

# SOME PROPERTIES OF NON-COMMUTATIVE REGULAR GRADED RINGS

by THIERRY LEVASSEUR

(Received 26 March, 1991)

**Introduction.** Let  $A$  be a noetherian ring. When  $A$  is commutative (of finite Krull dimension),  $A$  is said to be Gorenstein if its injective dimension is finite. If  $A$  has finite global dimension, one says that  $A$  is regular. If  $A$  is arbitrary, these hypotheses are not sufficient to obtain similar results to those of the commutative case. To remedy this problem, M. Auslander has introduced a supplementary condition. Before stating it, we recall that the grade of a finitely generated (left or right) module is defined by

$$j_A(M) = \inf\{i / \text{Ext}_A^i(M, A) \neq 0\} \in \mathbb{N} \cup \{+\infty\}.$$

The Auslander condition is: for all  $M$  as above, for all  $i \geq 0$  and every  $A$ -submodule  $N$  of  $\text{Ext}_A^i(M, A)$ , one has  $j_A(N) \geq i$ . A ring of finite (left and right) injective dimension (resp. finite homological global dimension) which satisfies the Auslander condition is called Auslander-Gorenstein (resp. Auslander-regular). On the other hand, assume that  $A$  is a graded algebra over a field  $k$ , of the form  $A = k \oplus A_1 \oplus A_2 \oplus \dots$ . Then M. Artin and W. Schelter have introduced in [1] a definition of regularity for  $A$ . (See Section 6.1 for a precise definition.) We shall then say that  $A$  is AS-regular.

Let  $A$  be a commutative regular affine  $k$ -algebra, and assume for simplicity that  $A$  is a domain. One knows that  $A$  is a Cohen-Macaulay ring, which is equivalent to:

$$\dim M + j_A(M) = \dim A \quad \text{for all non-zero finitely generated } A\text{-modules } M, \quad (\text{CM})$$

where  $\dim$  denotes the Krull dimension. When  $A$  is noncommutative, a natural substitute for the Krull dimension is the Gelfand-Kirillov dimension, denoted by  $\text{GKdim}$ . We then have many examples of Auslander-regular  $k$ -algebras which have the property (CM), where  $\dim$  is replaced by  $\text{GKdim}$ . The main examples occur when  $A$  is a filtered ring with a regular commutative graded associated ring, see [6]. The work of [3], [4] shows that the property (CM) plays an important role in the study of modules over AS-regular algebras.

The purpose of the present work is to compare the two notions of regularity introduced above and to study the property (CM) for graded noetherian  $k$ -algebras. The paper is organized as follows.

Sections 1 and 2 recall results about filtered and graded rings and Auslander-Gorenstein or regular rings.

Section 3 investigates the behaviour of the Auslander condition when we factor a graded  $k$ -algebra  $A = \bigoplus_{n \geq 0} A_n$  by a normal homogeneous non-zero divisor of positive degree.

In Section 4, we show that over an Auslander-Gorenstein ring of injective dimension  $\mu$ ,  $M \rightarrow \delta(M) = \mu - j_A(M)$  defines a dimension function in the sense of [17]. This is used to prove that an Auslander-regular  $k$ -algebra  $A = k \oplus A_1 \oplus A_2 \oplus \dots$  is a domain.

Section 5 studies the property (CM) over a graded  $k$ -algebra such that finitely generated graded modules have polynomial growth; we call this condition (PG). The main results are obtained in 5.10 and 5.13. Theorem 5.10 gives the following. Let  $A = \bigoplus_{n \geq 0} A_n$  be a finitely generated positively graded  $k$ -algebra with  $\dim_k A_0 < \infty$ ; suppose

$\Omega \in A_d, d > 0$ , is a normal non-zero divisor in  $A$  such that  $A/\Omega A$  is Auslander-Gorenstein and satisfies the condition (PG) and property (CM); then the same is true for  $A$ . In 5.13, a criterion is given for a graded  $k$ -algebra of injective dimension 2 to be an Auslander-Gorenstein algebra satisfying property (CM).

In Section 6, we compare the two notions of regularity given above. Theorem 6.3 shows that an Auslander-regular graded  $k$ -algebra  $A = k \oplus A_1 \oplus A_2 \oplus \dots$  such that  $n \rightarrow \dim_k A_n$  has polynomial growth is AS-regular. In 6.6, we consider the algebras  $B(X, \sigma, \mathcal{L})$  constructed in [5] from an elliptic curve  $X$ , an automorphism  $\sigma$  of  $X$  and an invertible sheaf  $\mathcal{L}$  of degree at least 3 on  $X$ . Using 5.13, we show that  $B(X, \sigma, \mathcal{L})$  is Auslander-Gorenstein of dimension 2 and has the property (CM). As a corollary (see 6.7), we deduce that the Sklyanin algebra as defined in [18] is Auslander-regular and satisfies the property (CM).

In [15], this last fact is used to obtain detailed results about graded modules over the Sklyanin algebra.

**1. Preliminaries.**

1.1. For the rest of the paper, we fix a commutative field  $k$ . The dimension of a  $k$ -vector space is denoted by  $\dim_k$ .

Let  $A$  be a ring. We denote by  $\text{Mod}(A)$  (resp.  $\text{Mod}_f(A)$ ) the category of left or right  $A$ -modules (resp. finitely generated  $A$ -modules). We write  ${}_A M$  or  $M_A$  to indicate that  $M$  is a left or right  $A$ -module. The ring  $A$  is called noetherian if it is left and right noetherian.

Let  ${}_A M$  (resp.  $M_A$ ) be in  $\text{Mod}(A)$  and  $a \in A$ . We say that  $a$  is a non-zero-divisor (n.z.d) in  $M$  if  $ax = 0, x \in M$ , implies  $x = 0$  (resp.  $xa = 0$  implies  $x = 0$ ). If  $a \in A$  is a n.z.d. in  ${}_A A$  (resp.  $A_A$ ), we say that  $a$  is a right (resp. left) n.z.d. in  $A$ . If  $a$  is a left and right n.z.d., we simply say that  $a$  is a n.z.d.

1.2. Let  $M$  be an  $A$ -module. We let  $\text{pd}_A(M)$  denote the projective dimension of  $M$  and  $\text{injdim}_A(M)$  the injective dimension of  $M$ . We say that  $A$  has finite global dimension (resp. finite injective dimension) if the left and right global dimensions of  $A$  are finite and equal (resp. the modules  ${}_A A$  and  $A_A$  have finite injective dimensions which are equal). In that case we denote these numbers by  $\text{gldim}(A)$  (resp.  $\text{injdim}(A)$ ).

DEFINITION. Let  $M$  be in  $\text{Mod}(A)$ . The grade number of  $M$  is

$$j_A(M) = \inf\{i / \text{Ext}_A^i(M, A) \neq (0)\} \in \mathbb{N} \cup \{+\infty\}.$$

If no confusion can arise, we write  $j(M)$  for  $j_A(M)$ . Notice that  $j_A((0)) = +\infty$ . When  $A$  is noetherian,  $j_A(M) \leq \text{pd}_A(M)$ , and if furthermore  $\text{injdim}(A) = \mu < \infty$ , we have  $j_A(M) \leq \mu$  for all non-zero  $M \in \text{Mod}_f(A)$  (see Remark 2.2(1)).

1.3. An (increasing) filtration  $\{A(n)\}_{n \in \mathbb{Z}}$  on  $A$  is a sequence of additive subgroups of  $A$  such that:  $1 \in A(0), A(n)A(m) \subset A(n+m), A = \bigcup_{n \in \mathbb{Z}} A(n)$ . The associated graded ring, denoted by  $G(A)$  or  $\text{gr}(A)$ , is the direct sum  $\bigoplus_{n \in \mathbb{Z}} A(n)/A(n-1)$ . We have similar definitions for a decreasing filtration. When  $A$  is a  $k$ -algebra, we require that  $k \subset A(0)$ . If  $\bigcap_{n \in \mathbb{Z}} A(n) = (0)$ , the filtration is called separated. We refer to [17] for the notions of filtered modules, associated graded modules, etc.

1.4. Assume  $A = \bigoplus_{n \in \mathbb{Z}} A_n$  is a graded ring. We say that  $A$  is positively graded if  $A_n = (0)$  for all  $n < 0$ . Then  $\mathcal{M} = \bigoplus_{n > 0} A_n$  is called the augmentation ideal and  $A_0 \cong A/\mathcal{M}$  is a bimodule over  $A$ . If furthermore  $A$  is a  $k$ -algebra with  $A_0 = k$  then we write  $A = k \oplus A_1 \oplus A_2 \oplus \dots$ . We denote by  $\text{Mod}^{\mathbb{g}}(A)$  (resp.  $\text{Mod}_f^{\mathbb{g}}(A)$ ) the category of graded  $A$ -modules (resp. finitely generated graded  $A$ -modules). Let  $M = \bigoplus_m M_m \in \text{Mod}^{\mathbb{g}}(A)$ . For  $p \in \mathbb{Z}$ , the shifted module  $M[p]$  is defined in  $\text{Mod}^{\mathbb{g}}(A)$  by setting  $M[p]_m = M_{p+m}$ . Recall that if  $M, N \in \text{Mod}^{\mathbb{g}}(A)$  then  $\text{HOM}_A(M, N)$  is the  $\mathbb{Z}$ -graded group such that  $\text{HOM}_A(M, N)_p = \{\phi \in \text{Hom}_A(M, N) / \phi(M_m) \subset N_{m+p} \text{ for all } m\}$ . When  $M \in \text{Mod}_f^{\mathbb{g}}(A)$ , one has  $\text{HOM}_A(M, N) = \text{Hom}_A(M, N)$ ; thus the derived functors  $\text{EXT}_A^i(M, -)$  coincide with the usual  $\text{Ext}_A^i(M, -)$ . When not otherwise specified, a map  $M \rightarrow N$  between graded  $A$ -modules will be an element of  $\text{HOM}_A(M, N)_0$ .

The graded ring  $A$  has finite graded injective dimension  $\mu$  if  ${}_A A$  and  $A_A$  are both of injective dimension  $\mu$  in the category  $\text{Mod}^{\mathbb{g}}(A)$ . We then write  $\text{grinjdim}(A) = \mu$ . Recall that  $E \in \text{Mod}^{\mathbb{g}}(A)$  is injective in  $\text{Mod}^{\mathbb{g}}(A)$  if and only if the functor  $\text{HOM}_A(-, E)$  is exact, i.e.  $\text{EXT}_A^i(-, E) = 0$  for all  $i \geq 1$ . It follows that  $\text{grinjdim}(A) = \mu$  is equivalent to  $\text{EXT}_A^i(-, {}_A A) = \text{EXT}_A^i(-, A_A) = (0)$  for  $i > \mu$ ,  $\text{EXT}_A^\mu(-, {}_A A) \neq (0)$  and  $\text{EXT}_A^\mu(-, A_A) \neq (0)$ . Similarly we can define the graded global dimension of  $A$ , denoted by  $\text{grgldim}(A)$ ; details are left to the reader.

The following result is proved in [7, 2.18].

**PROPOSITION.** *Let  $A$  be a graded ring. Then  $A$  is noetherian if and only if it is graded noetherian, i.e. every left or right graded ideal is finitely generated.*

1.5. We recall here some well-known facts of Homological Algebra. Our reference is [9] to which the reader is referred for the details. We fix a noetherian ring  $A$ . Assume that to each  ${}_A M \in \text{Mod}_f(A)$  is attached a projective resolution  $R.(M)$  by finitely generated projective  $A$ -modules. We set  $E_A^*(M) = H^*(\text{Hom}_A(R.(M), A))$ . The abelian groups  $E_A^*(M)$  are finitely generated right  $A$ -modules and are naturally identified with the groups  $\text{Ext}_A^*(M, A)$ . A similar result holds for  $M_A \in \text{Mod}_f(A)$ . We shall often write  $E^*(M)$  in place of  $E_A^*(M)$  when there is no possible ambiguity. Recall that if  $M', M'' \in \text{Mod}_f(A)$  and  $f \in \text{Hom}_A(M', M)$ ,  $f' \in \text{Hom}_A(M'', M')$ , there exist morphisms  $\alpha^*(f): E_A^*(M) \rightarrow E_A^*(M')$ ,  $\alpha^*(f'): E_A^*(M') \rightarrow E_A^*(M'')$  such that  $\alpha^*(f \circ f') = \alpha^*(f') \circ \alpha^*(f): E_A^*(M) \rightarrow E_A^*(M'')$ .

**REMARKS.** (1) If  $\alpha^n(f)$  and  $\alpha^n(f')$  are injective for some  $n \in \mathbb{N}$  then we get compatible injections  $E_A^n(M) \subset E_A^n(M') \subset E_A^n(M'')$  given by the maps  $\alpha^n(f)$ ,  $\alpha^n(f')$ ,  $\alpha^n(f \circ f')$ .

(2) Assume that  $A$  is graded,  $M \in \text{Mod}_f^{\mathbb{g}}(A)$  and that  $R.(M)$  consists of finitely generated projective graded  $A$ -modules. Then the modules  $E_A^*(M)$  are naturally graded and, for all  $p \in \mathbb{Z}$ , we have  $E_A^*(M[p]) = E_A^*(M)[-p]$ .

**2. Auslander conditions.** In this section, we summarize the results we shall use about Auslander-Gorenstein, or Auslander-regular, rings. The main references are [6], [7], [8], [10], [14], [16].

2.1. DEFINITION. Let  $A$  be a noetherian ring.

(a) An  $A$ -module  $M$  satisfies the Auslander-condition if:  $\forall q \geq 0, j_A(N) \geq q$  for all  $A$ -submodules  $N$  of  $E_A^q(M)$ .

(b) The ring  $A$  is said to be Auslander-Gorenstein (resp. Auslander-regular) of dimension  $\mu$  if:  $\text{injdim}(A) = \mu < \infty$  (resp.  $\text{gldim}(A) = \mu < \infty$ ), and every  $M \in \text{Mod}_f(A)$  satisfies the Auslander-condition.

2.2. The principal tool for studying Auslander-Gorenstein rings is given by the following result whose proof can be found in [14].

THEOREM. Assume  $A$  is a noetherian ring with  $\mu = \text{injdim}(A) < \infty$ . Let  $M$  be in  $\text{Mod}_f(A)$ .

