

SOME PROPERTIES OF NORDEN-WALKER METRICS¹

ARIF SALIMOV AND MURAT ISCAN

Abstract

The main purpose of the present paper is to study almost Norden structures on 4-dimensional Walker manifolds. We discuss the integrability and Kähler (holomorphic) conditions for these structures. The curvature properties for Norden-Walker metrics is also investigated. Then examples Norden-Walker metrics are constructed from an arbitrary harmonic function of two variables.

1. Introduction

Let M_{2n} be a Riemannian manifold with a neutral metric, i.e., with a pseudo-Riemannian metric g of signature (n, n) . We denote by $\mathfrak{S}_q^p(M_{2n})$ the set of all tensor fields of type (p, q) on M_{2n} . Manifolds, tensor fields and connections are always assumed to be differentiable and of class C^∞ .

Let (M_{2n}, φ) be an almost complex manifold with almost complex structure φ . This structure is said to be integrable if the matrix $\varphi = (\varphi_j^i)$ is reduced to the constant form in a certain holonomic natural frame in a neighborhood U_x of every point $x \in M_{2n}$. In order that the almost complex structure φ be integrable, it is necessary and sufficient that there exists a torsion-free affine connection ∇ with respect to which the structure tensor φ is covariantly constant, i.e., $\nabla\varphi = 0$. Also, we know that the integrability of φ is equivalent to the vanishing of the Nijenhuis tensor $N_\varphi \in \mathfrak{S}_2^1(M_{2n})$ [8, p. 124]. If φ is integrable, then φ is a complex structure and moreover M_{2n} is a \mathbf{C} -holomorphic manifold $X_n(\mathbf{C})$ whose transition functions are holomorphic mappings.

1.1. Norden metrics

A metric g is a Norden metric [14] if

$$g(\varphi X, \varphi Y) = -g(X, Y)$$

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or equivalently

$$g(\varphi X, Y) = g(X, \varphi Y)$$

for any $X, Y \in \mathfrak{S}_0^1(M_{2n})$. Metrics of this kind have been also studied under the other names: pure, anti-Hermitian and B-metrics (see [4], [5], [9], [15], [19], [21]). If (M_{2n}, φ) is an almost complex manifold with Norden metric g , we say that (M_{2n}, φ, g) is an almost Norden manifold. If φ is integrable, we say that (M_{2n}, φ, g) is a Norden manifold.

1.2. Holomorphic (almost holomorphic) tensor fields

Let t^* be a complex tensor field on a C-holomorphic manifold $X_n(\mathbf{C})$. The real model of such a tensor field is a tensor field on M_{2n} of the same order irrespective of whether its vector or covector arguments is subject to the action of the affiner structure φ . Such tensor fields are said to be pure with respect to φ . They were studied by many authors (see, e.g., [9], [16], [17], [19], [20], [21], [23]). In particular, for a $(0, q)$ -tensor field ω , the purity means that for any $X_1, \dots, X_q \in \mathfrak{S}_0^1(M_{2n})$, the following conditions should hold:

$$\omega(\varphi X_1, X_2, \dots, X_q) = \omega(X_1, \varphi X_2, \dots, X_q) = \dots = \omega(X_1, X_2, \dots, \varphi X_q).$$

We define an operator

$$\Phi_\varphi : \mathfrak{S}_q^0(M_{2n}) \rightarrow \mathfrak{S}_{q+1}^0(M_{2n})$$

applied to the pure tensor field ω by (see [23])

$$\begin{aligned} (\Phi_\varphi \omega)(X, Y_1, Y_2, \dots, Y_q) &= (\varphi X)(\omega(Y_1, Y_2, \dots, Y_q)) - X(\omega(\varphi Y_1, Y_2, \dots, Y_q)) \\ &\quad + \omega((L_{Y_1} \varphi)X, Y_2, \dots, Y_q) + \dots + \omega(Y_1, Y_2, \dots, (L_{Y_q} \varphi)X), \end{aligned}$$

where L_Y denotes the Lie differentiation with respect to Y .

