,$$

from which by integration on the manifold M we have

$$0 = \int_M (\gamma, \Delta\gamma)\eta - \|\nabla\gamma\|^2 + \frac{1}{5!}\int_M \frac{1}{2}Q_6(\gamma)\eta ,$$

which, by means of (2.1) and (5.5), takes the form

$$\|d\gamma\|^2 + \|\delta\gamma\|^2 - \|\nabla\gamma\|^2 + 6k\,\|\gamma\|^2 \leq 0\,,$$

or

$$\|\nabla\gamma\|^2 \geq 6k\,\|\gamma\|^2\,. \tag{6.4}$$

From the inequalities (6.1), (6.3) and (6.4) we derive the inequality

$$\int_M [75\,|\alpha|^6\,k - 20\,|\alpha|^4\,\theta(1-k) + 162k\,|\gamma|^2]\eta \leq 0\,.$$

We denote by F the following function

$$F = 75\,|\alpha|^6\,k - 20\,|\alpha|^4\,\theta(1-k) + 162k\,|\gamma|^2\,,$$

which, by means of (5.3) and (5.6), takes the form, at the point P,

$$\begin{aligned} F = {} & 75k(\alpha_{12}^2 + \alpha_{34}^2 + \alpha_{56}^2)^3 - 20(1-k)(\alpha_{12}^2 + \alpha_{34}^2 + \alpha_{56}^2)^2 \\ & \cdot(\alpha_{12}\alpha_{34} + \alpha_{34}\alpha_{56} + \alpha_{56}\alpha_{12}) + 162k(\alpha_{12}\alpha_{34}\alpha_{56})^2\,. \end{aligned} \tag{6.5}$$

It is easy to show the inequalities

$$\begin{aligned} & \alpha_{12}\alpha_{34} + \alpha_{34}\alpha_{56} + \alpha_{56}\alpha_{12} \leq \alpha_{12}^2 + \alpha_{34}^2 + \alpha_{56}^2\,, \\ & 27(\alpha_{12}\alpha_{34}\alpha_{56})^2 \leq (\alpha_{12}^2 + \alpha_{34}^2 + \alpha_{56}^2)^3\,. \end{aligned} \tag{6.6}$$

From (6.5) and the inequalities (6.6) we conclude that if $k > 20/101$, then $F \geq 0$, where the equality holds if $\alpha_{12} = \alpha_{34} = \alpha_{56} = 0$.

From the above we derive

Theorem II. *Let M be a compact orientable negative k-pinched manifold of dimension 7. If $k > 20/101$, then $K^2(M, \boldsymbol{R}) = 0$.*

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