(a) There exists a convergent spectral sequence in  $\text{Mod}_f(A)$ :

$$E_2^{p,-q}(M) = E_A^p(E_A^q(M)) \Rightarrow \mathbb{H}^{p-q}(M)$$

with  $\mathbb{H}^{p-q}(M) = 0$  if  $p \neq q$  and  $\mathbb{H}^0(M) = M$ . The resulting finite filtration of  $M$  by  $A$ -submodules is called the  $b$ -filtration. It has the form:

$$(0) = F^{\mu+1}M \subset F^\mu M \subset \dots \subset F^1 M \subset F^0 M = M.$$

(b) When  $j_A(E_A^q(M)) \geq q$  for all  $q$ , there are exact sequences

$$0 \rightarrow \frac{F^p M}{F^{p+1} M} \rightarrow E_2^{p,-p}(M) \rightarrow Q(p) \rightarrow 0$$

for all  $p \in \{0, \dots, \mu\}$ , where  $Q(p)$  has a filtration by submodules such that each composition factor is a subquotient of some  $E_2^{p+i+1,-p-i}(M)$  with  $i \geq 1$ .

REMARKS. (1) From (a), it follows that  $j_A(M) \leq \mu$  when  $M \in \text{Mod}_f(A) \setminus \{0\}$ .

(2) Let  $M$  and  $M'$  be in  $\text{Mod}_f(A)$ . Suppose we have fixed projective resolutions  $R.(M)$  and  $R.(M')$  as in 1.5. Let  $f$  be in  $\text{Hom}_A(M', M)$ . By standard constructions in homological algebra, to each map of complexes  $\{f: R.(M') \rightarrow R.(M)\}$  above  $f$  is associated a map  $E.(f)$  between the associated spectral sequences  $E_2^*(M')$ ,  $E_2^*(M)$ , and their invariants. It is easily seen that the induced map  $\mathbb{H}^0(M') \rightarrow \mathbb{H}^0(M)$  coincides with  $f$ . The homomorphisms  $E_2^{p,-q}(f): E_2^{p,-q}(M') \rightarrow E_2^{p,-q}(M)$  come from the maps  $\alpha^q(f): E_A^q(M) \rightarrow E_A^q(M')$  described in 1.5. In particular they are of the form  $\alpha^p(\alpha^q(f))$ . It follows that if  $\alpha^n(f)$  is an isomorphism then  $E_2^{p,-n}(f)$  is an isomorphism for all  $p$ .

2.3. THEOREM. Let  $A$  be an Auslander-Gorenstein ring. If  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  is an exact sequence in  $\text{Mod}_f(A)$  then we have  $j_A(M) = \inf\{j_A(M'), j_A(M'')\}$ .

This theorem is proved in [7, 1.8]. By analogy with similar properties for dimensions, one can say that the grade number is exact (on short exact sequences). We shall discuss this property in Section 4.

2.4. DEFINITION. Let  $A$  be a noetherian ring and  $M$  be in  $\text{Mod}_f(A) \setminus \{0\}$ . Then  $M$  is called pure if  $j_A(N) = j_A(M)$  for all non-zero submodules  $N$  of  $M$ . When  $n = j_A(M)$  we say that  $M$  is  $n$ -pure.

For an Auslander-Gorenstein ring the  $b$ -filtration of a module has a nice interpretation.

**THEOREM.** *Assume  $A$  is Auslander-Gorenstein and let  $M$  be a non-zero finitely generated  $A$ -module. Then if  $n = j_A(M)$  we have:*

- (a)  $E_A^n(M)$  is  $n$ -pure and  $E_A^p(E_A^p(M))$  is  $(0)$  or  $p$ -pure;
- (b)  $F^p M$  is the largest submodule  $X$  of  $M$  such that  $j_A(X) \geq p$ ;
- (c)  $n = \text{Max}\{p \mid M = F^p M\}$ .

For the proof see [7, 14, 16, 8].

**REMARK.** Obviously  $A$  is 0-pure, i.e.  $L^* = E_A^0(L) \neq (0)$  for all non-zero left or right ideals  $L$  of  $A$ .

**3. Graded Auslander-Gorenstein rings.** In this section  $A = \bigoplus_{m \in \mathbb{Z}} A_m$  is a graded ring.

3.1. Assume  $A$  is noetherian, let  $M$  be in  $\text{Mod}_f^{\mathbb{Z}}(A)$ . As recalled in 1.5, the modules  $E_A^p(M)$  are naturally graded. Therefore the spectral sequence  $E_2^*(M)$  of Theorem 2.2 is in  $\text{Mod}_f^{\mathbb{Z}}(A)$ . Working with graded modules in Definition 2.1, one can obviously define the notion of a “graded-Auslander-Gorenstein, or regular, ring”. Fortunately we have the following result.

**THEOREM** ([10, Theorem 0.1]). *The noetherian graded ring  $A$  is Auslander-Gorenstein (resp. regular) if and only if  $A$  is graded-Auslander-Gorenstein (resp. regular).*

3.2. Let  $B$  be a ring and  $\{B(\nu)\}_\nu$  be a separated filtration on  $B$  with associated graded ring  $G(B)$ . The Rees-ring attached to  $\{B(\nu)\}_\nu$  is  $R(B) := \bigoplus_{\nu} B(\nu)$ . Assume  $R(B)$  is noetherian. Recall that a filtration  $\Gamma = \{M(m)\}_m$  on  $M \in \text{Mod}(B)$  is called good if  $R(M) := \bigoplus_m M(m)$  belongs to  $\text{Mod}_f(R(B))$ . Any  $M \in \text{Mod}_f(B)$  can be equipped with a good filtration, the associated graded module  $G(M)$  is in  $\text{Mod}_f^{\mathbb{Z}}(G(B))$ . The filtration  $\{B(\nu)\}_\nu$ , or the ring  $B$  itself if there is no ambiguity, is called Zariskian if  $R(B)$  is noetherian and good filtrations on elements of  $\text{Mod}_f(B)$  are separated. For instance if the filtration is discrete and  $G(B)$  is noetherian then  $B$  is Zariskian. We recall the following result.

**PROPOSITION** ([7, 3.1, 3.4]). *Let  $B$  be a filtered ring and  $M \in \text{Mod}_f(B)$ . Assume  $R(B)$  is noetherian and let  $\Gamma$  be a good filtration on  $M$  with associated graded module  $G(M)$ .*

(i) *There exists a canonical good filtration on  $E_B^p(M)$  such that  $G(E_B^p(M))$  is a subquotient of  $E_{G(B)}^p(G(M))$  in  $\text{Mod}_f^{\mathbb{Z}}(G(B))$ . If  $B$  is Zariskian one has  $j_B(M) \geq j_{G(B)}(G(M))$ .*

(ii) *Suppose  $B$  is Zariskian and  $G(B)$  is Auslander-Gorenstein. Then  $j_B(M) = j_{G(B)}(G(M))$ .*

3.3. We return to  $A = \bigoplus_m A_m$  graded noetherian. In [10, Theorem 0.2], it is shown that  $\text{gldim}(A)$  (resp.  $\text{injdim}(A)$ ) is finite if and only if  $\text{grgldim}(A)$  (resp.  $\text{grinjdim}(A)$ ) is finite. Furthermore we have bounds:

$$\text{grinjdim}(A) \leq \text{injdim}(A) \leq \text{grinjdim}(A) + 1.$$

When  $A = \bigoplus_{m \geq 0} A_m$  is positively graded, we can filter  $A$  by setting  $A(\nu) = \sum_{i=0}^{\nu} A_i$ . It is clear that  $G(A) = \bigoplus_{\nu} A(\nu)/A(\nu-1)$  and  $A$  are equal. Thus  $\{A(\nu)\}_\nu$  is Zariskian. Notice that if

$M = \bigoplus_{i=-d}^{+\infty} M_i$  is a left-limited graded module, the filtration  $\left\{M(\nu) := \sum_{i=-d}^{\nu} M_i\right\}_{\nu}$  is good, the modules  $G(M)$  and  $M$  coincide in  $\text{Mod}^{\mathbb{R}}(A)$ . In this situation we have the following (presumably well-known) result (we include a proof for completeness).

LEMMA. Let  $A = \bigoplus_{m \geq 0} A_m$  be a positively graded noetherian ring. Then  $\text{injdim}(A) < \infty$  if and only if  $\text{grinjd}\dim(A) < \infty$ , in which case these two numbers are equal.

Proof. We filter  $A$  by  $\{A(\nu)\}_{\nu}$  as above. Suppose  $\mu = \text{injdim}(A) < \infty$  and let  $M$  be in  $\text{Mod}_f(A)$  such that  $N := E_A^{\mu}(M) \neq (0)$ . Choose any good filtration on  $M$ . By Proposition 3.2(i) and the previous remarks, we know that  $G(N)$  is a non-zero subquotient of  $E_{G(A)}^{\mu}(G(M)) = E_A^{\mu}(G(M))$  for some good filtration on  $N$ . Hence  $E_A^{\mu}(G(M)) \neq (0)$ . Since  $G(M) \in \text{Mod}_f^{\mathbb{R}}(A)$ , we get  $\text{grinjd}\dim(A) \geq \mu = \text{injdim}(A)$ . The lemma then follows from the results of [10] quoted above.

3.4. Let  $B$  be any ring and let  $\Omega$  be a normal element of  $B$ , i.e.  $\Omega B = B\Omega$ . Assume  $\Omega$  is a n.z.d. in  $B$ . Then we can define an automorphism  $\sigma$  of  $B$  by the rule:  $b\Omega = \Omega\sigma(b)$ . If  ${}_B M \in \text{Mod}(B)$ , we shall denote by  ${}^{\sigma}M \in \text{Mod}(B)$  the abelian group  $M$  with left  $B$ -action given by  $b.x = \sigma(b)x$ . When  $M_B \in \text{Mod}(B)$  we make a similar definition to get  $M^{\sigma} \in \text{Mod}(B)$ . The following assertions are easy to check.

- (1) The left multiplication by  $\Omega$  gives an element of  $\text{Hom}_B({}^{\sigma}M, M)$ .  $\text{Ker}_M \Omega^p = \{x \in M / \Omega^p x = 0\}$  and  $\Omega^p M$  are submodules of  $M$  for all  $p \geq 0$ .
- (2)  $\text{Hom}_B({}^{\sigma}M, B) \cong \text{Hom}_B(M, B)^{\sigma}$ . The isomorphism is given by

$$f \rightarrow \{f^{\sigma} : x \rightarrow \sigma(f(x))\}.$$

(3) The functor  $F : \text{Mod}(B) \rightarrow \text{Mod}(B)$ ,  $F(M) = {}^{\sigma}M$ , is an equivalence of categories and restricts to an equivalence of categories  $F : \text{Mod}_f(B) \rightarrow \text{Mod}_f(B)$ .

(4)  $E_B^i({}^{\sigma}M) \cong E_B^i(M)^{\sigma}$ .

We can filter  $B$  by the  $\Omega$ -adic filtration  $\{\Omega^{\nu}B\}_{\nu \geq 0}$  (it is a decreasing filtration). The following is well known.

PROPOSITION. (a) The associated graded ring  $G(B) = \bigoplus_{\nu} \Omega^{\nu}B / \Omega^{\nu+1}B$  is isomorphic to the graded ring  $(B/\Omega B)[X, \sigma]$ .

(b) (Rees-Lemma) If  $M \in \text{Mod}(B/\Omega B)$  and  $p \geq 0$  we have:  $E_B^{p+1}(M) \cong E_{B/\Omega B}^p(M)$ .

REMARKS. (1) In (a), the graded ring  $(B/\Omega B)[X, \sigma]$  is the Ore-extension defined by  $\bar{b}X = X\sigma(b)$ , where  $X$  is an indeterminate of degree 1.

(2) The  $\Omega$ -adic filtration is not Zariskian in general (it is generally not separated).

(3) The Rees-Lemma shows that  $j_{B/\Omega B}(M) = j_B(M) - 1$ . It is very easy to deduce from it that, if  $B$  is Auslander-Gorenstein of dimension  $\mu$  then  $B/\Omega B$  is Auslander-Gorenstein of dimension at most  $\mu - 1$ .

3.5. Let  $A = \bigoplus_{m \geq 0} A_m$  be positively graded. Assume  $\Omega \in A_d$ ,  $d > 0$ , is a normal n.z.d. in  $A$ . Then it is not difficult to prove the following proposition.

PROPOSITION. (a)  $A$  is noetherian if and only if  $A/\Omega A$  is noetherian.

(b)  $A$  is a domain if  $A/\Omega A$  is a domain.

The proof of (a) is based on  $G(A) = (A/\Omega A)[X, \sigma]$ , cf. 3.3, and the following: if  $L$  is a left or right graded ideal of  $A$ , one has  $L = \bigcap_{q \geq 0} (L + \Omega^q A)$ .

Assume  $A$  is noetherian. Even if the  $\Omega$ -adic filtration is not Zariskian, one has the following well-known result (see [16, 4.6.7]).

LEMMA. *The  $\Omega$ -adic filtration satisfies:*

(1) *the Rees ring  $\bigoplus_{\nu} \Omega^\nu A$  is noetherian, in particular good filtrations induce good filtrations on submodules;*

(2) *good filtrations on elements of  $\text{Mod}_f^g(A)$  are separated.*

For the convenience of the reader we sketch a proof of this lemma.

(1) Put  $F_\nu A = \Omega^{-\nu} A$  if  $\nu \leq 0$  and  $F_\nu A = A$  if  $\nu \geq 0$ . The Rees ring is  $R = \bigoplus_{\nu \in \mathbb{Z}} F_\nu A$ . Let  $X$  be an indeterminate. Define  $\theta: R \rightarrow A[X, X^{-1}]$  by  $\theta\left(\sum_{\nu} a_{\nu}\right) = \sum_{\nu} a_{\nu} X^{\nu}$ . Then  $\theta$  gives an isomorphism from  $R$  onto  $A[\Omega X^{-1}, X]$  and one easily sees that this ring is noetherian. The second assertion is obvious.

(2) Let  $M$  be an object of  $\text{Mod}_f^g(A)$ . Assume that  $\{F_n M\}_n$  is a good filtration on  $M$ . Using [7, 2.20], one obtains

$$\bigcap_n F_n M \subset \bigcap_{j \geq 1} \Omega^j M = \{m \in M \mid (1 - a)m = 0 \text{ for some } a \in \Omega A\}.$$

Since  $M$  is graded and  $\Omega$  is an element of positive degree we get that  $\bigcap_{j \geq 1} \Omega^j M = (0)$ .

