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Research Article

Some Properties of Numerical Solutions for Semilinear Stochastic Delay Differential Equations Driven by G-Brownian Motion

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This paper is concerned with the numerical solutions of semilinear stochastic delay differential equations driven by G-Brownian motion (G-SLSDDEs). The existence and uniqueness of exact solutions of G-SLSDDEs are studied by using some inequalities and the Picard iteration scheme first. Then the numerical approximation of exponential Euler method for G-SLSDDEs is constructed, and the convergence and the stability of the numerical method are studied. It is proved that the exponential Euler method is convergent, and it can reproduce the stability of the analytical solution under some restrictions. Numerical experiments are presented to confirm the theoretical results.

1. Introduction

Many models in many branches of science and industry, such as economics, finance, biology, and medicine, reveal stochastic effects and are introduced as stochastic differential equations (SDEs). Some phenomena in various fields such as population dynamics [1], optimal pricing in economics [2], thermal noise [3], and spread of virus [4] show stochastic behaviors. However, since most SDEs cannot be solved explicitly, numerical approximations which are on the basis of incorporating the stochastic factor in the classical numerical approximations for DDEs have become an important tool in the study of SDEs. A lot of numerical results for the SDEs have been obtained; please see the works of Chassagneux et al. [5], Higham et al. [6, 7], Banihashemi et al. [8], Babaei et al. [9], Liu and Mao [10], and so on.

The phenomenon of stiffness appears in the process of applying a certain numerical method to ODEs and SDEs. It is known that the stiffness makes standard explicit integrators useless. Nevertheless, the implicit scheme does not perform well for the step size reduction which is forced by accuracy requirements: the method tends to resolve all the oscillations in the solutions and hence leads to its numerical inefficiency. Due to the cost of computing the Jacobian and the exponential or related function of Jacobian, many works are directed at the semilinear stochastic problems and exploring the exponential integrators to approximate the semilinear stochastic problems as they can solve exactly the linear part and maintain some qualitative behaviors (including stability) of the exact solutions. We refer the readers to the works of Higham et al. [11], Hochbruck and Ostermann [12], Bouc and Pardoux [13], Maset and Zennaro [14], Pardoux [15], Altman [16], and Yuan [17].

In the real world, we are often faced with two kinds of uncertainties, that is, probabilistic uncertainty and model uncertainty. Model uncertainty is due to incomplete information, vague data, imprecise probability, and so forth. Many researchers investigate the characteristics of model uncertainty in order to provide a framework for theory and applications. Hou et al. [18] developed the stability analysis for discrete-time uncertain time-delay systems governed by an infinite-state Markov chain. They derived some sufficient conditions for the exponential stability in mean square with conditioning via linear matrix inequalities and established the equivalence among asymptotical stability in mean square, stochastic stability, and exponential stability in mean square. Yi et al. [19] investigated the stabilization of a class of chaotic systems with both model uncertainty and external disturbance. They developed a new UDE-based control method by combining the dynamic feedback control method and the uncertainty and disturbance estimator- (UDE-) based control method. Peng [20] gave the notions of G-expectation and G-Brownian motion on sublinear expectation space which provide the new perspective for the stochastic calculus under model uncertainty, which has aroused great interest. Based on the fundamental theory of time-consistent G-expectation, Peng [21] introduced the so-called G-Gaussian distribution and the G-Brownian motion and used them to set up the associated Itô integral. Since then, many works have been carried out on the stochastic calculus with respect to the G-Brownian motion. One can see the works of Denis et al. [22], Dolinsky et al. [23], Faizullah et al. [24–26], Ullah and Faizullah [27], Fadina and Herzberg [28], Hu and Peng [29], Li et al. [30], Ren et al. [31], Yin and Ren [32], Yang and Zhu [33], and Zhang and Chen [34]. It can be found that most researches focus on linear and nonlinear SDEs, SDDEs, and NSDDEs with G-Brownian motion; there are a few numerical analysis results for semilinear SDDEs with G-Brownian motion (G-SLSDDEs). To fill this gap, we investigate the numerical solutions of G-SLSDDEs and give some results in the present paper.

The remainder of the paper is organized as follows. In Section 2, we introduce some basic notations, assumptions, and properties which will be used in this paper. We devote Section 3 to presenting the existence and uniqueness of the exact solution for the G-SLSDDEs. In Section 4, we are in a position to establish the exponential integrators for G-SLDDEs and we derive conclusions about the convergence and the exponential mean-square stability of the exact solution for the G-SLSDDEs. We are also successful in establishing the exponential Euler method and proving that the exponential Euler method can preserve the mean-square stability of the exact solution. In Section 5, numerical simulations are presented to demonstrate the theoretical results; and a short conclusion is given in Section 6.

2. Preliminaries

We recall some basic definitions and notions from [20, 21]. Let Ω be a (nonempty) basic space and let $\mathscr{H}$ be a linear space of real-valued functions defined on Ω such that the constant $C \in \mathscr{H}$ and if $X_1, X_2, \ldots, X_n \in \mathscr{H}$, then $\varphi(X_1, X_2, \ldots, X_n) \in \mathscr{H}$ for each $\varphi \in \mathbb{C}_{l.\mathrm{Lip}}(R^n)$, where $\mathbb{C}_{l.\mathrm{Lip}}(R^n)$ is the space of linear function φ defined as follows:

$$\mathbb{C}_{l.\mathrm{Lip}}(R^n) = \{\varphi\colon R^n \longrightarrow R \mid \exists C \in R^+, m \in N, \text{s.t. } |\varphi(x) - \varphi(y)| \le C(1 + |x|^m + |y|^m)|x - y|\}, \quad (1)$$

for $x, y \in R^n$. We consider that $\mathscr{H}$ is the space of random variables.