When φ is a complex structure on M_{2n} and the tensor field $\Phi_\varphi \omega$ vanishes, the complex tensor field ω^* on $X_n(\mathbf{C})$ is said to be holomorphic (see [9], [19], [23]). Thus a holomorphic tensor field ω^* on $X_n(\mathbf{C})$ is realized on M_{2n} in the form of a pure tensor field ω , such that

$$(\Phi_\varphi \omega)(X, Y_1, Y_2, \dots, Y_q) = 0$$

for any $X, Y_1, \dots, Y_q \in \mathfrak{S}_0^1(M_{2n})$. Therefore such a tensor field ω on M_{2n} is also called holomorphic tensor field. When φ is an almost complex structure on M_{2n} , a tensor field ω satisfying $\Phi_\varphi \omega = 0$ is said to be almost holomorphic.

1.3. Holomorphic Norden (Kähler-Norden) metrics

In a Norden manifold a Norden metric g is called a *holomorphic* if

$$(1) \quad (\Phi_\varphi g)(X, Y, Z) = 0$$

for any $X, Y, Z \in \mathfrak{S}_0^1(M_{2n})$.

By setting $X = \partial_k$, $Y = \partial_i$, $Z = \partial_j$ in the equation (1), we see that the components $(\Phi_\varphi g)_{kij}$ of $\Phi_\varphi g$ with respect to a local coordinate system x^1, \dots, x^n may be expressed as follows:

$$(\Phi_\varphi g)_{kij} = \varphi_k^m \partial_m g_{ij} - \varphi_i^m \partial_k g_{mj} + g_{mj} (\partial_i \varphi_k^m - \partial_k \varphi_i^m) + g_{im} \partial_j \varphi_k^m.$$

If (M_{2n}, φ, g) is a Norden manifold with holomorphic Norden metric g , we say that (M_{2n}, φ, g) is a *holomorphic Norden manifold*.

In some aspects, holomorphic Norden manifolds are similar to Kähler manifolds. The following theorem is analogue to the next known result: An almost Hermitian manifold is Kähler if and only if the almost complex structure is parallel with respect to the Levi-Civita connection.

THEOREM 1 [6] (For paracomplex version see [18]). *For an almost complex manifold with Norden metric g , the condition $\Phi_\varphi g = 0$ is equivalent to $\nabla\varphi = 0$, where ∇ is the Levi-Civita connection of g .*

A *Kähler-Norden* manifold can be defined as a triple (M_{2n}, φ, g) which consists of a manifold M_{2n} endowed with an almost complex structure φ and a pseudo-Riemannian metric g such that $\nabla\varphi = 0$, where ∇ is the Levi-Civita connection of g and the metric g is assumed to be Nordenian. Therefore, there exists a one-to-one correspondence between *Kähler-Norden* manifolds and Norden manifolds with a *holomorphic metric*. Recall that in such a manifold, the Riemannian curvature tensor is pure and holomorphic, also the curvature scalar is locally holomorphic function (see [6], [15]).

Remark 1. We know that the integrability of the almost complex structure φ is equivalent to the existing a torsion-free affine connection with respect to which the equation $\nabla\varphi = 0$ holds. Since the Levi-Civita connection ∇ of g is a torsion-free affine connection, we have: If $\Phi_\varphi g = 0$, then φ is integrable. Thus, almost Norden manifold with conditions $\Phi_\varphi g = 0$ and $N_\varphi \neq 0$, i.e., *almost holomorphic Norden manifolds (analogues of the almost Kähler manifolds with closed Kähler form) does not exist.*

In the present paper, we shall focus our attention to the Norden manifolds in dimension four. Using the Walker metric we construct a new Norden-Walker metrics together with a proper almost complex structure [11].

2. Norden-Walker metrics

2.1. Walker metric g

A neutral metric g on a 4-manifold M_4 is said to be Walker metric if there exists a 2-dimensional null distribution D on M_4 , which is parallel with respect to g . From Walker's theorem [22], there is a system of coordinates with respect to which g takes the local canonical form

$$(2) \quad g = (g_{ij}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{pmatrix},$$

where a, b, c are smooth functions of the coordinates (x, y, z, t) . The paralel null 2-plane D is spanned locally by $\{\partial_x, \partial_y\}$, where $\hat{\partial}_x = \frac{\partial}{\partial x}$, $\hat{\partial}_y = \frac{\partial}{\partial y}$.