3.6. THEOREM. *Let  $A = \bigoplus_{m \geq 0} A_m$  be positively graded,  $\Omega \in A_d$ ,  $d > 0$ ,  $a$  normal n.z.d. in  $A$ ,  $G(A) = \bigoplus_m \Omega^m A / \Omega^{m+1} A$ . Then:*

(1) *if  $A$  is noetherian and  $\text{injdim}(A/\Omega A) < \infty$ , we have  $\text{injdim}(A) = \text{injdim}(G(A)) = \text{injdim}(A/\Omega A) + 1$ ;*

(2)  *$A/\Omega A$  is Auslander-Gorenstein of dimension  $\nu$  if and only if  $A$  is Auslander-Gorenstein of dimension  $\nu + 1$ .*

*Proof.* (1) By the Rees-Lemma, we know  $\text{injdim}(A/\Omega A) \leq \text{injdim}(A) - 1$ . Recall that, if  $R$  is any noetherian ring with  $\text{injdim}(R) < \infty$  and  $\sigma$  is an automorphism of  $R$ , we have that  $\text{injdim}(R[X, \sigma]) = \text{injdim}(R) + 1$  (the proof for  $\sigma$  the identity automorphism can be used unchanged). It follows that  $\text{injdim}(G(A)) = \text{injdim}(A/\Omega A) + 1 \leq \text{injdim}(A)$ . We thus have to show that  $\text{injdim}(A) \leq \text{injdim}(G(A))$ . Suppose  $M \in \text{Mod}_f^g(A)$  and let  $\Gamma$  be any good filtration on  $M$  (with respect to the  $\Omega$ -adic filtration). Since  $R(A)$  is noetherian, Proposition 3.2 (i) implies that  $G(E_A^p(M))$  is a subquotient of  $E_{G(A)}^p(G(M))$ . Since the module  $E_A^p(M)$  is graded, the good filtration on  $E_A^p(M)$  is separated. Therefore  $G(E_A^p(M)) = (0)$  implies  $E_A^p(M) = (0)$ . Notice in passing that we have proved the following result.

SUBLEMMA. *If  $M \in \text{Mod}_f^g(A)$  then  $j_A(M) \geq j_{G(A)}(G(M))$ .*

Put  $\gamma = \text{injdim}(G(A))$ . By the above sublemma,  $E_A^p(M) = (0)$  for all  $p > \gamma$  and  $M \in \text{Mod}_f^g(A)$ . Hence  $\text{grinjdim}(A) \leq \gamma$  and we get equality by Lemma 3.3.

(2) Suppose  $A/\Omega A$  is Auslander-Gorenstein of dimension  $\nu$ . By Proposition 3.5(a), we get  $A$  noetherian. From (1), we know  $\text{injdim}(A) = \nu + 1$ . By Theorem 3.1, we only have to show the “graded-Auslander-condition”: if  $M \in \text{Mod}_f^g(A)$  and  $N$  is a graded submodule of  $E_A^p(M)$ , we have  $j_A(N) \geq p$ .

Choose a good filtration on  $M$ . Then the canonical good filtration on  $E_A^p(M)$  induces a good separated filtration on  $N$ , for which  $G(N)$  is a submodule of  $G(E_A^p(M))$ . By the sublemma of part (1), we get  $j_A(N) \geq j_{G(A)}(G(N))$ . Since  $A/\Omega A$  is Auslander-Gorenstein,  $G(A) \cong (A/\Omega A)[X, \sigma]$  is also Auslander-Gorenstein by [10, Theorem 4.2]. Therefore

$$p \leq j_{G(A)}(E_{G(A)}^p(G(M))) \leq j_{G(A)}(G(E_A^p(M)))$$

(notice that  $G(E_A^p(M))$  is a subquotient of  $E_{G(A)}^p(G(M))$ ).

By these remarks, we have

$$j_{G(A)}(G(E_A^p(M))) \leq j_{G(A)}(G(N)) \leq j_A(N).$$

Hence  $p \leq j_A(N)$ , proving that  $A$  is Auslander-Gorenstein. We already noticed the converse in Remark 3.4 (3).

REMARK. The reader should compare Theorem 3.6 with [16, Theorem 4.6.15] and [10, Theorem 4.3], where similar results are proved, which have been at the origin of this theorem.

**4. The dimension function for an Auslander-Gorenstein ring.**

4.1. Let  $A$  be a noetherian ring. Assume we are given a decreasing chain  $(M_i)_{i \in \mathbb{N}}$  of submodules of  $M_0 = M \in \text{Mod}_f(A)$ . We have natural projections

$$f_{i+j,i} : M/M_{i+j} \rightarrow M/M_i, g_i : M \rightarrow M/M_i$$

which satisfy  $f_{i+2,i} = f_{i+1,i} \circ f_{i+2,i+1}, g_i = f_{i+1,i} \circ g_{i+1}$ . Using the notation of 1.5, we deduce

$$\alpha^*(f_{i+2,i}) = \alpha^*(f_{i+2,i+1}) \circ \alpha^*(f_{i+1,i}), \alpha^*(g_i) = \alpha^*(g_{i+1}) \circ \alpha^*(f_{i+1,i}). \tag{4.1.1}$$

LEMMA. Suppose  $n \in \mathbb{N}$  is such that, for all  $i \geq 0, E_A^{n-1}(M_i) = E_A^{n-1}(M_i/M_{i+1}) = (0)$ . Then there exists an increasing chain of submodules of  $E_A^n(M)$ :

$$E_A^n(M/M_1) \subset \dots \subset E_A^n(M/M_i) \subset \dots \subset E_A^n(M),$$

where each inclusion  $E_A^n(M/M_i) \subset E_A^n(M/M_{i+1})$  is given by  $\alpha^n(f_{i+1,i})$  and is the restriction of the injective map  $\alpha^n(g_i) : E_A^n(M/M_i) \rightarrow E_A^n(M)$ .

*Proof.* By [9, p. 92], we have connecting morphisms  $\alpha^n(f_{i+1,i}) : E^n(M/M_i) \rightarrow E^n(M/M_{i+1})$  and  $\alpha^n(g_i) : E^n(M/M_i) \rightarrow E^n(M)$ . The hypothesis states that these maps are injective. The lemma then follows from 1.5 and (4.1.1).

Under the hypothesis of the previous lemma, we have the next corollary.

COROLLARY. (a) There exists  $q \in \mathbb{N}$  such that  $\alpha^n(f_{q+j,q}) : E^n(M/M_q) \rightarrow E_A^n(M/M_{q+j})$  is an isomorphism for all  $j \geq 0$ .

(b) Assume  $\mu = \text{injdim}(A) < \infty$ . The maps  $E_2^{n-n}(f_{i+1,i})$  between the spectral sequences  $E_2^{*,*}(M/M_{i+1})$  and  $E_2^{*,*}(M/M_i)$ , defined in 2.2, are isomorphisms for all  $i \geq q$ .



*Proof.* (a) follows from  $E_A^n(M) \in \text{Mod}_f(A)$  and  $A$  noetherian.  
 (b) is a consequence of Remark 2.2(2).

4.2. THEOREM. Assume  $A$  is an Auslander-Gorenstein ring. Let  $M$  be in  $\text{Mod}_f(A)$  with  $j_A(M) \geq n$  and let  $(M_i)_{i \in \mathbb{N}}$  be a decreasing chain of submodules of  $M$ . Then there exists  $q \in \mathbb{N}$  such that  $j_A(M_i/M_{i+1}) \geq n + 1$  for all  $i \geq q$ .

*Proof.* Recall that  $j_A(X) \geq j_A(M)$  for any subquotient  $X$  of  $M$  (see Theorem 2.3). Hence we may assume  $M = M_0$  and apply Lemma 4.1 to the chain  $(M_i)_{i \in \mathbb{N}}$ . Put  $N_i = M/M_i$ ; since  $j(N_i) \geq n$ , we have  $N_i = F^n N_i$  in the b-filtration of  $N_i$  (see Theorem 2.4(c)). The map  $f_{i+1,i}: N_{i+1} \rightarrow N_i$  gives, by restriction, an  $A$ -module map between  $F^p N_{i+1}$  and  $F^p N_i$  for all  $p$ . (See Remark 2.2 (2) or Theorem 2.4 (b).)

Let  $q \in \mathbb{N}$  be given by Corollary 4.1. If  $i \geq q$ , the morphism  $\alpha := E_2^{n,-n}(f_{i+1,i})$  is an isomorphism and, by Remark 2.2 (2), we get a commutative diagram

$$\begin{array}{ccc} 0 \rightarrow \frac{F^n N_{i+1}}{F^{n+1} N_{i+1}} & \rightarrow & E_2^{n,-n}(N_{i+1}) \\ & \downarrow \bar{f} & \downarrow \alpha \\ 0 \rightarrow \frac{F^n N_i}{F^{n+1} N_i} & \rightarrow & E_2^{n,-n}(N_i) \end{array}$$

where the vertical map  $\bar{f}$  is induced by  $f_{i+1,i}$ . Consider  $X := (M_i/M_{i+1} + F^{n+1}N_{i+1})/F^{n+1}N_{i+1}$ . It is a submodule of  $N_{i+1}/F^{n+1}N_{i+1} = F^n N_{i+1}/F^{n+1}N_{i+1}$ . Since  $f_{i+1,i}(M_i/M_{i+1}) = (0)$ , we have  $\bar{f}(X) = 0$ . But as  $\alpha$  is an isomorphism, it follows that  $X = (0)$ ; that is,  $M_i/M_{i+1} \subset F^{n+1}N_{i+1}$ . By Theorem 2.4(b), this means  $j(M_i/M_{i+1}) \geq n + 1$ .

REMARK. Theorem 4.2 can be deduced from [7, 1.17], where it is shown that a chain  $M = M_0 \supset M_1 \supset \dots \supset M_p$  with  $j_A(M_i/M_{i+1}) = j_A(M)$  for all  $i$  has at most  $\varepsilon(M)$  terms for some fixed integer  $\varepsilon(M)$ . We shall come back to this number in (4.6.5).

4.3. It follows from [7, 1.19] that if  $M$  is a finitely generated module over an Auslander-Gorenstein ring  $A$  and if  $\phi \in \text{End}_A(M)$  is injective then  $j_A(M/\phi(M)) \geq j_A(M) + 1$ . We are going to prove that result using 4.2.

THEOREM. Let  $A$  be an Auslander-Gorenstein ring,  $\sigma$  an automorphism of  $A$  and  $M \in \text{Mod}_f(A)$ . If  $\phi \in \text{Hom}_A({}^\sigma M, M)$  is injective then we have  $j_A(M/\phi(M)) \geq j_A(M) + 1$ .

*Proof.* Recall that  ${}^\sigma M$  has been defined in 3.4. Obviously we may assume  $M \neq (0)$  and we know  $j(M/\phi(M)) \geq j(M)$  by Theorem 2.3. Suppose  $j(M/\phi(M)) = j(M)$  and set

$$\phi_0(M) := M, \phi_{i+1}(M) := \phi({}^\sigma \phi_i(M)) \text{ for all } i \geq 0.$$

Each  $\phi_i(M)$  is an  $A$ -submodule of  $M$  and  $\phi_1(M) = \phi(M)$ . If  $M \not\supseteq \phi_i(M)$ , an easy induction gives  $\phi_{i+1}(M) \subsetneq \phi_i(M)$  for all  $i$ . Furthermore there exist isomorphisms in  $\text{Mod}_f(A)$

$$\varphi_i: {}^\sigma \left[ \frac{M}{\phi_i(M)} \right] \rightarrow \frac{\phi_i(M)}{\phi_{i+1}(M)}.$$

Denoting by  $\bar{x}$  the class of an element in a quotient module, the  $\varphi_i$ 's are defined as

follows. For  $i = 1$ , we put  $\varphi_i(\bar{x}) = \overline{\phi(x)}$ . Assuming  $\varphi_i$  has been defined, consider

$$\overline{\phi_{i+1}}: \left[ \frac{\phi_i(M)}{\phi_{i+1}(M)} \right]^\sigma \rightarrow \frac{\phi_{i+1}(M)}{\phi_{i+2}(M)}, \quad \overline{\phi_{i+1}(\bar{x})} = \overline{\phi(x)}.$$

Then put  $\varphi_{i+1} := \overline{\phi_{i+1}} \circ \varphi_i$ ; with a slight abuse of notation, we have  $\varphi_{i+1}(\bar{x}) = \overline{\phi(\varphi_i(\bar{x}))}$ . It is easily seen that the  $\varphi_i$ 's are bijective  $A$ -linear maps. Using 3.4 (4), we get  $E_A^*(\phi_i(M)/\phi_{i+1}(M)) \cong E_A^*(M/\phi(M))$ . Hence  $j(\phi_i(M)/\phi_{i+1}(M)) = j(M/\phi(M)) = j(M)$  for all  $i$ . The chain  $\{\phi_i(M)\}_{i \geq 0}$  contradicts Theorem 4.2, hence the result.

4.4. COROLLARY. Let  $A = \bigoplus_{m \geq 0} A_m$  be a positively graded Auslander-Gorenstein ring,  $\Omega \in A_d$ ,  $d > 0$ , a normal n.z.d. in  $A$ . Suppose  $M \in \text{Mod}_f^{\mathbb{R}}(A)$  such that  $\Omega$  is a n.z.d. in  $M$ . Then  $j_A(M/\Omega M) = j_A(M) + 1$ .

*Proof.* Define  $\sigma \in \text{Aut}(A)$  by  $a\Omega = \Omega\sigma(a)$ . Notice that  $\sigma(A_m) = A_m$  for all  $m$ . Therefore the module  ${}^\sigma M$  defined in 3.4 is graded with  $({}^\sigma M)_m = M_m$ . The hypothesis says that there is an exact sequence in  $\text{Mod}_f(A)$

$$0 \rightarrow {}^\sigma M \xrightarrow{\Omega} M \rightarrow M/\Omega M \rightarrow 0. \tag{*}$$

The multiplication by  $\Omega$  is a graded map of degree  $d > 0$ . It induces right multiplication by  $\Omega$  from  $E_A^p(M)$  to  $E_A^p({}^\sigma M)$  for all  $p$ . Suppose  $E_A^{p+1}(M/\Omega M) = (0)$ , the long exact sequence coming from (\*) gives  $E_A^p({}^\sigma M) = E_A^p(M)\Omega$ . By the remark above,  $E_A^p(M) = E_A^p(M)^\sigma$  as graded groups; since  $E_A^p(M)^\sigma = E_A^p({}^\sigma M)$ , see 3.4 (5), the graded Nakayama's lemma forces  $E_A^p(M) = (0)$ . Thus  $j(M/\Omega M) \leq j(M) + 1$ . To get the reverse inequality, apply Theorem 4.3 to  $\Omega \in \text{Hom}_A({}^\sigma M, M)$ .

4.5. DEFINITION. Let  $A$  be an Auslander-Gorenstein ring of dimension  $\mu$ . Define the dimension of  $M \in \text{Mod}_f(A)$  by  $\delta(M) = \mu - j(M)$ .