Definition 1 (see [20]; sublinear expectation). A function $\widehat{E}\colon \mathscr{H} \longrightarrow R$ is called sublinear expectation if, $\forall X, Y \in \mathscr{H}, C \in R$, and $\lambda \ge 0$, it satisfies the following properties:

(1) **Monotonicity**: if $X \ge Y$, then $\widehat{E}[X] \ge \widehat{E}[Y]$

(2) **Constant preserving**: $\widehat{E}[C] = C$

(3) **Subadditivity**: $\widehat{E}[X + Y] \ge \widehat{E}[X] + \widehat{E}[Y]$ or $\widehat{E}[X] - \widehat{E}[Y] \ge \widehat{E}[X - Y]$

(4) **Positive homogeneity**: $\widehat{E}[\lambda X] = \lambda \widehat{E}[X]$

Also, if $\widehat{E}[X] = \widehat{E}[-X] = 0$, then $\widehat{E}[C + \lambda X + \ Y] = C + \widehat{E}[Y]$.

The triple $(\Omega, \mathscr{H}, \widehat{E})$ is called a sublinear expectation space. If (7) and (18) are satisfied, the aforementioned function $\widehat{E}\colon \mathscr{H} \longrightarrow R$ is called a nonlinear expectation and the triple $(\Omega, \mathscr{H}, \widehat{E})$ is relevantly called a nonlinear expectation space. For the details of the notions of G-normal distribution, G-expectation, G-conditional expectation, and G-Brownian motion, see Chapters 2 and 3 of Peng [21].

Definition 2 (see [21]; G-normal distribution). Let $(\Omega, \mathscr{H}, \widehat{E})$ be a sublinear expectation space, and $X \in \mathscr{H}$ with

$$\overline{\sigma}^2 = \widehat{E}\left[X^2\right], \underline{\sigma}^2 = -\widehat{E}\left[-X^2\right]. \quad (2)$$

Then, X is said to be G-distributed or $N(0; [\overline{\sigma}^2, \underline{\sigma}^2])$-distributed, if, $\forall a, b \ge 0$, we have

$$aX + bY \sim \sqrt{a^2 + b^2} X, \quad (3)$$

where each $Y \in \mathscr{H}$ is an independent copy of X; that is, $Y \sim X$, and Y is independent from X.

Definition 3 (see [21]; G-Brownian motion). The sublinear expectation $\widehat{E}\colon L_{ip}(\Omega) \longrightarrow R$ is called a G-expectation if the corresponding canonical process $\{\omega(t)\}_{t \ge 0}$ on the sublinear expectation space $(\Omega, L_{ip}(\Omega), \widehat{E})$ is a G-Brownian motion; that is, for $0 \le s < t$, it satisfies the following conditions:

(1) $\omega(0) = 0$

(2) The increment $\omega(t + s) - \omega(s)$ is independent of $\omega(t_1), \omega(t_2), \ldots, \omega(t_n)$ for each $n \in Z^+$, and $0 \le t_1 \le \cdots \le t_n \le t$

(3) The increment $\omega(t + s) - \omega(s)$ is $N(0, [s\underline{\sigma}^2, s\overline{\sigma}^2])$−distributed

For each fixed $T \ge 0$, set $\Omega_T = \{\omega_{\cdot \wedge T}\colon \omega \in \Omega\}$.

$$L_{ip}(\Omega_T) = \left\{\varphi(\omega(t_1), \omega(t_2), \ldots, \omega(t_n))\colon t_1, t_2, \ldots, t_n \in [0, T], \varphi \in \mathbb{C}_{l.Lip}(R^n), n \in N\right\}, \quad (4)$$

where $L_{ip}(\Omega_t) \subseteq L_{ip}(\Omega_T)$ for $t \leq T$ and $L_{ip}(\Omega_T) = \cup_{m=1}^{\infty} L_{ip}(\Omega_m)$. The completion of $L_{ip}(\Omega)$ under the norm $|X|_p = (\widehat{E}[|X|^p])^{1/p}$ for $p \geq 1$ is denoted by $L_G^p(\Omega)$ and $L_G^p(\Omega_t) \subset L_G^p(\Omega_T) \subset L_G^p(\Omega)$ for $0 \leq t \leq T < \infty$. An important proposition about conditional G-expectation $\widehat{E}[\cdot|\Omega_t]$, $t \in [0, T]$ is presented as follows.

Proposition 1 (see [21]). *ie conditional expectation* $\widehat{E}[\cdot|\Omega_t]$, $t \in [0, T]$ *holds for each* $X, Y \in L_G^1(\Omega_t)$:

(1) If $X \geq Y$, *then* $\widehat{E}[X|\Omega_t] \geq \widehat{E}[Y|\Omega_t]$

(2) $\widehat{E}[\eta|\Omega_t] = \eta$, *for each* $t \in [0, \infty)$ *and* $\eta \in L_G^1(\Omega_t)$

(3) $\widehat{E}[X] - \widehat{E}[Y] \leq \widehat{E}[X - Y|\Omega_t]$

(4) $\widehat{E}[\eta X|\Omega_t] = \eta^+ \widehat{E}[X|\Omega_t] + \eta^- \widehat{E}[-X|\Omega_t]$ *for each bounded* $\eta \in L_G^1(\Omega_t)$

(5) $\widehat{E}[\widehat{E}[X|\Omega_t]|\Omega_s] = \widehat{E}[X|\Omega_{t\wedge s}]$; *in particular,* $\widehat{E}[\widehat{E}[X|\Omega_t]] = \widehat{E}[X]$