In [11, Fact 1], a proper almost complex structure with respect to g is defined as a g -orthogonal almost complex structure J so that J is a standard generator of a positive $\frac{\pi}{2}$ rotation on D , i.e., $J\hat{\partial}_x = \hat{\partial}_y$ and $J\hat{\partial}_y = -\hat{\partial}_x$. Then for the Walker metric g , such a proper almost complex structure J is determined uniquely as

$$(3) \quad \begin{pmatrix} 0 & -1 & -c & \frac{1}{2}(a-b) \\ 1 & 0 & \frac{1}{2}(a-b) & c \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

In [1], for such a proper almost complex structure J on Walker 4-manifold M , an almost Norden structure (g^{N+}, J) is constructed, where g^{N+} is a metric on M , with properties $g^{N+}(JX, JY) = -g^{N+}(X, Y)$. In fact, as one of these examples, such a metric takes the form (see Proposition 6 in [1]):

$$g^{N+} = \begin{pmatrix} 0 & -2 & 0 & -b \\ -2 & 0 & -a & -2c \\ 0 & -a & 0 & \frac{1}{2}(1-ab) \\ -b & -2c & \frac{1}{2}(1-ab) & -2c \end{pmatrix}.$$

We may call this an almost Norden-Walker metric. The construction of such a structure in [1] is to find a Norden metric for a given almost complex structure, which is different form the Walker metric.

The purpose of the present paper is to find also an almost Norden-Walker structure (g, F) , where the metric is nothing but the Walker metric g , with an appropriate almost complex structure F , to be determined. That is, for a fixed metric g , we will find an almost complex structure F which satisfy $g(FX, FY) = -g(X, Y)$.

In [1], for a given almost complex structure, a metric is constructed. Our method is, however, for a given metric, an almost complex structure is constructed.

2.2. Almost Norden-Walker manifolds

Let F be an almost complex structure on a Walker manifold M_4 , which satisfies

- i) $F^2 = -I$,
- ii) $g(FX, Y) = g(X, FY)$ (Nordenian property),
- iii) $F\partial_x = \partial_y, F\partial_y = -\partial_x$ (F induces a positive $\frac{\pi}{2}$ -rotation on D).

We easily see that these three properties define F non-uniquely, i.e.,

$$\begin{cases} F\partial_x = \partial_y, \\ F\partial_y = -\partial_x, \\ F\partial_z = \alpha\partial_x + \frac{1}{2}(a+b)\partial_y - \partial_t, \\ F\partial_t = -\frac{1}{2}(a+b)\partial_x + \alpha\partial_y + \partial_z \end{cases}$$

and F has the local components

$$F = (F_j^i) = \begin{pmatrix} 0 & -1 & \alpha & -\frac{1}{2}(a+b) \\ 1 & 0 & \frac{1}{2}(a+b) & \alpha \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

with respect to the natural frame $\{\partial_x, \partial_y, \partial_z, \partial_t\}$, where $\alpha = \alpha(x, y, z, t)$ is an arbitrary function.

We must note that the proper almost complex structure J as in (3) is determined uniquely. In our case of the almost Norden-Walker structure, the almost complex structure F just obtained contains an arbitrary function $\alpha(x, y, z, t)$.

Our purpose is to find a nontrivial almost Norden-Walker structure with the Walker metric g explicitly.

Therefore, we now put $\alpha = c$. Then g defines a unique almost complex structure

$$(4) \quad \varphi = (\varphi_j^i) = \begin{pmatrix} 0 & -1 & c & -\frac{1}{2}(a+b) \\ 1 & 0 & \frac{1}{2}(a+b) & c \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

The triple (M_4, φ, g) is called almost Norden-Walker manifold. In conformity with the terminology of [2], [3], [10], [11], [13] we call φ the proper almost complex structure.

Remark 2. From (4) we immediately see that in the case $a = -b$ and $c = 0$, φ is integrable.

2.3. Integrability of φ

We consider the general case for integrability.

The almost complex structure φ on almost Norden-Walker manifolds is integrable if and only if

$$(5) \quad (N_\varphi)_{jk}^i = \varphi_j^m \partial_m \varphi_k^i - \varphi_k^m \partial_m \varphi_j^i - \varphi_m^i \partial_j \varphi_k^m + \varphi_m^i \partial_k \varphi_j^m = 0.$$

From (4) and (5) find the following integrability condition.