Note that  $j(M) \in \{0, \dots, \mu\} \cup \{\infty\}$ ; hence  $\delta(M) \in \{0, \dots, \mu\} \cup \{-\infty\}$ . The term dimension given to  $\delta(M)$  is justified by the next result.

PROPOSITION. The function  $\delta: \text{Mod}_f(A) \rightarrow \{0, \dots, \mu, -\infty\}$  is an exact dimension function. Furthermore it is finitely partitive. This means:

- (i)  $\delta((0)) = -\infty$ ,
- (ii) if  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  is exact then  $\delta(M) = \text{Max}\{\delta(M'), \delta(M'')\}$ ,
- (iii) if  $PM = (0)$  for some prime  $P$  and  $M$  is a torsion  $A/P$ -module then  $\delta(M) \leq \delta(A/P) - 1$  (similarly for  $M$  a right module),
- (iv) if  $M = M_0 \supset M_1 \supset \dots \supset M_i \supset \dots$  is a chain of submodules then  $\delta(M_i/M_{i+1}) \leq \delta(M) - 1$  for  $i \gg 0$ .

*Proof.* We follow the terminology of [17, 6.8.4 and 8.7.3]. The properties (i), (ii) and (iv) are clear from 1.2, 2.3, 4.2. To prove (iii), notice that it's enough to show  $\delta(A/P + Aa) \leq \delta(A/P) - 1$  for a non-zero-divisor  $\bar{a}$  in  $A/P$ . This follows from Theorem 4.3 applied to  $\phi: A/P \rightarrow A/P$ ,  $\phi(\bar{x}) = \bar{x}\bar{a}$ .

4.6. We assume  $A$  Auslander-Gorenstein of dimension  $\mu$ . We are going to list consequences and properties coming from the dimension  $\delta$ .

(4.6.1)  $M \in \text{Mod}_r(A)$  is called *homogeneous* if  $\delta(N) = \delta(M)$  for all non-zero submodules  $N$  of  $M$ . If  $\delta(M) = s$ , we shall simply say  $M$  is  $s$ -homogeneous. Notice that  $M$   $s$ -homogeneous is the same as  $M$   $(\mu - s)$ -pure, see 2.4. In particular, by Remark 2.4,  $A$  is  $\mu$ -homogeneous as a left or right module.

(4.6.2)  $M \in \text{Mod}_r(A)$  is said to be *critical* if  $\delta(M/N) < \delta(M)$  for all non-zero submodules  $N$  of  $M$ . If  $\delta(M) = s$ , we say  $M$  is  $s$ -critical. This is equivalent to  $j_A(M/N) > j_A(M) = s$  by definition.

(4.6.3) Any non-zero  $M \in \text{Mod}_r(A)$  has a critical submodule: use the proof of [17, 6.2.10].

(4.6.4) Let  $M$  be in  $\text{Mod}_r(A)$ . A *critical composition series* for  $M$  is a chain  $M = M_t \supset M_{t+1} \supset \dots \supset M_0 = (0)$  such that each factor  $M_i/M_{i+1}$  is critical. Notice that  $\delta(M) = \text{Max}_{1 \leq i \leq t} \{\delta(M_i/M_{i-1})\}$ . Any  $M \in \text{Mod}_r(A)$  has a critical composition series. One can

find a critical series with  $\delta(M_i/M_{i-1}) \geq \delta(M_{i-1}/M_{i-2})$  for all  $i$ . Furthermore, two such critical composition series have the same length. This follows from (4.6.3) together with properties of  $\delta$  using standard arguments (see [17, 6.2.20, 6.2.21] with  $\delta$  in place of  $K$ ).

(4.6.5) Let  $M \in \text{Mod}_r(A)$  with  $\delta(M) = s$ . If  $(*) M = M_t \supset M_{t-1} \supset \dots \supset M_0 = (0)$  is a critical composition series of  $M$ , we set

$$\varepsilon_s(M) = \# \{s\text{-critical factors in } (*)\}, \quad \varepsilon_\mu(M) = \# \{\mu\text{-critical factors in } (*)\}.$$

Then the following are easy to prove:

the numbers  $\varepsilon_s(M)$  and  $\varepsilon_\mu(M)$  depend only on  $M$ ;

the function  $M \rightarrow \varepsilon_s(M)$  is additive on short exact sequences in

$\{M \in \text{Mod}_r(A) / \delta(M) = s\}$ ;

the function  $M \rightarrow \varepsilon_\mu(M)$  is additive on short exact sequences in  $\text{Mod}_r(A)$ .

The number  $\varepsilon_\mu(M)$  will be called the  $\delta$ -critical length of  $M$ .

(4.6.6) The b-filtration of  $M \in \text{Mod}_r(A)$  obtained in 2.3, 2.4 can be reinterpreted as follows. Put  $n = j_A(M)$ ,  $\delta(M) = s = \mu - n$ . The submodule  $F^p M$  is the largest submodule  $X$  of  $M$  such that  $\delta(X) \leq \mu - p$ . The module  $E_\lambda^n(M)$  is  $s$ -homogeneous. We have exact sequences

$$0 \rightarrow \frac{F^p M}{F^{p+1} M} \rightarrow E_\lambda^p(E_\lambda^n(M)) \rightarrow Q(p) \rightarrow 0,$$

where  $E_\lambda^p(E_\lambda^n(M))$  is  $(0)$  or  $(\mu - p)$ -homogeneous and  $\delta(Q(p)) \leq \mu - p - 2$ .

(4.6.7) It is not difficult to prove that the number  $\varepsilon_s(M)$  defined in (4.6.4) is equal to the number  $\varepsilon(M)$  obtained in [7, 1.17].

4.7. In this section, we suppose that  $A = \bigoplus_{m \geq 0} A_m$  is a positively graded Auslander-

Gorenstein ring of dimension  $\mu$ . Let  $\delta$  be its dimension as defined in 4.5. Let  $M$  be in  $\text{Mod}_r^g(A)$ . We say that  $M$  is *graded-critical* (resp. *graded-homogeneous*) if  $\delta(M/N) < \delta(M)$  (resp.  $\delta(N) = \delta(M)$ ) for all non-zero graded submodules  $N$  of  $M$ . These definitions are not of great interest since we have the next lemma.

LEMMA. *The module  $M \in \text{Mod}_r^g(A)$  is graded-critical (resp. graded-homogeneous) if and only if it is critical (resp. homogeneous).*

*Proof.* Set  $M = \bigoplus_m M_m$ . Recall that  $\left\{ A(n) := \bigoplus_{j \leq n} A_j \right\}_n$  is a filtration with associated graded ring equal to  $A$ . If  $\Gamma$  is a good filtration on  $M' \in \text{Mod}_t(A)$  (w.r.t to  $\{A(n)\}_n$ ), we have  $\delta(\text{gr}_\Gamma(M')) = \delta(M')$  by Proposition 3.2 (ii). Filter the module  $M$  by

$$\Gamma = \left\{ M(n) := \sum_{i \leq n} M_i \right\}_n.$$

It is a good filtration and we obviously have  $\text{gr}_\Gamma(M) = M$ . If  $N$  is any non-zero submodule of  $M$ ,  $\Gamma$  induces good separated filtrations on  $N$  and  $M/N$  still denoted by  $\Gamma$ . Then the exact sequence  $0 \rightarrow \text{gr}_\Gamma(N) \rightarrow M = \text{gr}_\Gamma(M) \rightarrow \text{gr}_\Gamma(M/N) \rightarrow 0$  in  $\text{Mod}_t^g(A)$  shows that  $\delta(N) = \delta(M)$  if  $M$  is graded-homogeneous and  $\delta(M/N) < \delta(M)$  if  $M$  is graded-critical. The converse is obvious.

**PROPOSITION.** *Any non-zero  $M \in \text{Mod}_t^g(A)$  has a critical series composed of graded submodules.*

*Proof.* Notice first that  $M$  contains a graded-critical submodule (repeat the proof of [17, 6.2.10] with graded modules and  $\delta$  in place of  $K$ ). It follows as usual that  $M$  has a graded-critical composition series. Applying the lemma above, we conclude that it is a critical composition series in  $\text{Mod}_t(A)$ .

**4.8. THEOREM.** *Let  $A = \bigoplus_{m \geq 0} A_m$  be a positively graded Auslander-regular  $k$ -algebra with  $A_0 = k$ . Then  $A$  is a domain.*

*Proof.* We follow [3, Section 3]. Recall that  $A$  is  $\mu$ -homogeneous, see (4.6.1). We first prove that  $\varepsilon_\mu(A) = 1$ . Any  $M \in \text{Mod}_t^g(A)$  has a finite resolution by graded free modules of the form  $P_i = \bigoplus_{j=1}^{d_i} A[m_i, j]$ , see [17, 12.2.9]. By additivity of  $\varepsilon_\mu(-)$  (see (4.6.4)), we get  $\varepsilon_\mu(M) = \sum_i (-1)^i \varepsilon_\mu(P_i) = \sum_i d_i \varepsilon_\mu(A) \in \mathbb{Z} \varepsilon_\mu(A)$ . Now choose  $M \in \text{Mod}_t^g(A)$   $\mu$ -critical (for instance a graded-critical ideal of  $A$ ). We have  $\varepsilon_\mu(M) = 1 \in \mathbb{Z} \varepsilon_\mu(A)$  and therefore  $\varepsilon_\mu(A) = 1$ . Let  $a \in A$ ,  $a \neq 0$ . We must show that  $a$  is a right n.z.d., the proof being identical on the left. Consider the exact sequence  $0 \rightarrow Aa \rightarrow A \rightarrow (A/Aa) \rightarrow 0$ . Since  $A$  is  $\mu$ -homogeneous we have  $\delta(Aa) = \mu$ , hence  $\varepsilon_\mu(Aa) \geq 1$ . Since  $\varepsilon_\mu(A) = 1 = \varepsilon_\mu(Aa) + \varepsilon_\mu(A/Aa)$  we have  $\varepsilon_\mu(A/Aa) = 0$ . Hence  $\delta(A/Aa) \leq \mu - 1$  which means  $j(A/Aa) \geq 1$ , i.e.  $\text{Hom}_A(A/Aa, A) \cong \{b \in A \mid ab = 0\} = (0)$ . Thus  $a$  is a right n.z.d.

**5. The Cohen-Macaulay property.** The rings in this section are  $k$ -algebras over the fixed field  $k$ . The material concerning Gelfand-Kirillov dimension comes from [13].

**5.1.** We first recall the definition of polynomial growth for a function  $d: \mathbb{Z} \rightarrow \mathbb{N}$ ,  $d \neq 0$ . Suppose  $d(n) = 0$  for  $n \ll 0$ ,  $d$  increasing for  $n \geq n_0$ . Then  $d$  has polynomial growth of degree  $\rho \in \mathbb{N}$  if there exist  $K, m \in \mathbb{N}$  such that

$$d(n) \leq Kn^\rho, \quad n^\rho \leq d(mn) \text{ for almost all } n.$$

We shall write  $G(d) = P_\rho$  when this holds. We then have  $\rho = \overline{\lim}_n \log_n d(n)$ ,  $\log_n d(n) := \log d(n) / \log n$ .

5.2. We shall use the following properties of Gelfand-Kirillov dimension, denoted by  $\text{GKdim}$ . Let  $A$  be a  $k$ -algebra filtered by  $\{A(i)\}_i$  (increasing) such that  $\dim_k A(i) < \infty$  for all  $i$  (in particular it is discrete). Assume that  $\text{gr}(A) = \bigoplus A(i)/A(i-1)$  is a finitely generated  $k$ -algebra.

(5.2.1) Let  $M \in \text{Mod}_f(A)$ ,  $\Gamma = \{M(i)\}_i$  a filtration on  $M$  such that  $\dim_k M(i) < \infty$  and  $\text{gr}_\Gamma(M) \in \text{Mod}_f^{\text{gr}}(\text{gr}(A))$ . Then, setting  $f_\Gamma(n) = \dim_k M(n)$ , we have  $\text{GKdim}_A M = \text{GKdim}_{\text{gr}(A)} \text{gr}_\Gamma(M) = \overline{\lim}_n \log_n f_\Gamma(n)$ . See [13, 6.6].

(5.2.2) If  $M \in \text{Mod}_f(A)$ , we have  $\text{GKdim}_A M = 0$  if and only if  $\dim_k M < \infty$  and  $\text{GKdim}_A M \geq 1$  if  $\dim_k M = +\infty$ .

(5.2.3) Assume  $\text{gr}(A)$  noetherian. Then  $\text{GKdim}$  is exact in  $\text{Mod}_f(A)$ :  $\text{GKdim}_A M = \text{Max}\{\text{GKdim}_A M', \text{GKdim}_A M''\}$  for every exact sequence  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  in  $\text{Mod}_f(A)$ ; see [13, 6.14].

5.3. Assume  $B$  is a  $k$ -algebra. One defines an  $s$ -homogeneous or  $s$ -critical  $B$ -module (for  $\text{GKdim}$ ) as in (4.6.1) or (4.6.2), with  $\text{GKdim}$  replacing  $\delta$ . Combining 6.8.15 and 8.3.16 of [17], one obtains the following theorem.

**THEOREM.** *Suppose  $B$  is a noetherian  $k$ -algebra such that  $\text{GKdim}$  is exact and  $B$  is (left and right) homogeneous (for  $\text{GKdim}$ ). Then  $B$  has a (left and right) Artinian quotient ring. Furthermore the set  $S$  of n.z.d. in  $B$  satisfies*

$$S = \{b \in B / \text{GKdim } B / bB < \text{GKdim } B\} = \{b \in B / \text{GKdim } B / Bb < \text{GKdim } B\}.$$

5.4. Let  $A = \bigoplus_{n \geq 0} A_n$  be a finitely generated positively graded  $k$ -algebra with  $\dim_k A_0 < \infty$ . Then  $\dim_k A_n < \infty$  for all  $n$  and  $\left\{A(n) := \bigoplus_{i=0}^n A_i\right\}_{n \geq 0}$  is a positive filtration with  $\text{gr}(A) = A$ . Hence the results recalled in 5.2 apply to  $\text{Mod}_f(A)$  (assuming  $A$  noetherian for (5.2.3)). We introduce a condition named (PG) for polynomial growth.

**DEFINITION.** The algebra  $A$  satisfies (PG) if, for every  $M = \bigoplus_n M_n$  in  $\text{Mod}_f^{\text{gr}}(A) \setminus \{0\}$ , one has  $G(f_M) = P_s$  for some  $s \in \mathbb{N}$ . Here  $f_M(n) = \dim_k M(n)$ ,  $M(n) = \sum_{i \leq n} \dim_k M_i$ .