For $T \in R^+$, *a partition* π_T *of* $[0, T]$ *is a finite-ordered subset* $\pi = \{t_1, \ldots, t_N\}$ *such that* $0 = t_0 < t_1 < \cdots < t_N = T$. *Let* $p \geq 1$ *be fixed. Define*

$$M_G^{p,0}(0,T) = \eta(t) = \eta(t,\omega) = \sum_{j=1}^{N} \xi_{j-1}(\omega) I_{[t_{j-1},t_j)}(t); \xi_{j-1} \in L_G^p\left(\mathscr{F}_{t_{j-1}}\right), t_{j-1} < t_j, \quad j = 1, 2, \ldots, N, t_0 = 0, t_N = T, N \geq 1, \tag{5}$$

where $L_G^p(\mathscr{F}_t) = \left\{\xi \in L_G^1(\mathscr{F}_t) \colon \widehat{E}(|\xi|^p) < \infty\right\}$. For $\eta(t) = \sum_{j=0}^{N-1} \xi_j(\omega) I_{[t_j,t_{j+1})}(t) \in M_G^{p,0}(0,T)$, set

$$\widehat{E}_T(\eta) := \frac{1}{T}\int_0^T \widehat{E}[\eta(t)]\mathrm{d}t = \frac{1}{T}\sum_{j=0}^{N-1} \widehat{E}\left(\xi_j\right)\left(t_{j+1} - t_j\right). \tag{6}$$

Then, $\widehat{E}_T \colon M_G^{p,0}(0,T) \mapsto R$ forms a sublinear expectation. For each $p \geq 1$, $M_G^p(0,T)$ denotes the completion of $M_G^{p,0}(0,T)$ under the norm

$$\|\eta\|_{M_G^{p,0}(0,T)} := \frac{1}{T}\left(\int_0^T \widehat{E}[|\eta(t)|^p]\mathrm{d}t\right)^{1/p}. \tag{7}$$

Definition 4 (see [20]; Itô integral) For each $\eta \in M_G^{p,0}(0,T)$ with the form

$$\eta(t) = \sum_{j=0}^{N-1} \xi_j(\omega) I_{[t_j,t_{j+1})}(t), \tag{8}$$

the Itô integral of G-Brownian motion is defined as

$$I(\eta) = \int_0^T \eta(s)\mathrm{d}\omega^a(s) := \sum_{j=0}^{N-1} \xi_j\left(\omega^a\left(t_{j+1}\right) - \omega^a\left(t_j\right)\right). \tag{9}$$

The mapping $I \colon M_G^{2,0}(0,T) \mapsto L_G^2(\mathscr{F}_T)$ can be continuously extended to $I \colon M_G^2(0,T) \mapsto L_G^2(\mathscr{F}_T)$. For each $\eta \in M_G^2(0,T)$, the stochastic integral is defined by

$$\int_0^T \eta(s)\mathrm{d}\omega^a(s) := I(\eta). \tag{10}$$

Definition 5 (see [20]; quadratic variation process). Let $\pi_t^N = \{t_0^N, t_1^N, \ldots, t_N^N\}$, let $N = 1, 2, \ldots$, be a sequence of partitions of $[0, t]$, and set $\mu(\pi_t^N) = \max_{1\leq i\leq N}|t_i^N - t_{i-1}^N|$. An increasing process $\{\langle\omega^a\rangle(t), t \geq 0\}$ with $\langle\omega^a\rangle(0) = 0$, defined by

$$\langle\omega^a\rangle(t) = \lim_{\mu(\pi_t^N)} \sum_{k=0}^{N-1}\left(\omega^a\left(t_{k+1}^N\right) - \omega^a\left(t_k^N\right)\right)^2 = \left(\omega^2(t)\right)^2 - 2\int_0^t \omega^a(s)\mathrm{d}\omega^a(s), \tag{11}$$

is called the quadratic variation process of G-Brownian motion. Furthermore, for each fixed $s \geq 0$,

$$\langle\omega^a\rangle(t+s) - \langle\omega^a\rangle(s) = \left\langle\left(\omega^s\right)^a\right\rangle(t), \tag{12}$$

where $\omega^s(t) = \omega(s+t) - \omega(s)$, $t \geq 0$, $(\omega^s)^a(t) = (a, \omega^s(t))$, and $(x, y) = \sum_{i=1}^d x_i y_i$ for $x, y \in R^d$.

Let $\{\omega\}_{t\geq 0}$ be a 1-dimensional G-Brownian motion with $G(a) = 1/2\widehat{E}[a\omega_1^2] = 1/2(\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-)$, where $\widehat{E}[\omega_1^2] = \overline{\sigma}^2, -\widehat{E}[-\omega_1^2] = \underline{\sigma}^2$, and $0 \leq \underline{\sigma} \leq \overline{\sigma}$. As for the definitions of Itô integral and the quadratic variation process with G-Brownian motion, we present two propositions in the following.

Proposition 2 (see [35]). *For any* $0 \leq t \leq T < \infty$, *let the quadratic variation of G-Brownian motion* $\langle\omega\rangle_t = \int_0^t u_s ds$. *Then, one obtains*

$$\underline{\sigma}^2(T - t) \leq \langle\omega\rangle_T - \langle\omega\rangle_t \leq \overline{\sigma}^2(T - t)\ q.s. \tag{13}$$

Moreover, $\underline{\sigma}^2 dt \leq \langle\omega\rangle_t = u_t dt \leq \overline{\sigma}^2 dt\ q.s.$; *and* $G(a) \geq 1/2au_t\ q.s.$

Proposition 3 (see [21]). *For any* $0 \leq t \leq T < \infty$,

(1) $\widehat{E}[\int_0^T \eta_t d\omega_t] = 0$; $\widehat{E}|\int_0^T \eta_t d\langle\omega\rangle_t| \leq \overline{\sigma}^2 \widehat{E}[\int_0^T |\eta_t| dt]$, $\forall \eta_t \in M_G^1(0,T)$

(2) $\widehat{E}[(\int_0^T \eta_t d\omega_t)^2] = \widehat{E}[\int_0^T \eta_t^2 d\langle\omega\rangle_t]$, $\forall \eta_t \in M_G^2(0,T)$

(3) $\widehat{E}[\int_0^T |\eta_t|^p dt] \le \int_0^T \widehat{E}[|\eta_t|^p]dt$, $\forall \eta_t \in M_G^2(0,T)$, $p \le 1$

Lemma 1 (see [36]).