THEOREM 2. *The proper almost complex structure φ on almost Norden-Walker manifolds is integrable if and only if the following PDEs hold:*

$$(6) \quad \begin{cases} a_x + b_x + 2c_y = 0, \\ a_y + b_y - 2c_x = 0. \end{cases}$$

From this theorem, we easily see that if $a = -b$ and $c = 0$, then φ is integrable (see Remark 2).

Let (M_4, φ, g) be a Norden-Walker manifolds ($N_\varphi = 0$) and $a = b$. Then the equation (6) reduces to

$$(7) \quad \begin{cases} a_x = -c_y, \\ a_y = c_x, \end{cases}$$

from which follows

$$(8) \quad \begin{aligned} a_{xx} + a_{yy} &= 0, \\ c_{xx} + c_{yy} &= 0, \end{aligned}$$

e.g., the functions a and c are harmonic with respect to the arguments x and y . Thus we have

THEOREM 3. *If the triple (M_4, φ, g) is Norden-Walker and $a = b$, then a and c are all harmonic with respect to the arguments x, y .*

2.4. Example of Norden-Walker metric

We now apply the Theorem 3 to establish the existence of special types of Norden-Walker metrics. In our arguments, the harmonic function plays an important part.

Let $a = b$ and $h(x, y)$ be a harmonic function of variables x and y , for example $h(x, y) = e^x \cos y$. We put

$$a = a(x, y, z, t) = h(x, y) + \alpha(z, t) = e^x \cos y + \alpha(z, t)$$

where α is an arbitrary smooth function of z and t . Then, a is also harmonic with respect to x and y . We have

$$\begin{aligned} a_x &= e^x \cos y, \\ a_y &= -e^x \sin y. \end{aligned}$$

From (7), we have PDEs for c to satisfy as

$$\begin{aligned} c_x &= a_y = -e^x \sin x, \\ c_y &= -a_x = -e^x \cos y. \end{aligned}$$

For these PDEs, we have solutions

$$c = -e^x \sin y + \beta(z, t),$$

where β is arbitrary smooth function of z and t . Thus the Norden-Walker metric has components of the form

$$g = (g_{ij}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & e^x \cos y + \alpha(z, t) & -e^x \sin y + \beta(z, t) \\ 0 & 1 & -e^x \sin y + \beta(z, t) & e^x \cos y + \alpha(z, t) \end{pmatrix}$$

3. Holomorphic Norden-Walker (Kähler-Norden-Walker) metrics

Let (M_4, φ, g) be an almost Norden-Walker manifold. If

$$(9) \quad (\Phi_{\varphi g})_{kij} = \varphi_k^m \partial_m g_{ij} - \varphi_i^m \partial_k g_{mj} + g_{mj} (\partial_i \varphi_k^m - \partial_k \varphi_i^m) + g_{im} \partial_j \varphi_k^m = 0,$$

then by virtue of Theorem 1 φ is integrable and the triple (M_4, φ, g) is called a holomorphic Norden-Walker or a Kähler-Norden-Walker manifold. Taking account of Remark 1, we see that almost Kähler-Norden-Walker manifold with condition $\Phi_{\varphi g} = 0$ and $N_{\varphi} \neq 0$ does not exist.