**REMARKS.** (1)  $\Gamma := \{M(n)\}_n$  is a good filtration on  $M$  (with respect to the filtration  $\{A(n)\}_n$ ). This is because  $M \cong \text{gr}_\Gamma(M)$  as a  $\text{gr}(A)$ -module. Thus  $\text{GKdim}_A M = \overline{\lim}_n \log_n f_n(n) = s$  by (5.2.1).

(2) If  $I$  is a graded ideal of  $A$  then, if  $A$  satisfies (PG),  $A/I$  satisfies (PG).

5.5. There is an example of an algebra satisfying (PG) which is of particular importance for the applications. Suppose  $A = k \oplus A_1 \oplus A_2 \oplus \dots$  is noetherian. If  $M = \bigoplus_n M_n \in \text{Mod}_f^{\text{gr}}(A)$ , the Hilbert series is defined by  $h_M(t) := \sum_{n \in \mathbb{Z}} (\dim_k M_n) t^n$ . Suppose that, for all  $M \in \text{Mod}_f^{\text{gr}}(A)$ , one has  $h_M(t) := q_M(t)/p_A(t)$ ,  $p_A(t) \in \mathbb{Z}[t]$ ,  $q_M(t) \in \mathbb{Z}[t, t^{-1}]$ , where the roots of  $p_A(t)$  are roots of unity. Then [4, 2.21] shows that  $\dim_k M_n = an^{s-1}/(s-1)! + (\text{terms of degree less than } s-1 \text{ in } n)$ , where  $a \in \mathbb{N}$  and  $s$  is the order of the pole at  $t = 1$  of  $h_M(t)$ . We have  $(1-t^r)h_M(1) \in \mathbb{N}$ ,  $s = \text{GKdim}_A M$  and the algebra satisfies

(PG). When  $p_A(t) = (1 - t)^\mu$ ,  $\mu \in \mathbb{N}$ ,  $h_M(t)$  takes the form  $f_M(t)/(1 - t)^\nu$ ,  $f_M(t) \in \mathbb{Z}[t, t^{-1}]$  with  $e(M) := f_M(1) \in \mathbb{N}$ .

5.6. We return to  $A = \bigoplus_{n \geq 0} A_n$  as in 5.4. Assume that  $A$  is noetherian. The following result should be well known. We include a proof for completeness.

PROPOSITION. *Let  $\Omega \in A_d$ ,  $d > 0$ , be a normal element. Then  $A$  satisfies (PG) if and only if  $A/\Omega A$  satisfies (PG).*

*Proof.* By remark 5.4 (2), we only have to show one implication. So assume that  $A / \Omega A$  satisfies (PG). Let  $M = \bigoplus_n M_n \in \text{Mod}_f^{\mathbb{Z}}(A)$ , non-zero. We may assume  $M_n = 0$  for all  $n < q_0 \leq 0$  (we don't impose  $M_{q_0} \neq (0)$ ). Let  $p \in \mathbb{N}$  and recall that  $\text{Ker}_M \Omega^p = \{x \in M / \Omega^p x = 0\}$  is a graded submodule of  $M$ . Since  $\{\text{Ker}_M \Omega^p\}_p$  is an increasing chain of submodules we can define  $n(M)$  to be the smallest integer such that  $\text{Ker}_M \Omega^{n(M)} = \text{Ker}_M \Omega^{n(M)+i}$  for all  $i \geq 0$ . We set  $\bar{M} = M/\Omega M$ ,  $M' = M/\text{Ker}_M \Omega$ . With obvious notations, it is easily seen that  $x' \in \text{Ker}_{M'} \Omega^p$  if and only if  $x \in \text{Ker}_M \Omega^{p+1}$ . Hence  $\text{Ker}_{M'} \Omega^p = \text{Ker}_M \Omega^{p+1} / \text{Ker}_M \Omega$  and we have that  $n(M') \leq n(M) - 1$  if  $n(M) \geq 1$ . We shall prove by induction on  $n(M)$  than  $n \xrightarrow{f_n} \sum_{i \leq n} \dim_k M_i$  has polynomial growth.

1. Assume that  $n(M) = 0$ , i.e.  $\Omega$  is a n.z.d. in  $M$ . We have exact sequences

$$0 \rightarrow M_{n-d} \xrightarrow{\Omega} M_n \rightarrow \bar{M}_n \rightarrow 0,$$

where  $\bar{M}_n := M_n/\Omega M_{n-d}$  gives the grading on  $\bar{M}$ . It is easy to deduce that  $f_{\bar{M}}(n) = f_M(n) - f_M(n - d)$ . Notice that  $\bar{M}_n = (0)$  for  $n < q_0$ ,  $\bar{M}_{q_0} = M_{q_0}$  and set  $E(n) := [(n - q_0)/d]$  (integral part). Then we have  $f_M(n) = \sum_{i=0}^{E(n)} f_{\bar{M}}(n - id)$ . Suppose  $G(f_{\bar{M}}) = P_\rho$ ,  $\rho \in \mathbb{N}$ , so that there exist  $n_0 \in \mathbb{N}^*$ ,  $k, m \in \mathbb{N}^*$  such that  $f_{\bar{M}}(n) \leq Kn^\rho$  and  $f_{\bar{M}}(mn) \geq n^\rho$  for all  $n \geq n_0 > 0$ . If  $n \gg 0$ , we get

$$f_M(n) \leq \sum_{q=q_0}^n f_{\bar{M}}(q) \leq C + K \sum_{q=n_0}^n q^\rho \leq C + K(n - n_0 + 1)n^\rho \leq K'n^{\rho+1},$$

where  $C = \sum_{q=q_0}^{n_0-1} f_{\bar{M}}(q)$  and  $K' \in \mathbb{N}$ .

Let  $n \gg 0$  and notice that  $m(n - n_0) \leq mn - q_0/d$ . We have

$$f_M(mnd) \geq \sum_{i=0}^{m(n-n_0)} f_{\bar{M}}(mnd - id) \geq \sum_{j=0}^{n-n_0} f_{\bar{M}}(mnd - mjd).$$

If  $0 \leq j \leq n - n_0$ , one has  $nd - jd \geq nd - nd + n_0d \geq n_0$ . Hence  $f_{\bar{M}}(md(n - j)) \geq d^\rho(n - j)^\rho$  by the choice of  $m$  and  $n_0$ . Therefore  $f_M(mnd) \geq \sum_{j=0}^{n-n_0} d^\rho(n - j)^\rho \geq \sum_{j=0}^{n-n_0} (n - j)^\rho$ . Since  $\phi(n) := n_0^\rho + (n_0 + 1)^\rho + \dots + n^\rho$  is a polynomial in  $n$  of degree  $\rho + 1$ , we can find  $m' \in \mathbb{N}^*$  such that  $f_M(mm'nd) \geq \phi(m'n) \geq n^{\rho+1}$  if  $n \gg 0$ . This proves the case  $n(M) = 0$ .

2. Assume  $n(M) > 0$ . Set  $K := \text{Ker}_M \Omega = \bigoplus_n K_n$ ; hence  $K \neq (0)$  and we have  $f_M(n) = f_K(n) + f_{M'}(n)$ . Observe that  $K \in \text{Mod}_f^{\#}(A/\Omega A)$  and that we can apply the induction to  $M'$ . Thus  $G(f_K) = P_{\rho'}$  and  $G(f_{M'}) = P_{\rho''}$  for  $\rho', \rho'' \in \mathbb{N}$ . Set  $\rho = \text{Max}\{\rho', \rho''\}$ . It is easy to show that  $G(f_M) = P_{\rho}$ .

5.7. We continue with  $A = \bigoplus_{n \geq 0} A_n$  as in 5.6 and we suppose that  $A$  satisfies (PG). Then we have the following result.

LEMMA. Let  $\Omega \in A_d$ ,  $d > 0$ , be a normal element. Assume  $\Omega$  is a n.z.d in  $M \in \text{Mod}_f^{\#}(A) \setminus (0)$ . Then  $\text{GKdim}_A M/\Omega M = \text{GKdim}_A M - 1$ .

Proof. As in the proof of 5.6, we set  $M = \bigoplus_{i \geq q_0} M_i$ ,  $q_0 \leq 0$ ,  $\bar{M} = M/\Omega M = \bigoplus_i \bar{M}_i$ . Hence  $\bar{M}_i = (0)$  if  $i < q_0$  and  $\bar{M}_{q_0} = M_{q_0}$ . Recall that  $f_M(n) = \sum_{i=0}^{E(n)} f_{\bar{M}}(n - id)$ ,  $E(n) := [(n - q_0)/d]$ , (see part 1 in the proof of 5.6). Set  $s = \text{GKdim } M$ . By [13, 5.1], we know  $\text{GKdim } \bar{M} \leq s - 1$ . Set  $\rho = \text{GKdim } \bar{M}$ , so that  $f_{\bar{M}}(n) \leq Kn^{\rho}$  for some  $K \in \mathbb{N}$  and all  $n \geq n_0 > 0$ . Define  $a(n)$  to be the greatest integer  $q \leq E(n)$  such that  $n - qd \geq n_0$ , and put  $c = \sum_{\rho=q_0}^{n_0-1} f_{\bar{M}}(\rho)$ . Then if  $n \gg 0$ , we have

$$f_M(n) \leq c + \sum_{i=0}^{a(n)} f_{\bar{M}}(n - id) \leq K[a(n) + 1]n^{\rho} + c.$$

But  $a(n) \leq E(n) \leq n + c'$  for some  $c' \in \mathbb{N}$  independent of  $n$ , hence  $f_M(n) \leq Kn^{\rho+1} + c''$  for some constant  $c''$ . This implies  $s = \lim_n \log_n f_M(n) \leq \rho + 1$ .

5.8. Let  $(R, \mathcal{M})$  be a noetherian commutative local ring. Recall that  $R$  is Cohen-Macaulay if and only if, for all non-zero  $M \in \text{Mod}_f(R)$ , we have  $\text{Kdim } M + j_R(M) = \text{Kdim } R$ , where  $\text{Kdim}$  denotes the Krull dimension. Thus it is natural to make the following definition.

DEFINITION. Let  $A$  be a noetherian  $k$ -algebra with  $\text{GKdim } A = \omega \in \mathbb{N}$ . We say that  $A$  satisfies the Cohen-Macaulay property if  $\text{GKdim}_A M + j_A(M) = \omega$  for all

$$M \in \text{Mod}_f(A) \setminus (0).$$

REMARKS. (1) For short we shall say that  $A$  is CM.

(2) Notice that, if  $A$  is CM, we must have  $j_A(M) < \infty$  and  $\text{GKdim}_A M \in \mathbb{N}$  for all  $M \neq (0)$  in  $\text{Mod}_f(A)$ .

(3) Recall that we have seen that over an Auslander-Gorenstein ring of dimension  $\mu$ ,  $\delta(M) = \mu - j_A(M)$  defines a dimension function on  $\text{Mod}_f(A)$  (see 4.5). Thus, if  $A$  is also CM, we get  $\delta(M) + (\omega - \mu) = \text{GKdim}_A M$ . See remark (5) below.

(4) Let  $0 \rightarrow M_1 \rightarrow M \rightarrow M_2 \rightarrow 0$  be exact in  $\text{Mod}_f(A)$ . We always have  $\text{GKdim } M \geq \text{Max}\{\text{GKdim } M_i\}$  and  $j_A(M) \geq \inf_i \{j_A(M_i)\}$ . Therefore if  $A$  is CM we see that  $\text{GKdim}_A$  and  $j_A(\ )$  are exact, i.e. we have equalities in these inequalities.

(5) Suppose  $\text{char } k = 0$ . Then the Weyl algebra  $A_{\mu}(k) = k[p_1, \dots, p_{\mu}, q_1, \dots, q_{\mu}]$ ,  $p_i q_j - q_j p_i = \delta_{ij}$ ,  $[p_i, p_j] = [q_i, q_j] = 0$ , is Auslander-regular of dimension  $\mu$  and satisfies

the Cohen-Macaulay property (here  $\omega = 2\mu$ ). See [6, Chap. 2, Section 7.1], and [14, Théorème 4.4] for a generalization.

When  $A$  is graded, one can define a “graded CM property” by taking  $M \in \text{Mod}_r^{\mathbb{Z}}(A) \setminus (0)$  in Definition 5.8.

LEMMA. *Let  $A = \bigoplus_{n \geq 0} A_n$  be a positively graded ring such that  $\dim_k A_0 < \infty$ . Assume  $A$  is Auslander-Gorenstein and satisfies the “graded CM-property”. Then  $A$  is CM.*

*Proof.* Consider the Zariskian filtration  $\left\{ A(n) := \sum_{i \leq n} A_i \right\}_n$  on  $A$ . We recalled in Proposition 3.2(ii) and (5.2.1) that  $j_A(N) = j_A(\text{gr}_\Gamma(N))$ , and  $\text{GKdim}_A N = \text{GKdim}_A \text{gr}_\Gamma(N)$  for any good filtration  $\Gamma$  on  $N \in \text{Mod}_r(A)$ . This proves the lemma.

5.9. The following proposition shows that the CM-property makes an Auslander-Gorenstein algebra very close to a commutative one.

PROPOSITION. *Let  $A$  be a noetherian  $k$ -algebra with  $\text{injdim}(A) = \mu < \infty$  and  $\text{GKdim } A = \omega \in \mathbb{N}$ . Assume  $A$  is CM. Then  $A$  is Auslander-Gorenstein if and only if  $E_A^p(E_A^q(M)) = (0)$  for all  $p < q$ , and all  $M \in \text{Mod}_r(A)$ . Furthermore when this holds we have*

- (i)  $\text{GKdim } M$  is exact and finitely partitive in  $\text{Mod}_r(A)$ ,
- (ii)  $A$  is homogeneous for  $\text{GKdim}$  and has a left and right Artinian quotient ring.

*Proof.* Assume  $E^p(E^q(M)) = (0)$  if  $p < q$ . Let  $N$  be a submodule of  $E^q(M)$ . By the property CM we have  $\text{GKdim } E^q(M) \leq \omega - q$  and, since  $\text{GKdim } N \leq \text{GKdim } E^q(M)$  always holds, we get  $j(N) \geq q$ . Then (i) and (ii) follow from Remark 5.8 (3), 4.5, (4.6.1) and 5.3.

REMARKS. (1) By means of (4.6.5) and  $\delta(M) + (\omega - \mu) = \text{GKdim } M$ , the  $b$ -filtration can be interpreted in terms of  $\text{GKdim}$ . We leave the translation to the reader.