(1) Let $p \ge 1$, $a, \widehat{a} \in R^d$, $\eta \in M_G^p([0,T])$, and $0 \le s \le t \le T$. Then,

$$\widehat{E}\left(\sup_{s\le u\le t}\left|\int_s^u \eta_r \mathrm{d}\langle w^a, w^{\widehat{a}}\rangle_r\right|^p\right) \le \left(\frac{\sigma_{(a+\widehat{a})(a+\widehat{a})^T} + \sigma_{(a-\widehat{a})(a-\widehat{a})^T}}{4}\right)^p \cdot |t-s|^{p-1}\int_s^t \widehat{E}\left[|\eta_t|^p\right]\mathrm{d}u. \tag{14}$$

(2) Let $p \ge 2$, $\eta \in M_G^p([0,T])$, and $0 \le s \le t \le T$. Then,

$$\widehat{E}\left(\sup_{s\le u\le t}\left|\int_s^u \eta_r \mathrm{d}w_r\right|^p\right) \le C_p \widehat{E}\left(\left|\int_s^t |\eta_u|^2 \mathrm{d}u\right|^{p/2}\right) \le C_p |t-s|^{p/2-1}\int_s^t \widehat{E}\left[|\eta_u|^p\right]\mathrm{d}u, \tag{15}$$

where C_p is a positive constant independent of η.

Lemma 2 (see [37]). *Doob martingale inequality). Let $\{X(t)\}_{t\ge 0}$ be an R^n-value G-martingale and let $[a,b]$ be a bounded interval of R^+; if $p > 1$, $X(t) \in L_G^p(\Omega, R^n)$, then*

$$\widehat{E}\left(\sup_{a\le t\le b}|X(t)|^p\right) \le \left(\frac{p}{p-1}\right)^p \widehat{E}(X(b))^p. \tag{16}$$

3. Existence and Uniqueness Theorem

Throughout this paper, unless otherwise specified, $a \wedge b$ and $a \vee b$ denote $\max\{a,b\}$ and $\min\{a,b\}$, respectively. Let $\tau > 0$ and $BC([-\tau,0];R^d)$ denote the family of all bound continuous R^d-valued functions φ defined on $[-\tau,0]$ with norm $\|\varphi\| = \sup_{-\tau\le\theta\le 0}|\varphi(\theta)|$. For $p > 0$ and $t \ge 0$, let $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\ge 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}\}_{t\ge 0}$ satisfying the usual conditions. Let ω_t be a one-dimensional G-Brownian motion with $\omega_t \sim N(0, [\underline{\sigma}^2 t, \overline{\sigma}^2 t])$.

In this paper, we focus on the G-SLSDDEs with the following form:

$$\begin{aligned}
&\mathrm{d}y(t) = (Ay(t) + f(t, y(t), y(t-\tau))\mathrm{d}t + g(t, y(t), y(t-\tau))\mathrm{d}\omega_t + h(t, y(t), y(t-\tau))\mathrm{d}\langle\omega\rangle_t, \quad t \in [0,T],\\
&y(t) = y_0, \quad t \in [-\tau, 0],
\end{aligned} \tag{17}$$

with initial condition

$$y_0 = \xi = \left\{\varphi\colon \varphi \text{ is } \mathcal{F}_0 - \text{measurable, } BC\left([-\tau,0];R^d\right) - \text{value random variable, such that } \varphi \in M_G^2\left([-\tau,0];R^d\right)\right\}, \tag{18}$$

where $A \in R^{d\times d}$, $f, g, h\colon R \times R^d \times R^d \longrightarrow R^d$, as well as $f, g, h \in M_G^2([0,T];R^d)$, $\forall T \ge 0$, and $\{\langle\omega\rangle(t), t \ge 0\}$ is the quadratic variation process of G-Brownian motion $\{\langle\omega\rangle(t), t \ge 0\}$.

The G-SLSDDEs (7) with initial value (18) can be written in the following equivalent form:

$$y(t) = y_0 + \int_0^t [Ay(u) + f(u, y(u), y(u-\tau))]\mathrm{d}u + \int_0^t g(u, y(u), y(u-\tau))\mathrm{d}\omega(u) + \int_0^t h(u, y(u), y(u-\tau))\mathrm{d}\langle\omega\rangle(u). \tag{19}$$

To ensure the existence and uniqueness of the solutions, we assume that f, g, and h satisfy the following Lipschitz condition:

(H1) Lipschitz condition: There exists a positive constant L_1, for all $x_1, y_1, x_2, y_2 \in R^d$, $t \geq 0$, such that

$$|f(t,x_1,y_1) - f(t,x_2,y_2)|^2 \vee |g(t,x_1,y_1) - g(t,x_2,y_2)|^2 \vee |g(t,x_1,y_1) - g(t,x_2,y_2)|^2 \leq L_1\left(|x_1 - x_2|^2 + |y_1 - y_2|^2\right). \quad (20)$$

According to (20), it is easy to obtain the following linear growth condition:

$$|f(t,x,y)|^2 \leq 2|f(t,x,y) - f(t,0,0)|^2 + 2|f(t,0,0)|^2 \leq L_1\left(|x|^2 + |y|^2\right) + 2|f(t,0,0)|^2 \leq G_1\left(1 + |x|^2 + |y|^2\right). \quad (21)$$

Similarly,

$$|g(t,x,y)| \leq G_1\left(1 + |x|^2 + |y|^2\right), |h(t,x,y)| \leq G_1\left(1 + |x|^2 + |y|^2\right), \quad (22)$$

where $G_1 = L_1 \vee 2|f(t,0,0)|^2 \vee 2|g(t,0,0)|^2$.