Substituting (2) and (4) in (9), we obtain

$$\begin{aligned} (\Phi_{\varphi g})_{xzz} &= a_y, & (\Phi_{\varphi g})_{xzt} &= (\Phi_{\varphi g})_{xtz} = \frac{1}{2}(b_x - a_x) + c_y, \\ (\Phi_{\varphi g})_{xlt} &= b_y - 2c_x, & (\Phi_{\varphi g})_{yzz} &= -a_x, \\ (\Phi_{\varphi g})_{yzt} &= (\Phi_{\varphi g})_{ytz} = \frac{1}{2}(b_y - a_y) - c_x, \\ (\Phi_{\varphi g})_{ytl} &= -b_x - 2c_y, \\ (\Phi_{\varphi g})_{zxx} &= (\Phi_{\varphi g})_{zzx} = (\Phi_{\varphi g})_{lxt} = (\Phi_{\varphi g})_{ltx} = c_x, \\ (\Phi_{\varphi g})_{zxt} &= (\Phi_{\varphi g})_{ztx} = -(\Phi_{\varphi g})_{txz} = -(\Phi_{\varphi g})_{tzx} = \frac{1}{2}(a_x + b_x), \\ (\Phi_{\varphi g})_{zyz} &= (\Phi_{\varphi g})_{zzy} = (\Phi_{\varphi g})_{lyt} = (\Phi_{\varphi g})_{lty} = c_y, \\ (\Phi_{\varphi g})_{zyt} &= (\Phi_{\varphi g})_{zty} = -(\Phi_{\varphi g})_{lyz} = -(\Phi_{\varphi g})_{tzy} = \frac{1}{2}(a_y + b_y), \\ (\Phi_{\varphi g})_{zzz} &= ca_x - a_t + 2c_z + \frac{1}{2}(a + b)a_y, \\ (\Phi_{\varphi g})_{zzt} &= (\Phi_{\varphi g})_{ztz} = cc_x + b_z + \frac{1}{2}(a + b)c_y, \\ (\Phi_{\varphi g})_{ztl} &= cb_x + a_t - 2c_z + \frac{1}{2}(a + b)b_y, \\ (\Phi_{\varphi g})_{lzz} &= ca_y - b_z - \frac{1}{2}(a + b)a_x, \\ (\Phi_{\varphi g})_{lzt} &= (\Phi_{\varphi g})_{ltz} = cc_y - a_t + 2c_z - \frac{1}{2}(a + b)c_x, \\ (\Phi_{\varphi g})_{ltl} &= cb_y + b_z - \frac{1}{2}(a + b)b_x. \end{aligned}$$

From these equations we have

THEOREM 4. *The triple (M_4, φ, g) is Kähler-Norden-Walker if and only if the following PDEs hold:*

$$(10) \quad a_x = a_y = c_x = c_y = b_x = b_y = b_z = 0, \quad a_t - 2c_z = 0.$$

COROLLARY. *The triple (M_4, φ, g) with metric*

$$g = (g_{ij}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a(z) & 0 \\ 0 & 1 & 0 & b(t) \end{pmatrix}$$

is always Kähler-Norden-Walker.

4. Curvature properties of Norden-Walker manifolds

If R and r are respectively the curvature and the scalar curvature of the Walker metric, then the components of R and r have, respectively, expressions (see [11], Appendix A and C)

$$(11) \quad \begin{aligned} R_{xzxz} &= -\frac{1}{2}a_{xx}, & R_{xzxt} &= -\frac{1}{2}c_{xx}, & R_{xzyz} &= -\frac{1}{2}a_{xy}, & R_{xzyt} &= -\frac{1}{2}c_{xy}, \\ R_{xzzt} &= \frac{1}{2}a_{xt} - \frac{1}{2}c_{xz} - \frac{1}{4}a_y b_x + \frac{1}{4}c_x c_y, & R_{xtxt} &= -\frac{1}{2}b_{xx}, & R_{xtyz} &= -\frac{1}{2}c_{xy}, \\ R_{xtyt} &= -\frac{1}{2}b_{xy}, & R_{xtzt} &= \frac{1}{2}c_{xt} - \frac{1}{2}b_{xz} - \frac{1}{4}(c_x)^2 + \frac{1}{4}a_x b_x - \frac{1}{4}b_x c_y + \frac{1}{4}b_y c_x, \\ R_{yzyz} &= -\frac{1}{2}a_{yy}, & R_{yzyt} &= -\frac{1}{2}c_{yy}, \\ R_{yzzt} &= \frac{1}{2}a_{yt} - \frac{1}{2}c_{yz} - \frac{1}{4}a_x c_y + \frac{1}{4}a_y c_x - \frac{1}{4}a_y b_y + \frac{1}{4}(c_y)^2, & R_{ylyt} &= -\frac{1}{2}b_{yy}, \\ R_{ylyz} &= \frac{1}{2}c_{yt} - \frac{1}{2}b_{yz} - \frac{1}{4}c_x c_y + \frac{1}{4}a_y b_x, \\ R_{zlyt} &= c_{zt} - \frac{1}{2}a_{tt} - \frac{1}{2}b_{zz} - \frac{1}{4}a(c_x)^2 + \frac{1}{4}aa_x b_x + \frac{1}{4}ca_x b_y - \frac{1}{2}cc_x c_y - \frac{1}{2}a_t c_x \\ &\quad + \frac{1}{2}a_x c_t - \frac{1}{4}a_x b_z + \frac{1}{4}ca_y b_x + \frac{1}{4}ba_y b_y - \frac{1}{4}b(c_y)^2 - \frac{1}{2}b_z c_y \\ &\quad + \frac{1}{4}a_y b_t + \frac{1}{4}a_z b_x + \frac{1}{2}b_y c_z - \frac{1}{4}a_t b_y \end{aligned}$$