(2) Assume  $A = \bigoplus_{n \geq 0} A_n$  is a positively graded noetherian ring with  $\text{injdim}(A) < \infty$  and  $A$  satisfies the “graded CM-property”. Then the proof above shows that  $A$  is graded-Auslander-Gorenstein if and only if  $E_A^p(E^q(M)) = (0)$  for all  $p < q$  and  $M \in \text{Mod}_r^{\mathbb{Z}}(A)$ . By Theorem 3.1 and Lemma 5.8, we deduce that  $A$  is Auslander-Gorenstein and CM, if  $\dim_k A_0 < \infty$ .

5.10. Let  $(R, \mathcal{M})$  be a commutative noetherian local ring. It is well known that if  $\Omega \in \mathcal{M}$  is a n.z.d. then  $R$  is Gorenstein or CM if and only if  $R/\Omega R$  is such. Therefore the following result is not surprising.

THEOREM. *Let  $A = \bigoplus_{n \geq 0} A_n$  be a finitely generated positively graded  $k$ -algebra with  $\dim_k A_0 < \infty$ . Suppose  $\Omega \in A_d$ ,  $d > 0$ , is a normal n.z.d. in  $A$ . Then  $B = A/\Omega A$  is Auslander-Gorenstein (of dimension  $\nu$ ), satisfies the property CM and condition (PG) if and only if  $A$  does so (with  $\text{injdim}(A) = \nu + 1$ ).*

*Proof.* Assume  $A$  is Auslander-Gorenstein, CM and satisfies (PG). Then by Remarks 3.4 (3) and 5.4 (2),  $B$  has the required properties.

To prove the converse we start with a general result.



**SUBLEMMA.** *Let  $A$  be an Auslander-Gorenstein  $k$ -algebra for which  $\text{GKdim}$  is exact. Let  $0 \rightarrow M_1 \rightarrow M \rightarrow M_2 \rightarrow 0$  be exact in  $\text{Mod}_i(A)$  with  $\text{GKdim}_A M_i + j_A(M_i) = \omega \in \mathbb{N}$  for  $i = 1, 2$ . Then  $\text{GKdim}_A M + j_A(M) = \omega$ .*

*Proof.* We have

$$\text{GKdim } M = \text{Max}_{i=1,2} \{ \text{GKdim } M_i \} = \text{Max}_{i=1,2} \{ \omega - j_A(M_i) \} = \omega - \inf_{i=1,2} \{ j_A(M_i) \} = \omega - j_A(M)$$

since  $A$  is Auslander-Gorenstein.

By Theorem 3.6, Proposition 5.6, Lemma 5.8, we only have to show that  $\text{GKdim } M + j(M) = \omega + 1$  for all  $M \in \text{Mod}_i^{\neq}(A) \setminus (0)$ , where  $\omega = \text{GKdim } B$ . We adopt the notation of the proof of Proposition 5.6:  $\bar{M} = M/\Omega M$ ,  $M' = M/\text{Ker}_M \Omega$ , where  $\text{Ker}_M \Omega^p = \{x \in M / \Omega^p x = 0\}$ ,  $n(M)$  is the smallest integer  $q$  such that  $\text{Ker}_M \Omega^q = \text{Ker}_M \Omega^{q+1}$ . We noticed that  $n(M') \leq n(M) - 1$  and we induct on  $n(M)$ . If  $M = \text{Ker}_M \Omega$ , i.e.  $M' = (0)$  then  $M \in \text{Mod}_i^{\neq}(B)$ . Hence  $\text{GKdim}_B M + j_B(M) = \omega$ ,  $\text{GKdim}_B M = \text{GKdim}_A M$ ,  $j_B(M) = j_A(M) - 1$  (Rees Lemma, Proposition 3.4(b)). Thus  $\text{GKdim}_A M + j_A(M) = \omega + 1$ . If  $n(M) = 0$ ,  $\Omega$  is a n.z.d in  $M$ . By Lemma 5.7,  $\text{GKdim}_A M/\Omega M = \text{GKdim}_A M - 1$ ,  $j_A(M/\Omega M) = j_B(M/\Omega M) + 1$  by the Rees Lemma and  $j_A(M/\Omega M) = j_A(M) + 1$  by Corollary 4.4. Since  $\text{GKdim}_A M/\Omega M = \text{GKdim}_B M/\Omega M = \omega - j_B(M/\Omega M)$  we obtain  $\text{GKdim}_A M + j_A(M) = \omega + 1$ . If  $n(M) \geq 1$ , we may assume  $M' \neq (0)$ . Consider the exact sequence  $0 \rightarrow \text{Ker}_M \Omega \rightarrow M \rightarrow M' \rightarrow 0$ . By the sublemma, induction and the results above, we obtain  $\text{GKdim}_A M + j_A(M) = \omega + 1$ .

An easy induction yields the following corollary.

**COROLLARY.** *Let  $A$  be a finitely generated positively graded  $k$ -algebra with  $\dim_k A_0 < \infty$ . Suppose  $\Omega = \{\Omega_1, \dots, \Omega_l\}$  is a regular normalizing sequence of homogeneous elements of positive degree in  $A$ . Put  $B := A/\Omega A$  and suppose that  $B$  is Auslander-Gorenstein of dimension  $\nu$ , satisfies the CM-property and condition (PG). Then  $A$  is Auslander-Gorenstein of dimension  $\mu = \nu + l$ , satisfies the CM-property and condition (PG).*

**REMARKS.** (1)  $\Omega$  regular normalizing means that, for all  $i$ ,  $\Omega_{i+1}$  is a normal n.z.d in  $A/(\Omega_1, \dots, \Omega_i)A$ .

(2) Obviously if we suppose  $\text{gldim}(A) < \infty$  in the corollary, we obtain  $A$  Auslander-regular.

5.11. Let  $(R, \mathcal{M})$  be a commutative noetherian local ring. To attempt to prove that  $R$  is Gorenstein one can proceed as follows. Find a regular sequence  $(\Omega_1, \dots, \Omega_l)$  contained in  $\mathcal{M}$  with  $l = \text{Kdim } R$ , then  $R/(\Omega_1, \dots, \Omega_l)$  is an Artinian ring and one has to prove that its socle is a 1-dimensional vector space. In the non-commutative setting it is in general impossible to find a regular normalizing sequence  $(\Omega_1, \dots, \Omega_l)$  such that  $\text{GKdim } A/(\Omega_1, \dots, \Omega_l) = 0$ . But, we shall see in Section 6 some examples where this factor ring has  $\text{GKdim } 2$ . For this reason we give in 5.13 a criterion to ensure that such an algebra is Auslander-Gorenstein. We begin with an easy lemma.

**LEMMA.** *Let  $B$  be a noetherian  $k$ -algebra. Assume  $\text{GKdim}$  exact in  $\text{Mod}_i(B)$ ,  $B$  is  $\nu$ -homogeneous, and the quotient ring  $Q$  of  $B$  is self-injective (i.e.  $\text{injdim}(Q) = 0$ ). Then for all  $M \in \text{Mod}_i(B)$  we have:*

- (a)  $\text{GKdim}_B M \leq \nu - 1$  if and only if  $M^* = E_B^0(M) = (0)$ ;
- (b) for all  $q \geq 1$ ,  $\text{GKdim } E_B^q(M) \leq \nu - 1$  and  $E_B^0(E_B^q(M)) = (0)$ .

*Proof.* (a) Denote by  $S$  the set of n.z.d. in  $B$ . Thus  $Q = S^{-1}B = BS^{-1}$  by Theorem 5.3. Suppose  $\text{GKdim}_B M < \nu$ . Let  $\theta \in \text{Hom}_B(M, B)$ . Then  $\text{GKdim}_B \theta M \leq \text{GKdim}_B M < \nu$ , so  $\theta M = 0$  since  $B$  is  $\nu$ -homogeneous. Conversely if  $M^* = (0)$  then  $\text{Hom}_Q(S^{-1}M, Q) \cong \text{Hom}_B(M, B) \otimes_B Q = (0)$ . Since  $Q$  is self-injective, we know  $S^{-1}M \cong (S^{-1}M)^{**}$ , the bidual over  $Q$ . Hence  $S^{-1}M = (0)$  and  $\text{GKdim } M \leq \nu - 1$  follows easily.

(b) The first assertion is consequence of  $\text{injdim}(Q) = 0$  and  $E_B^q(M) \otimes_B Q \cong E_Q^q(S^{-1}M)$ . The second follows from (a) and the first.

5.12. In this section we assume that  $B = k \oplus B_1 \oplus B_2 \oplus \dots$  is a noetherian positively graded  $k$ -algebra. We set  $\mathcal{M} = \bigoplus_{i \geq 1} B_i$  and consider  $k = B/\mathcal{M}$  as a left or right graded  $B$ -module. We shall write  $E_B^i({}_B k) = \delta_{i,\nu} k_B$  or  $E_B^i(k_B) = \delta_{i,\nu} k$  to mean that this module is (0) for  $i \neq \nu$  and isomorphic to  $k$  if  $i = \nu$ . Notice that  $E_B^i(k) \in \text{Mod}_i^e(B)$ ; hence we in fact have  $E_B^i(k) \cong k[m]$  for some  $m \in \mathbb{Z}$  (in  $\text{Mod}_i^e(B)$ ).

LEMMA. Assume  $\nu = \text{injdim}(B) < \infty$ . Then

(a) if  $E_B^i({}_B k) = \delta_{i,\nu} k_B$  and  ${}_B M \in \text{Mod}_i^e(B)$  is such that  $\dim_k M < \infty$ , we have  $E_B^i(M) = (0)$  for  $i < \nu$  and  $\dim_k E_B^\nu(M) = \dim_k M$ ;

(b) if  $E_B^i(k_B) = \delta_{i,\nu} k_B$  and  $E_B^i(k_B) = \delta_{i,\nu} k$  and we let  ${}_B N \in \text{Mod}_i(B)$  with  $\text{pd}_B(N) < \infty$  then we have  $\text{Tor}_{\nu-i}^B(k_B, N) \cong \text{Ext}_{i(B)}^B(k_B, N)$  for all  $i \geq 0$ .

*Proof.* (a) We argue by induction on  $\dim_k M$ . It is true for  $\dim_k M = 1$  by hypothesis. For any  $M \in \text{Mod}_i^e(B)$  there exists an exact sequence in  $\text{Mod}_i^e(B)$ :

$$0 \rightarrow M' \rightarrow M \rightarrow {}_B k[m] \rightarrow 0$$

for some  $m \in \mathbb{Z}$ . Then use induction and the long exact sequence in cohomology.

(b) We recall the following two facts.

(b.1) Let  $B$  be any noetherian ring with  $\text{injdim}(B) = \nu < \infty$  and let  ${}_B N \in \text{Mod}_i(B)$  with  $\text{pd}_B(N) < \infty$ . Then  $\text{injdim}_B(N) \leq \nu$ . This can be proved by using the spectral sequence

$$\text{Tor}_p^B(\text{Ext}_B^q(M, B), N) \Rightarrow \text{Ext}_B^{q-p}(M, N), \text{ where } {}_B M \in \text{Mod}_i(B) \text{ (see [12]).}$$

(b.2) There exists a convergent spectral sequence

$$E_2^{p,-q} = \text{Ext}_B^p(\text{Ext}_B^q(k_B, B), N) \Rightarrow \text{Tor}_{-(p-q)}(k_B, N)$$

(see [12] and use (b.1)).

We now prove the assertion. Since  $\text{Ext}_B^q(k_B, B) \cong \delta_{q,\nu} k$ , the spectral sequence of (b.2) degenerates to isomorphisms  $E_2^{p,-\nu} \cong \text{Ext}_B^p({}_B k, N) \cong E_\infty^{p,-\nu} = \text{Tor}_{\nu-p}(k_B, N)$ .

5.13. THEOREM. Let  $B = k \oplus B_1 \oplus B_2 \oplus \dots$  be a noetherian graded algebra such that:

(a)  $\text{GKdim}_B M \in \mathbb{N}$  for all  $M \in \text{Mod}_i^e(B)$ ;

(b)  $\text{injdim}(B) = \text{GKdim } B = 2$ ,  $B$  is 2-homogeneous with a self-injective quotient ring;

(c)  $E_B^i({}_B k) = \delta_{i,2} k_B$  and  $E_B^i(k_B) = \delta_{i,2} k$ .

Then  $B$  is Auslander-Gorenstein and satisfies the property CM.

*Proof.* We are going to prove that  $E^p(E^q(M)) = (0)$  for  $p < q$  and  $M \in \text{Mod}_i(B)$ , and that  $B$  is CM. The theorem will then follow from Proposition 5.9.

By Lemma 5.11 (b), we have  $E^0(E^q(M)) = 0$  if  $q \geq 1$ . Since  $\text{injdim}(B) = 2$ , it remains to show  $E^1(E^2(M)) = (0)$ . Consider the spectral sequence  $E_2^{p,-q} = E^p(E^q(M)) \Rightarrow \mathbb{H}^{p-q}$  of

Theorem 2.2. We get  $E_\infty^{1,-2} = (0)$  (because  $\mathbb{H}^n = (0)$  if  $n \neq 0$ ) and for all  $r \geq 2$  we have maps

$$E_r^{1-r,r-3} \xrightarrow{d_r} E_r^{1,-2} \xrightarrow{d_r} E_r^{r+1,-1-r}.$$

But since  $1 - r < 0$  and  $1 + r > 2$  we have  $E_r^{1-r,r+3} = E_r^{r+1,-1-r} = (0)$ . Hence  $E^1(E^2(M)) = E_2^{1,-2} = E_3^{1,-2} = \dots = E_\infty^{1,-2} = (0)$ .

We shall break the proof of the property CM into four steps. First notice that one can reduce to the graded CM-property by Remark 5.9 (2).

1. By hypothesis GKdim takes values in  $\mathbb{N}$  on  $\text{Mod}_f^g(B)$ ; hence  $\text{GKdim}_B M = 0, 1$  or  $2$  for all  $M \in \text{Mod}_f^g(B)$ . If  $\text{GKdim}_B M = 0$  then  $\dim_k M < \infty$  and  $j_B(M) = 2$  by Lemma 5.12. If  $\text{GKdim}_B M = 2$ , we have  $E^0(M) = M^* \neq (0)$  by Lemma 5.11, i.e.  $j_B(M) = 0$ . Therefore the case  $\text{GKdim}_B M = 1$  is the only difficult case and all that follows is to deal with this case.