Theorem 1. *Assume that f, g, and h satisfy the Lipschitz condition (20), and there is a nonnegative constant $\lambda > 0$ such that*

$$\langle z, f(t,z,0)\rangle \leq -\lambda|z|^2, \quad (23)$$

hold for all $t \geq 0$, $z \in R^d$. Then there exists a unique solution $y = \{y(t), t \in [0,\infty)\}$ of equation (7) and the solution belongs to $M_G^2(R^+; R^d)$. Moreover, the solution $y(t)$ satisfies the condition

$$\widehat{E}\left(\sup_{0\leq s\leq T}|y_n(s)|^2\right) \leq \left[\left(4 + 8|A|^2 + 16G_1 + 32\overline{\sigma}^2G_1 + 8G_1\overline{\sigma}^4T\right)\widehat{E}|y_0|^2 + 8G_1T^2 + 16\overline{\sigma}^2G_1T + 4G_1\overline{\sigma}^4T^2\right] \cdot e^{\left(8|A|^2+16G_1+32\overline{\sigma}^2G_1+8G_1\overline{\sigma}^4T\right)}. \quad (24)$$

Proof. The proof is rather technical, and we shall divide the whole proof into several steps. ☐

Step 1. Boundedness. For every $T > 0$ and integer $n \geq 1$, define the stopping time

$$\tau_n = T \wedge \inf\{t \in [0,T]: |y(t)| \geq n\}. \quad (25)$$

Clearly, $\tau_n \uparrow T$ almost surely. Set $y_n(t) = y(t\wedge\tau_n)$ for $t \in [0,T]$. Then $y_n(t)$ satisfies the equation

$$\begin{aligned} y_n(t) = y_0 &+ \int_0^t\left(Ay_n(s) + f(s,y_n(s),y_n(s-\tau))I_{[[0,\tau_n]]}(s)\mathrm{d}s\right. \\ &+ \int_0^t g(s,y_n(s),y_n(s-\tau))I_{[[0,\tau_n]]}(s)\mathrm{d}\omega(s) + \int_0^t h(s,y_n(s),y_n(s-\tau))I_{[[0,\tau_n]]}(s)\mathrm{d}\langle\omega\rangle(s). \end{aligned} \quad (26)$$

Using the elementary inequality $|a+b+c+d|^2 \leq 4(|a|^2 + |b|^2 + |c|^2 + |d|^2)$, we have

$$\begin{aligned} |y_n(t)|^2 \leq 4|y_0|^2 &+ 4\left|\int_0^t\left(Ay_n(s) + f(s,y_n(s),y_n(s-\tau))I_{[[0,\tau_n]]}(s)\right)\mathrm{d}s\right|^2 \\ &+ 4\left|\int_0^t g(s,y_n(s),y_n(s-\tau))I_{[[0,\tau_n]]}(s)\mathrm{d}\omega(s)\right|^2 + 4\left|\int_0^t h(s,y_n(s),y_n(s-\tau))I_{[[0,\tau_n]]}(s)\mathrm{d}\langle\omega\rangle(s)\right|^2. \end{aligned} \quad (27)$$

Therefore,

$$\begin{aligned}\sup_{0\le s\le t}\left|y_n(s)\right|^2 \le 4\left|y_0\right|^2 &+ 4\sup_{0\le s\le t}\left|\int_0^t \left(Ay_n(s) + f\left(s, y_n(s), y_n(s-\tau)\right)I_{[[0,\tau_n]]}(s)\right)\mathrm{d}s\right|^2 \\ &+ 4\sup_{0\le s\le t}\left|\int_0^t g\left(s, y_n(s), y_n(s-\tau)I_{[[0,\tau_n]]}(s)\mathrm{d}\omega(s)\right)\right|^2 \\ &+ 4\sup_{0\le s\le t}\left|\int_0^t h\left(s, y_n(s), y_n(s-\tau)\right)I_{[[0,\tau_n]]}(s)\mathrm{d}\langle\omega\rangle(s)\right|^2.\end{aligned} \tag{28}$$

Taking the G-expectation on both sides, it gives

$$\begin{aligned}\widehat{E}\left(\sup_{0\le s\le t}\left|y_n(s)\right|^2\right) \le 4\widehat{E}\left(\left|y_0\right|^2 + \sup_{0\le s\le t}\left|\int_0^t \left(Ay_n(s) + f\left(s, y_n(s), y_n(s-\tau)\right)I_{[[0,\tau_n]]}(s)\right)\mathrm{d}s\right|^2\right) \\ + 4\widehat{E}\left(\sup_{0\le s\le t}\left|\int_0^t g\left(s, y_n(s), y_n(s-\tau)\right)I_{[[0,\tau_n]]}(s)\mathrm{d}\omega(s)\right|^2\right) \\ + 4\widehat{E}\left(\sup_{0\le s\le t}\left|\int_0^t h\left(s, y_n(s), y_n(s-\tau)\right)I_{[[0,\tau_n]]}(s)\mathrm{d}\langle\omega\rangle(s)\right|^2\right).\end{aligned} \tag{29}$$