and

$$(12) \quad r = a_{xx} + 2c_{xy} + b_{yy}.$$

Suppose that the triple (M_4, φ, g) is Kähler-Norden-Walker. Then from the last equation in (10) and (11), we see that

$$R_{zlyt} = c_{zt} - \frac{1}{2}a_{tt} = -\frac{1}{2}(a_t - 2c_z)_t = 0.$$

From (10) we easily see that the other components of R in (11) directly all vanish. Thus we have

THEOREM 5. *If a Norden-Walker manifold (M_4, φ, g) is Kähler-Norden-Walker, then M_4 is flat.*

Remark 3. We note that a Kähler-Norden manifold is non-flat, in such manifold curvature tensor pure and holomorphic [6].

Let (M_4, φ, g) be a Norden-Walker manifold with the integrable proper structure φ , i.e., $N_\varphi = 0$. If $a = b$, then from proof of the Theorem 3 we see that the equation (7) hold. If $c = c(y, z, t)$ and $c = c(x, z, t)$, then $c_{xy} = (c_x)_y = (c_y)_x = 0$. In these cases, by virtue of (7) we find $a = a(x, z, t)$ and $a = (y, z, t)$ respectively. Using of $c_{xy} = 0$ and $a_{xx} + b_{yy} = 0$ (see (8)), we from (12) obtain $r = 0$. Thus we have

THEOREM 6. *If (M_4, φ, g) is a Norden-Walker non-Kähler manifold with metrics*

$$g = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a(x, z, t) & c(y, z, t) \\ 0 & 1 & c(y, z, t) & a(x, z, t) \end{pmatrix}, \quad \tilde{g} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a(y, z, t) & c(x, z, t) \\ 0 & 1 & c(x, z, t) & a(y, z, t) \end{pmatrix}$$

then M_4 is scalar flat.

5. On a relation between the Goldberg conjecture of almost Norden-Walker and Kähler-Norden-Walker manifolds

Let (M_{2n}, φ, g) be an almost Norden manifold, and choose a φ -compatible 2-form $\Omega_\varphi(X, Y) = h(\varphi X, Y)$ on M_{2n} , where $h(X, Y) = \tilde{g}(X, Y) + \tilde{g}(\varphi X, \varphi Y)$ is Hermitian metric for any Riemannian metric \tilde{g} , which exists provided M_{2n} is paracompact [7, p. 60]. Then we can propose an almost Norden version of Goldberg conjecture as follows [12]: (G_1) if M_{2n} is compact and (G_2) g is Einstein, and (G_3) if a φ -compatible 2-form is closed, then φ must be integrable.

We now define two subfamilies in the set of all compact Norden-Walker 4-manifolds:

$$KNW = \{(M_4, \varphi, g) : \Phi_\varphi g = 0\},$$

$$GNW = \{(M_4, \varphi, g) : M_4 \text{ with conditions } (G_2), (G_3)\}.$$

THEOREM 7. *Let $M_4 \in KNW$. Then M_4 must be a manifold with condition (G_2) of GNW , and φ is integrable.*

Proof. Suppose that $M_4 \in KNW$. Then from Theorem 5 we see that M_4 is flat, and therefore g is Einstein. By virtue of Remark 1, we see that φ is integrable, if $\Phi_\varphi g = 0$. Thus the proof is completed.

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Arif Salimov
BAKU STATE UNIVERSITY
DEPARTMENT OF ALGEBRA AND GEOMETRY
BAKU, 370145
AZERBAIJAN

ATATURK UNIVERSITY
FACULTY OF ARTS AND SCIENCES
DEPARTMENT OF MATHEMATICS
ERZURUM
TURKEY
E-mail: asalimov@atauni.edu.tr

Murat Iscan
ATATURK UNIVERSITY
FACULTY OF ARTS AND SCIENCES
DEPARTMENT OF MATHEMATICS
ERZURUM
TURKEY
E-mail: miscan@atauni.edu.tr