2. We recall the Auslander Approximation Theorem in  $\text{Mod}_f^g(B)$ , see [11, Proposition 3.8]. The Auslander Approximation Theorem states the following. Let  $B$  be a graded noetherian ring. Let  $M \in \text{Mod}_f^g(B)$  and suppose there exists  $r \in \mathbb{N}$  such that  $E_B^p(E_B^q(M)) = (0)$  for all  $p < q \leq r$ . Then there exists  $M_1 \in \text{Mod}_f^g(B)$ ,  $\text{pd}_B(M_1) \leq r$ , a graded morphism  $f: M \rightarrow M_1$  such that  $\text{Ext}^i(f): E_B^i(M_1) \rightarrow E_B^i(M)$  is an isomorphism for all  $i \in \{1, \dots, r\}$ .

The Auslander Approximation Theorem applies to any  $M \in \text{Mod}_f^g(B)$ ,  $B$  as in the theorem, since we showed  $E^p(E^q(M)) = (0)$  if  $p < q \leq 2$ . Hence we get  $f: M \rightarrow M_1$ ,  $\text{pd}_B(M_1) \leq 2$ ,  $\alpha^i(f): E_B^i(M_1) \xrightarrow{\cong} E_B^i(M)$  isomorphisms for  $i = 1, 2$ .

3. When  ${}_B M \in \text{Mod}_f^g(B)$ , we define the torsion submodule by  $T(M) := \{x \in M \mid \exists i \geq 0, \mathcal{M}^i x = 0\}$  and we put  $\bar{M} = M/T(M)$ . Recall that the socle of  $M$  is  $\text{Soc}(M) = \text{Hom}_B({}_B k, M) = \{x \in M \mid \mathcal{M}x = 0\}$ . We do the same for  $M_B \in \text{Mod}_f^g(B)$ .

*Claim.* (i)  $T(M) \neq (0)$  if and only if  $\text{Soc}(M) \neq (0)$  if and only if  $E_B^2(M) \neq (0)$ .

(ii)  $E_B^i(M) \cong E_B^i(\bar{M})$ ,  $i = 0, 1$ .

(iii)  $E_B^2(\bar{M}) = (0)$  and  $E_B^2(M) \cong E_B^2(T(M))$  is a finite dimensional vector space.

*Proof of the claim.* Clearly  $T(M) \neq (0)$  if and only if  $\text{Soc}(M) \neq (0)$ . Notice that  $\dim_k T(M) < \infty$ ; hence  $E^i(T(M)) = (0)$  if  $i = 0, 1$  and  $\dim_k E^2(T(M)) = \dim_k T(M)$  by Lemma 5.12(a). The long exact sequence

$$\dots \rightarrow E^{i-1}(T(M)) \rightarrow E^i(\bar{M}) \rightarrow E^i(M) \rightarrow E^i(T(M)) \rightarrow \dots$$

proves (ii). Since  $T(M) = (0)$ , (i) implies (iii). If  $T(M) \neq (0)$ , we have  $E^2(T(M)) \neq (0)$  and hence  $E^2(M) \neq (0)$  follows from the surjection  $E^2(M) \rightarrow E^2(T(M))$ . So it remains only to show that  $E^2(M) \neq (0)$  implies  $T(M) \neq (0)$ , i.e.  $\text{Soc}(M) \neq (0)$ . Let  $f: M \rightarrow M_1$  be given by step 2. Remark that  $\text{pd}_B(M_1) = 2$  (because  $E^2(M_1) \cong E^2(M) \neq (0)$ ) and, since it is a graded module, we have  $\text{Tor}_2^B(k_B, M_1) \neq (0)$ . By Lemma 5.12(b), we have  $\text{Tor}_{2-i}^B(k_B, M_1) \cong \text{Ext}_B^i({}_B k, M_1)$  for all  $i$ . In particular  $\text{Soc}(M_1) = \text{Hom}_B({}_B k, M_1) \cong \text{Tor}_2^B(k_B, M_1) \neq (0)$ . Recall that  $f$  induces a map between the spectral sequences  $E_2^{p,-q}(M_1)$ ,  $E_2^{p,-q}(M)$  and their invariants (see Remark 2.1 (2)). Hence we obtain a commutative diagram:

$$\begin{array}{ccccccc} F^3 M = (0) & \rightarrow & F^2 M & \rightarrow & F^1 M & \rightarrow & F^0 M = M \\ & & \downarrow f & & \downarrow f & & \downarrow f \\ F^3 M_1 = (0) & \rightarrow & F^2 M_1 & \rightarrow & F^1 M_1 & \rightarrow & F^0 M_1 = M_1 \end{array}$$

Since  $\alpha^i(f): E^i(M_1) \xrightarrow{\cong} E^i(M)$ ,  $i = 1, 2$ , we have isomorphisms  $E_2^{i,-i}(f): E^i(E^i(M)) \rightarrow E^i(E^i(M_1))$  (Remark 2.1 (2)).

By Theorem 2.2 (b) and  $\text{injdim}(B) = 2$ , we conclude that  $f$  induces isomorphisms

$$f: F^2M \cong E^2(E^2(M)) \simeq F^2M_1 \cong E^2(E^2(M_1))$$

and

$$\tilde{f}: \frac{F^1M}{F^2M_1} \cong E^1(E^1(M)) \simeq \frac{F^1M_1}{F^2M_1} \cong E^1(E^1(M_1)).$$

It follows that  $f$  is an isomorphism from  $F^1M$  onto  $F^1M_1$ . Since  $E^2(M) \neq (0)$  and  $E^1(E^2(M)) = (0)$ ,  $i = 0, 1$ , we see that  $E^2(E^2(M)) \neq (0)$ , and therefore  $F^2M \cong E^2(E^2(M)) \neq (0)$ . Recall that we have  $0 \rightarrow M_1/F^1M_1 \rightarrow E^0(E^0(M_1)) = M_1^{**}$ . Hence  $\text{Soc}(M_1/F^1M_1) \subset \text{Soc}(M_1^{**}) = (0)$  since  $M_1^{**}$  is a submodule of a free  $B$ -module and  $\text{Soc}(B) = (0)$ . It follows that  $\text{Soc}(F^1M_1) = \text{Soc}(M_1)$ . Thus  $\text{Soc}(M) \supset \text{Soc}(F^1M) \cong \text{Soc}(F^1M_1) = \text{Soc}(M_1) \neq (0)$ , proving  $\text{Soc}(M) \neq (0)$ .

4. We now finish the proof of the property CM for  $B$ . Assume  $\text{GKdim}_B M = 1$ . Then  $E^0(M) = M^* = (0)$  by Lemma 5.11. Suppose  $E^1(M) = (0)$ ; we get  $E_2^{q \cdot -1}(M) = E^p(E^1(M)) = (0)$  for all  $p$ . Hence  $E_2^{p \cdot -q}(M) = (0)$  except for  $(p, q) = (2, 2)$ , and the spectral sequence degenerates to  $M \cong E^2(E^2(M))$ . By step 3, this implies  $\dim_k M < \infty$  and a contradiction. Thus  $j_B(M) = 1$ . If  $\text{GKdim}_B M = 2$ , we have  $E^0(M) = M^* \neq (0)$  by Lemma 5.11, i.e.  $j_B(M) = 0$ .

REMARKS. (1) The condition on the quotient ring of  $B$  is in particular satisfied when  $B$  is semi-prime.

(2) Let  $A = k \oplus A_1 \oplus A_2 \oplus \dots$  be a finitely generated graded algebra containing  $\Omega = \{\Omega_1, \dots, \Omega_l\}$ , a regular normalizing sequence of homogeneous elements of positive degree in  $A$ . Assume that  $B = A/\Omega A$  satisfies the hypothesis of the theorem above and condition (PG). We deduce from Corollary 5.10 that:  $A$  is Auslander-Gorenstein, satisfies the property CM and the condition (PG),  $\text{injdim}(A) = l + 2$ .

### 6. Artin-Schelter regular algebras.

6.1. In [1] is introduced a notion of ‘‘regularity’’ for graded algebras. To avoid any confusion we have chosen to call them AS-regular algebras. We shall compare this definition with Auslander-regularity. For technical reasons it is natural to enlarge the class of AS-regular algebras to AS-Gorenstein algebras.

DEFINITION. Let  $A = k \oplus A_1 \oplus A_2 \oplus \dots$  be a finitely generated  $k$ -algebra. We say that  $A$  is AS-regular (resp. AS-Gorenstein) of dimension  $\mu$  if:

- (i)  $\text{gldim}(A) = \mu < \infty$  (resp.  $\text{injdim}(A) = \mu < \infty$ );
- (ii) the function  $n \rightarrow \dim_k A_n$  has polynomial growth of degree  $d - 1 \in \mathbb{N} \cup \{-1\}$ ;
- (iii)  $A$  is Gorenstein in the following sense:  $E_A^i({}_A k) = \delta_{i,\mu} k_A$  and  $E_A^i(k_A) = \delta_{i,\mu} k$ .

REMARKS. (1) Assume  $A$  is a noetherian AS-regular algebra. Then  $A$  satisfies the condition (PG) defined in 5.4. In fact the Hilbert series of every  $M \in \text{Mod}_f^{\mathbb{Z}}(A)$  has the form  $h_M(t) = q_M(t)/p_A(t)$  as in 5.5.

(2) We have  $d = \text{GKdim } A$ .

(3) By the Rees lemma, if  $\{\Omega_1, \dots, \Omega_l\}$  is a regular normalizing sequence of homogeneous elements of positive degree in some AS-Gorenstein algebra then  $B = A/(\Omega_1, \dots, \Omega_l)$  is also AS-Gorenstein.

(4) The definition of an AS-regular algebra given in [3] requires that  ${}_A k$  has a free resolution in  $\text{Mod}_f^{\mathbb{Z}}(A)$ .

6.2. In [1] and [3], [4], the following natural questions are raised about AS-regular algebras.

(6.2.1) Is  $A$  noetherian?

(6.2.2) Is  $A$  a domain?

(6.2.3) Is it always true that  $\text{gldim}(A) = \text{GKdim } A$  (i.e.  $\mu = d$ )?

When  $A$  is AS-Gorenstein, one can ask similar questions, replacing (6.2.2) and (6.2.3) by

(6.2.4). Is  $A$   $\mu$ -homogeneous for  $\text{GKdim}$ ?

(6.2.5) Is it true that  $\text{injdim}(A) = \text{GKdim } A$ ?

The results from [1] and [3], [4] give (partial) answers to these questions; we summarize in the following theorem.

**THEOREM.** *Let  $A$  be an AS-regular algebra of dimension  $\mu$ .*

(i) *If  $\mu \leq 3$  and  $A$  is generated by  $A_1$  then  $A$  is a noetherian domain and  $\text{GKdim } A = \mu$ .*

(ii) *If  $A$  is noetherian and  $\text{GKdim } A = \mu \leq 4$  then  $A$  is a domain.*

(iii) *If  $\text{GKdim } A = \mu \leq 3$  and  $A$  is noetherian then  $A$  satisfies the property CM and the condition:  $E_A^i(E_A^j(M)) = (0)$  for all  $i < j$  and  $M \in \text{Mod}_f^{\mathbb{Z}}(A)$ .*

Using the results of 5.9, we deduce the next corollary.

**COROLLARY.** *Let  $A$  be an AS-regular algebra as in (iii) of the previous theorem. Then  $A$  is Auslander-regular and satisfies the property CM.*

We shall recover (particular) cases of this corollary in 6.7 and prove a similar result for some algebras of  $\text{gldim}(A) \leq 4$ .

6.3. In 4.8, we showed that a positively graded Auslander-regular  $k$ -algebra with  $A_0 = k$  is a domain. In view of Corollary 6.2, we may ask the following questions.

(6.3.1) Let  $A = k \oplus A_1 \oplus A_2 \oplus \dots$  be finitely generated over  $k$ . Do we have:  $A$  AS-regular (resp. AS-Gorenstein) if and only if  $A$  is Auslander-regular (resp. Auslander-Gorenstein) and  $A$  satisfies condition (PG)? Furthermore does  $A$  satisfy the property CM with  $\text{GKdim } A = \text{gldim}(A)$  (resp.  $\text{GKdim } A = \text{injdim } A$ )?

Notice that a positive answer to (6.3.1) gives positive answers to (6.2.j),  $j = 1, \dots, 5$ . One of the implications is easy, namely we have the following result.

**THEOREM.** *Let  $A = k \oplus A_1 \oplus A_2 \oplus \dots$  be an Auslander-Gorenstein  $k$ -algebra of dimension  $\mu$ . Then  $E_A^i({}_A k) = \delta_{i,\mu} k_A$ . In particular if  $n \rightarrow \dim_k A_n$  has polynomial growth  $A$  is AS-Gorenstein.*

*Proof.* Put  $n = j({}_A k)$ ,  $m = j(k_A)$ . We first have to show  $n = m$ . Set  $N_A := E_A^n({}_A k) \in \text{Mod}_f^{\mathbb{Z}}(A)$ . Since there exists a surjective graded morphism  $N_A \rightarrow k_A[m] \rightarrow 0$ , for some  $m \in \mathbb{Z}$ , we have  $j(N) \leq j(k_A) = m$ . But  $j(N) = n$  by Theorem 2.4. Hence  $n \leq m$  and, by symmetry,  $n = m$ . We now prove  $n = \mu$ . Let  $S_A \in \text{Mod}_f^{\mathbb{Z}}(A)$  be such that  ${}_A M = E^\mu(S) \neq (0)$ .  $S$  exists by Lemma 3.3. Since  $E^i(E^\mu(S)) = (0)$  if  $i < \mu$ , we have  $j(M) = \mu$ . There exists an exact sequence  $0 \rightarrow M' \rightarrow M \rightarrow {}_A k[m] \rightarrow 0$  in  $\text{Mod}_f^{\mathbb{Z}}(A)$ . It follows that  $\mu = j(M) = \inf\{j({}_A k), j(M')\}$ ; thus  $n = j({}_A k) = \mu$ . It remains to show that  $\dim_k E^\mu({}_A k) = 1$  (same proof for  $k_A$ ). The b-filtration for  ${}_A k$  takes the form:  $(0) \subset F^\mu {}_A k = \dots = {}_A k$ ; hence  ${}_A k = F^\mu {}_A k \cong E^\mu(E^\mu({}_A k))$  by Theorem 2.2. Put  $N_A = E^\mu({}_A k)$ , so that  $j(N) = \mu$ , and suppose that  $\dim_k N > 1$ . Consider an exact sequence in  $\text{Mod}_f^{\mathbb{Z}}(A)$ :  $0 \rightarrow N' \rightarrow N \rightarrow$

$k_A[m] \rightarrow 0$ . We get  $j(N') \geq j(N) = \mu$ ; hence  $j(N') = \mu$ . Therefore we have an exact sequence  $0 \rightarrow E^\mu(k_A[m]) \rightarrow E^\mu(N) \rightarrow E^\mu(N') \rightarrow 0$ . From  $E^\mu(N') \neq (0)$  and  $E^\mu(N) \cong_A k$ , it follows that  $E^\mu(k_A[m]) = E^\mu(k_A)[-m] = (0)$ , contradicting the fact that  $j(k_A) = \mu$ .