It follows from Hö lder's inequality and the linear growth condition (21) that

$$\begin{aligned}&\widehat{E}\left(\sup_{0\le s\le t}\left|\int_0^t \left(Ay_n(s) + f\left(s, y_n(s), y_n(s-\tau)\right)I_{[[0,\tau_n]]}(s)\right)\mathrm{d}s\right|^2\right) \\ &\le T\int_0^t \left(2|A|^2\widehat{E}\left|y_n(s)\right|^2 + 2G_1\left(1 + \widehat{E}\left|y_n(s)\right|^2 + \widehat{E}\left|y_n(s-\tau)\right|^2\right)\mathrm{d}s\right) \\ &\le 2G_1T^2 + \left(2|A|^2 + 4G_1\right)\int_0^t \widehat{E}\left(\sup_{-\tau\le v\le s}\left|y_n(v)\right|^2\right)\mathrm{d}s.\end{aligned} \tag{30}$$

Together with Proposition 3 and Doob martingale inequality, one can get

$$\begin{aligned}&\widehat{E}\left(\sup_{0\le s\le t}\left|\int_0^t g\left(s, y_n(s), y_n(s-\tau)\right)I_{[[0,\tau_n]]}(s)\mathrm{d}\omega(s)\right|^2\right) \\ &\le 4\overline{\sigma}^2\widehat{E}\left(\int_0^t \left|g\left(s, y_n(s), y_n(s-\tau)\right)\right|^2\mathrm{d}s\right) \\ &\le 4\overline{\sigma}^2G_1\widehat{E}\left(\int_0^t \left(1 + \left|y_n(s)\right|^2 + \left|y_n(s-\tau)\right|^2\right)\mathrm{d}s\right) \\ &\le 4\overline{\sigma}^2G_1T + 8\overline{\sigma}^2G_1\int_0^t \widehat{E}\left(\sup_{-\tau\le v\le s}\left|y_n(v)\right|^2\right)\mathrm{d}s.\end{aligned} \tag{31}$$

Similarly, it follows from Proposition 2 and Hö lder's inequality that one gains

$$
\begin{aligned}
&\widehat{E}\left(\sup_{0\le s\le t}\left|\int_0^t h(s, y_n(s), y_n(s-\tau)) I_{[[[0,\tau_n]]]}(s)\mathrm{d}\langle\omega\rangle(s)\right|^2\right)\\
&\quad\le \widehat{E}\left(\sup_{0\le s\le t}\left|\int_0^t h(s, y_n(s), y_n(s-\tau)) u_s \mathrm{d}s\right|^2\right)\\
&\quad\le \widehat{E}\left[\int_0^t u_s^2 \mathrm{d}s\left(\int_0^t \left|h(s, y_n(s), y_n(s-\tau))\right|^2 \mathrm{d}s\right)\right]\\
&\quad\le G_1\overline{\sigma}^4T^2 + 2G_1\overline{\sigma}^4 T\int_0^t \widehat{E}\left(\sup_{-\tau\le v\le s}\left|y_n(v)\right|^2\right)\mathrm{d}s.
\end{aligned}
\tag{32}
$$

Substituting (30)–(32) into (29), one obtains

$$
\begin{aligned}
\widehat{E}\left(\sup_{0\le s\le t}\left|y_n(s)\right|^2\right) &\le 4\widehat{E}\left|y_0\right|^2 + 8G_1T^2 + \left(8|A|^2 + 16G_1\right)\int_0^t \widehat{E}\left(\sup_{-\tau\le v\le s}\left|y_n(v)\right|^2\right)\mathrm{d}s\\
&\quad + 16\overline{\sigma}^2 G_1 T + 32\overline{\sigma}^2 G_1\int_0^t \widehat{E}\left(\sup_{-\tau\le v\le s}\left|y_n(v)\right|^2\right)\mathrm{d}s + 4G_1\overline{\sigma}^4T^2 + 8G_1\overline{\sigma}^4 T\int_0^t \widehat{E}\left(\sup_{-\tau\le v\le s}\left|y_n(v)\right|^2\right)\mathrm{d}s\\
&= 4\widehat{E}\left|y_0\right|^2 + 8G_1T^2 + 16\overline{\sigma}^2G_1T + 4G_1\overline{\sigma}^4T^2 + \left(8|A|^2 + 16G_1 + 32\overline{\sigma}^2G_1 + 8G_1\overline{\sigma}^4T\right)\int_0^t \widehat{E}\left(\sup_{-\tau\le v\le s}\left|y_n(v)\right|^2\right)\mathrm{d}s\\
&\le \left(4 + 8|A|^2 + 16G_1 + 32\overline{\sigma}^2G_1 + 8G_1\overline{\sigma}^4T\right)\widehat{E}\left|y_0\right|^2 + 8G_1T^2 + 16\overline{\sigma}^2G_1T\\
&\quad + 4G_1\overline{\sigma}^4T^2 + \left(8|A|^2 + 16G_1 + 32\overline{\sigma}^2G_1 + 8G_1\overline{\sigma}^4T\right)\int_0^t \widehat{E}\left(\sup_{0\le v\le s}\left|y_n(v)\right|^2\right)\mathrm{d}s.
\end{aligned}
\tag{33}
$$

An application of the Gronwall inequality yields that

$$
\widehat{E}\left(\sup_{0\le s\le t}\left|y_n(s)\right|^2\right) \le \left[\left(4 + 8|A|^2 + 16G_1 + 32\overline{\sigma}^2G_1 + 8G_1\overline{\sigma}^4T\right)\widehat{E}\left|y_0\right|^2 + 8G_1T^2 + 16\overline{\sigma}^2G_1T + 4G_1\overline{\sigma}^4T^2\right]\cdot e^{\left(8|A|^2+16G_1+32\overline{\sigma}^2G_1+8G_1\overline{\sigma}^4T\right)}.
\tag{34}
$$

Thus,

$$
\widehat{E}\left(\sup_{0\le s\le \tau_n}\left|y_n(s)\right|^2\right) \le \left[\left(4 + 8|A|^2 + 16G_1 + 32\overline{\sigma}^2G_1 + 8G_1\overline{\sigma}^4T\right)\widehat{E}\left|y_0\right|^2 + 8G_1T^2 + 16\overline{\sigma}^2G_1T + 4G_1\overline{\sigma}^4T^2\right]\cdot e^{\left(8|A|^2+16G_1+32\overline{\sigma}^2G_1+8G_1\overline{\sigma}^4T\right)}.
\tag{35}
$$

Finally, the required inequality (24) follows by letting $n \longrightarrow \infty$.

Step 2. Uniqueness. Let $y(t)$ and $\overline{y}(t)$ be two solutions of Equation (7). By the proof of boundedness, we know that both belong to $M_G^2(R^+;R^d)$, and

$$\begin{aligned} y(t)-\overline{y}(t) = {} & \int_0^t [A(y(s)-\overline{y}(s)) + f(s,y(s),y(s-\tau)) - f(s,\overline{y}(s),\overline{y}(s-\tau))]\mathrm{d}s \\ & + \int_0^t [g(s,y(s),y(s-\tau)) - g(s,\overline{y}(s),\overline{y}(s-\tau))]\mathrm{d}\omega(s) \\ & + \int_0^t [h(s,y(s),y(s-\tau)) - h(s,\overline{y}(s),\overline{y}(s-\tau))]\mathrm{d}\langle\omega\rangle(s). \end{aligned} \tag{36}$$

In the same way as the proof of the boundedness, we have

$$\widehat{E}\left(\sup_{0\le s\le t}|y(s)-\overline{y}(s)|^2\right) \le \left(8|A|^2 + 16L_1 + 32\overline{\sigma}^2 L_1 + 8L_1\overline{\sigma}^4 T\right)\int_0^t \widehat{E}\left(\sup_{0\le v\le s}|y(v)-\overline{y}(v)|^2\right)\mathrm{d}s. \tag{37}$$

An application of the Gronwall inequality yields that

$$\widehat{E}\left(\sup_{0\le t\le T}|y(t)-\overline{y}(t)|^2\right) = 0. \tag{38}$$

Hence, $y(t)=\overline{y}(t)$ for all $0\le t\le T$ almost surely. The uniqueness has been proved.

Step 3. Existence: Set $y_0(t)=y_0$, for $0\le t\le T$. Let $y_0^n=\varphi$, $n=1,2,\ldots$, and for $t\in[0,T]$, define the Picard iterations

$$\begin{aligned} y_n(t) = y_0 & + \int_0^t (Ay_{n-1}(s) + f(s,y_{n-1}(s),y_{n-1}(s-\tau)))\mathrm{d}s \\ & + \int_0^t g(s,y_{n-1}(s),y_{n-1}(s-\tau))\mathrm{d}\omega(s) + \int_0^t h(s,y_{n-1}(s),y_{n-1}(s-\tau))\mathrm{d}\langle\omega\rangle(s). \end{aligned} \tag{39}$$

Obviously, $y_0(t)\in M_G^2([0,T];R^d)$. Moreover, it is easy to see by induction that $y_n(t)\in M_G^2([0,T];R^d)$; in fact,

$$\begin{aligned} |y_n(t)|^2 \le 4E|y_0|^2 & + 4\left|\int_0^t (Ay_{n-1}(s) + f(s,y_{n-1}(s),y_{n-1}(s-\tau)))\mathrm{d}s\right|^2 \\ & + 4\left|\int_0^t g(s,y_{n-1}(s),y_{n-1}(s-\tau))\mathrm{d}\omega(s)\right|^2 + 4\left|\int_0^t h(s,y_{n-1}(s),y_{n-1}(s-\tau))\mathrm{d}\langle\omega\rangle(s)\right|^2. \end{aligned} \tag{40}$$

Taking the G-expectation on both sides, it follows from Propositions 2 and 3 and Doob martingale inequality (taking $p=2$) that

$$\begin{aligned} \widehat{E}\left(|y_n(t)|^2\right) \le 4\widehat{E}|y_0|^2 & + 4t\widehat{E}\int_0^t |Ay_{n-1}(s) + f(s,y_{n-1}(s),y_{n-1}(s-\tau))|^2\mathrm{d}s \\ & + 16\overline{\sigma}^2\widehat{E}\int_0^t |g(s,y_{n-1}(s),y_{n-1}(s-\tau))|^2\mathrm{d}s + 4\overline{\sigma}^4 t\widehat{E}\int_0^t |h(s,y_{n-1}(s),y_{n-1}(s-\tau))|^2\mathrm{d}s. \end{aligned} \tag{41}$$