6.4. We recall here results from [5] and [3], [4] which will be used to give an example of an Auslander-Gorenstein ring in the next section.

For simplicity we assume that  $k$  is algebraically closed and we denote by  $X$  an irreducible projective variety over  $k$ . The category of quasi-coherent (resp. coherent)  $\mathcal{O}_X$ -modules is denoted by  $\mathcal{O}_X\text{-Mod}$  (resp.  $\mathcal{O}_X\text{-Mod}_{\text{coh}}$ ). Let  $\sigma$  be a  $k$ -automorphism of  $X$ . When  $\mathcal{M} \in \mathcal{O}_X\text{-Mod}$  we set  $\mathcal{M}^\sigma := \sigma^* \mathcal{M}$ . An invertible sheaf is called  $\sigma$ -ample if: for all  $\mathcal{F} \in \mathcal{O}_X\text{-Mod}_{\text{coh}}$ , all  $q \geq 1$ ,  $H^q(X, \mathcal{F} \otimes \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \dots \otimes \mathcal{L}^{\sigma^{n-1}}) = 0$  if  $n \gg 0$ . From now on we fix a  $\sigma$ -ample invertible sheaf  $\mathcal{L}$  and, as in [5], we define

$$\begin{aligned} \mathcal{B}_0 &:= \mathcal{O}_X, \mathcal{B}_n := \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \dots \otimes \mathcal{L}^{\sigma^{n-1}} \text{ if } n > 0, \\ \mathcal{B}_n &:= \mathcal{L}^{-\sigma^{-1}} \otimes \dots \otimes \mathcal{L}^{-\sigma^{-n}} \text{ if } n < 0, \\ \mathcal{B} &= \bigoplus_{n \in \mathbb{Z}} \mathcal{B}_n, \quad B_n = H^0(X, \mathcal{B}_n), \quad B := B(X, \sigma, \mathcal{L}) := \bigoplus_{n \geq 0} B_n. \end{aligned}$$

Then  $\mathcal{B}$  is a sheaf of graded algebras and  $B$  is a graded domain. The main theorem of [5] proves the next theorem.

**THEOREM.** *The algebra  $B = k \oplus B_1 \oplus B_2 \oplus \dots$  is noetherian and there is an equivalence of categories:*

$$\text{Mod}^{\mathbb{B}}(B)/\text{Tors} \simeq \mathcal{O}_X\text{-Mod},$$

where  $\text{Tors}$  is the full subcategory of  $\text{Mod}^{\mathbb{B}}(B)$  consisting of modules which are direct limits of finite dimensional modules.

**REMARK.** The equivalence from  $\mathcal{O}_X\text{-Mod}$  onto  $\text{Mod}^{\mathbb{B}}(B)/\text{Tors}$  is induced by the functor  $\Gamma_*: \mathcal{O}_X\text{-Mod} \rightarrow \text{Mod}^{\mathbb{B}}(B)$  defined by  $\Gamma_*(\mathcal{M}) := \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{B}_n)$ . It takes coherent  $\mathcal{O}_X$ -modules to finitely generated  $B$ -modules.

6.5. We assume that the variety  $X$  of 6.4 is a smooth elliptic curve and we fix  $\sigma \in \text{Aut}_k(X)$ . Then any ample invertible sheaf  $\mathcal{L}$  is  $\sigma$ -ample and we have  $\text{GKdim } B = 2$ , see [5, 1.5] and Lemma 6.6.

The following proposition is a reformulation of results by A. Yekutieli, see [20], [21].

**PROPOSITION.** *The algebra  $B$  satisfies:  $\text{injdim}(B) = 2$ ,  $E_B^i(k_B) = \delta_{i,2} k$ ,  $E_{B(B)}^i(k) = \delta_{i,2} k_B$ .*

*Proof.* By [20, 4.3.1] and the fact that the canonical sheaf on  $X$  is isomorphic to  $\mathcal{O}_X$ , we have a resolution by injectives in  $\mathcal{O}_X\text{-Mod}$ :  $0 \rightarrow \mathcal{O}_X \xrightarrow{\delta} \mathcal{K}_X^{-1} \xrightarrow{\delta} \mathcal{K}_X^0 \rightarrow 0$ . Set  $J^0 = \Gamma_*(\mathcal{K}_X^{-1})$ ,  $J^1 = \Gamma_*(\mathcal{K}_X^0)$ ,  $J^2 = \Gamma_*(\mathcal{K}_X^0) / \delta(\Gamma_*(\mathcal{K}_X^{-1}))$ . Then, by the proof of [20, 4.5.3], one obtains:  $J^i$  is a graded bimodule over  $B$  injective on both sides,  $J^0$  and  $J^1$  are torsion free and  $J^2 \cong \text{HOM}_k(B, k)$  is the injective hull of  ${}_B k$ , or  $k_B$ , in  $\text{Mod}^{\mathbb{B}}(B)$ . Thus, applying  $\Gamma_*$  to the resolution above, we obtain an injective resolution  $0 \rightarrow B \rightarrow J^0 \rightarrow J^1 \rightarrow J^2 \rightarrow 0$  in  $\text{Mod}^{\mathbb{B}}(B)$ . This proves  $\text{grinjdim}(B) = \text{injdim}(B) \leq 2$ . By symmetry, the proposition will follow from  $E_{B(B)}^i(k) = \delta_{i,2} k_B$ . Since  $J^0, J^1$  are torsion free and  $J^2 \cong \text{HOM}_k(B, k)$ , we have:  $\text{Hom}_{B(B)}(J^i, k) = (0)$  if  $i = 0, 1$  and  $\text{Hom}_{B(B)}(J^2, k) = \text{Hom}_k(B, k)_0 \cong k_B$ . Since  $E_{B(B)}^i(k) = H^i(\text{HOM}_{B(B)}(k, J^*)) = H^i(\text{Hom}_B(k, J^*))$ , the proof is complete.

6.6. Assume  $(X, \sigma, \mathcal{L})$  is as in 6.5 and suppose furthermore that  $\mathcal{L}$  is very ample, i.e.  $\deg \mathcal{L} \geq 3$ . Denote by  $C$  the algebra  $B(X, \text{Id}_X, \mathcal{L})$ . Hence  $C = \bigoplus_{n \geq 0} C_n$ , where  $C_n = H^0(X, \mathcal{L}^{\otimes n})$ , is the homogeneous coordinate ring of the projective embedding  $X \rightarrow \mathbb{P} (H^0(X, \mathcal{L})^*)$ .

LEMMA. Let  $M = \bigoplus_m M_m$  be in  $\text{Mod}_f^g(B)$ . Then the function  $m \rightarrow \dim_k M_m$  is given by a polynomial for  $n \gg 0$ . In particular  $B$  satisfies the condition (PG).

*Proof.* Let  $\mathcal{M}$  be a coherent  $\mathcal{O}_X$ -module. We first notice that, if  $m \gg 0$ ,  $\dim_k H^0(X, \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{B}_m) = \dim_k H^0(X, \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m})$ . This can be proved as follows. Taking a finite locally free resolution of  $\mathcal{M}$ , one reduces to the case where  $\mathcal{M}$  is locally free. Then [2, Lemma 15] gives the result since  $\mathcal{B}_m$  and  $\mathcal{L}^{\otimes m}$  have the same degree. Now, by Theorem 6.4, we have equivalences

$$\text{Mod}^g(B)/\text{Tors} \cong \mathcal{O}_X\text{-Mod} \cong \text{Mod}^g(C)/\text{Tors}.$$

If  $\dim_k M < \infty$ , the lemma is obvious. Therefore we assume that  $M$  is torsion free and denote by  $\mathcal{M} \in \mathcal{O}_X\text{-Mod}_{\text{coh}}$  the corresponding  $\mathcal{O}_X$ -module. See  $N = \Gamma_*(\mathcal{M}) \in \text{Mod}_f^g(C)$ . By the previous remark,  $\dim_k M_m = \dim_k N_m$ . Then it is well known that  $m \rightarrow \dim_k N_m$  is given by a polynomial of degree  $\text{GKdim } N - 1$  for  $n \gg 0$ . Hence the lemma is proved.

REMARK. The Hilbert series of  $M \in \text{Mod}_f^g(B)$  is of the form  $h_M(t) = q_M(t)/(1-t)^2$ ,  $q_M(t) \in \mathbb{Z}[t, t^{-1}]$ . This is a consequence of the proof of the lemma and the truth of this assertion in  $\text{Mod}_f^g(C)$ . We have  $h_B(t) = (1 + (r-2)t + t^2)/(1-t)^2$  if  $r = \deg \mathcal{L}$ .

THEOREM. Let  $(X, \sigma, \mathcal{L})$  be a triple as above. Then the algebra  $B(X, \sigma, \mathcal{L})$  is Auslander-Gorenstein of dimension 2, satisfies the CM-property and the condition (PG).

*Proof.* Notice that  $\text{GKdim } B = 2$  and apply the lemma, Proposition 6.5 and Theorem 5.13.

6.7. We deduce, from Remark 5.13 (2), the following result.

COROLLARY. Let  $A = k \oplus A_1 \oplus A_2 \oplus \dots$  be a finitely generated graded  $k$ -algebra. Assume  $\Omega = \{\Omega_1, \dots, \Omega_l\}$  is a regular normalizing sequence of homogeneous elements of positive degree in  $A$  such that  $A/\Omega A \cong B(X, \sigma, \mathcal{L})$  for some triple  $(X, \sigma, \mathcal{L})$  as in 6.6. Then  $A$  is Auslander-Gorenstein of dimension  $l+2$  and satisfies the CM-property and condition (PG).

APPLICATIONS. (1) Let  $A$  be an AS-regular algebra of dimension 3 and type A as in [3]. Then  $A$  is Auslander-regular and satisfies the CM-property. This follows from the corollary since  $A/\Omega A \cong B(X, \sigma, \mathcal{L})$  for a central element of degree 3 and a triple  $(X, \sigma, \mathcal{L})$  as in 6.6 (with  $\deg \mathcal{L} = 3$ ). In particular, we recover [4, Theorem 4.1] in this case.

(2) Let  $S$  be the Sklyanin algebra as in [18, Theorem 5.4]. Then  $S$  is Auslander-regular of dimension 4 and satisfies the CM-property. The corollary applies to  $S$  since  $S/(\Omega_1, \Omega_2) \cong B(X, \sigma, \mathcal{L})$  for a regular centralizing sequence  $\{\Omega_1, \Omega_2\}$  of quadratic elements and a triple  $(X, \sigma, \mathcal{L})$  as in 6.6 (with  $\deg \mathcal{L} = 4$ ). Since  $\text{gldim}(S) = 4$  by [18, Theorem 0.3], we have that  $S$  is Auslander-regular.

(3) In [19], the construction of the Sklyanin algebra is generalized to obtain AS-regular algebras  $A$  of dimension 4 such that  $A/(\Omega_1, \Omega_2) \cong B(X, \sigma, \mathcal{L})$  for a regular normalizing sequence  $\{\Omega_1, \Omega_2\}$  of quadratic elements and a triple  $(X, \sigma, \mathcal{L})$  as in 6.6 (with  $\deg \mathcal{L} = 4$ ). Thus the corollary can be applied to these algebras.

## REFERENCES

1. M. Artin and W. F. Schelter, Graded algebras of global dimension 3, *Adv. in Math.* **66** (1987), 171–216.
2. M. F. Atiyah, Vector bundles over an elliptic curve, *Proc. London Math. Soc.* (3) **27** (1957), 414–452.
3. M. Artin, J. Tate and M. Van den Bergh, Some algebras associated to automorphisms of elliptic curves, *The Grothendieck Festschrift* (Birkhauser, 1990), 33–85.
4. M. Artin, J. Tate and M. Van den Bergh, Modules over regular algebras of dimension 3, preprint (1988).
5. M. Artin and M. Van den Bergh, Twisted homogeneous coordinate rings, *Algebra* **133** (1990), 249–271.
6. J. E. Björk, *Rings of differential operators* (North-Holland, 1979).
7. J. E. Björk, The Auslander condition on noetherian rings, *Séminaire Dubreil-Malliavin 1987–88*, Lecture Notes in Mathematics 1404 (Springer, 1989), 137–173.
8. J. E. Björk and E. K. Ekström, Filtered Auslander-Gorenstein rings, *Colloque en l'honneur de J. Dixmier* (Birkhauser, 1990), 424–448.
9. N. Bourbaki, *Algèbre, Chapitre 10* (Masson, 1980).
10. E. K. Ekström, The Auslander condition on graded and filtered noetherian rings, *Séminaire Dubreil-Malliavin 1987–1988*, Lecture Notes in Mathematics 1404 (Springer, 1989), 220–245.
11. R. M. Fossum, P. A. Griffith and I. Reiten, *Trivial extensions of abelian categories with applications to ring theory*, Lecture Notes in Mathematics 456 (Springer, 1975).
12. F. Ischebeck, Eine Dualität Zwischen den Funktoren Ext und Tor, *J. Algebra* **11** (1969), 510–531.
13. G. Krause and T. H. Lenagan, *Growth of algebras and Gelfand-Kirillov dimension*, Research Notes in Mathematics 116 (Pitman, 1985).
14. T. Levasseur, Complexe bidualisant en algèbre non commutative, *Séminaire Dubreil-Malliavin 1983–84*, Lecture Notes in Mathematics 1146 (Springer, 1985), 270–287.
15. T. Levasseur and S. P. Smith, Modules over the Sklyanin algebra, preprint (1991).
16. Li Huishi, *Non-commutative Zariskian rings* (Ph.Doc. thesis, U.I.A., Antwerp, 1989).
17. J. C. McConnell and J. C. Robson, *Non-commutative noetherian rings* (Wiley, 1987).
18. S. P. Smith and J. T. Stafford, Regularity of the four dimensional Sklyanin algebra, preprint (1989).
19. J. T. Stafford, Regularity of algebras related to the Sklyanin algebra, preprint (1990).
20. A. Yekutieli, *The residue complex and duality for some non-commutative rings* (Ph. Doc. thesis, MIT, 1990).
21. A. Yekutieli, Dualizing complexes over noncommutative graded algebras, preprint (1990).

DÉPARTEMENT DE MATHÉMATIQUES  
 UNIVERSITÉ DE BRETAGNE OCCIDENTALE  
 6, AVENUE VICTOR LE GORGEU  
 29287 BREST CEDEX  
 FRANCE