Using inequality (21) and the Cauchy inequality, we obtain

$$\begin{aligned}
\widehat{E}\left(\left|y_n(t)\right|^2\right) &\le 4\widehat{E}\left|y_0\right|^2 + 8t\widehat{E}\int_0^t \left|Ay_{n-1}(s)\right|^2 + \left|f\left(s, y_{n-1}(s), y_{n-1}(s-\tau)\right)\right|^2 \mathrm{d}s + 16\bar{\sigma}^2\widehat{E}\int_0^t \left|g\left(s, y_{n-1}(s), y_{n-1}(s-\tau)\right)\right|^2 \mathrm{d}s \\
&\quad + 4\bar{\sigma}^4 t\widehat{E}\int_0^t \left|h\left(s, y_{n-1}(s), y_{n-1}(s-\tau)\right)\right|^2 \mathrm{d}s \le 4E\left|y_0\right|^2 \\
&\quad + 8T\widehat{E}\int_0^t \left(G_1 + \left(|A|^2 + G_1\right)\left|y_{n-1}(s)\right|^2 + G_1\left|y_{n-1}(s-\tau)\right|^2\right)\mathrm{d}s + 16\bar{\sigma}^2\widehat{E}\int_0^t \left(G_1 + G_1\left|y_{n-1}(s)\right|^2 + G_1\left|y_{n-1}(s-\tau)\right|^2\right)\mathrm{d}s \\
&\quad + 4\bar{\sigma}^4 T\widehat{E}\int_0^t \left(G_1 + G_1\left|y_{n-1}(s)\right|^2 + G_1\left|y_{n-1}(s-\tau)\right|^2\right)\mathrm{d}s \\
&\le C_1 + \left(8T + 16\bar{\sigma}^2 + 4\bar{\sigma}^4 T\right)\widehat{E}\int_0^t \left(\left(|A|^2 + G_1\right)\left|y_{n-1}(s)\right|^2 + G_1\left|y_{n-1}(s-\tau)\right|^2\right)\mathrm{d}s \\
&\le C_1 + \left(8T + 16\bar{\sigma}^2 + 4\bar{\sigma}^4 T\right)\int_0^t \left(\left(|A|^2 + 2G_1\right)\widehat{E}\sup_{0\le r\le s}\left|y_{n-1}(r)\right|^2 + G_1\widehat{E}\left|y_0\right|^2\right)\mathrm{d}s \\
&\le C_2 + 4\left(|A|^2 + 2G_1\right)\left(2T + 4\bar{\sigma}^2 + \bar{\sigma}^4 T\right)\int_0^t \widehat{E}\sup_{0\le r\le s}\left|y_{n-1}(r)\right|^2 \mathrm{d}s \\
&\le C_2 + 4\left(|A|^2 + 2G_1\right)\left(2T + 4\bar{\sigma}^2 + \bar{\sigma}^4 T\right)\int_0^t \widehat{E}\left|y_{n-1}(s)\right|^2 \mathrm{d}s.
\end{aligned} \tag{42}$$

where $C_1 = 4\widehat{E}|y_0|^2 + (8T + 16\bar{\sigma}^2 + 4\bar{\sigma}^4 T)\mathrm{T}\mathrm{G}_1$ and $C_2 = C_1 + (8T + 16\bar{\sigma}^2 + 4\bar{\sigma}^4 T)G_1 T\widehat{E}|y_0|^2$.

Hence, for any $k \ge 1$, we derive that

$$\max_{1\le n\le k}\widehat{E}\left|y_n(t)\right|^2\Big) \le C_2 + 4\left(|A|^2 + 2G_1\right)\left(2T + 4\bar{\sigma}^2 + \bar{\sigma}^4 T\right)\int_0^t \max_{1\le n\le k}\widehat{E}\left|y_{n-1}(s)\right|^2 \mathrm{d}s. \tag{43}$$

Note that

$$\max_{1\le n\le k}\widehat{E}\left|y_{n-1}(s)\right|^2 \le \max\left\{\widehat{E}\left|y_0\right|^2, \max_{1\le n\le k}\widehat{E}\left|y_n(s)\right|^2\right\} \le \widehat{E}\left|y_0\right|^2 + \max_{1\le n\le k}\widehat{E}\left|y_n(s)\right|^2. \tag{44}$$

We have

$$\begin{aligned}
&\max_{1\le n\le k}\widehat{E}\left|y_n(t)\right|^2\Big) \\
&\quad \le C_2 + 4\left(|A|^2 + 2G_1\right)\left(2T + 4\bar{\sigma}^2 + \bar{\sigma}^4 T\right)\int_0^t \left(\widehat{E}\left|y_0\right|^2 + \max_{1\le n\le k}\widehat{E}\left|y_n(s)\right|^2\right)\mathrm{d}s \\
&\quad \le C_3 + 4\left(|A|^2 + 2G_1\right)\left(2T + 4\bar{\sigma}^2 + \bar{\sigma}^4 T\right)\int_0^t \max_{1\le n\le k}\widehat{E}\left|y_n(s)\right|^2 \mathrm{d}s,
\end{aligned} \tag{45}$$

where $C_3 = C_2 + 4T(|A|^2 + 2G_1)(2T + 4\bar{\sigma}^2 + \bar{\sigma}^4 T)\widehat{E}|y_0|^2$. It follows from the Gronwall inequality that

$$\max_{1\le n\le k}\widehat{E}\left|y_n(t)\right|^2 \le C_3 e^{4\left(|A|^2+2G_1\right)\left(2T+4\bar{\sigma}^2+\bar{\sigma}^4 T\right)}. \tag{46}$$

Since k is arbitrary, we can have

$$\widehat{E}\left|y_n(t)\right|^2\Big) \le C_3 e^{4\left(|A|^2+2G_1\right)\left(2T+4\bar{\sigma}^2+\bar{\sigma}^4 T\right)}, \quad \text{for all } 0 \le t \le T, n \ge 1. \tag{47}$$

This shows that $y_n(t) \in M_G^2([0,T]; R^d)$.

Using the elementary inequality, we obtain

$$\begin{aligned} \left|y_1(t) - y_0(t)\right|^2 &= \left|y_1(t) - y_0\right|^2 \\ &\le 3\left|\int_0^t (Ay_0(s) + f(s, y_0, y_0(s-\tau)))\mathrm{d}s\right|^2 + 3\left|\int_0^t g(s, y_0(s), y_0(s-\tau))\mathrm{d}\omega(s)\right|^2 \\ &\quad + 3\left|\int_0^t h(s, y_0(s), y_0(s-\tau))\mathrm{d}\langle\omega\rangle(s)\right|^2. \end{aligned} \tag{48}$$

Taking the G-expectation on (48), it follows from Propositions 2 and 3 and the properties of G-Itô integral that

$$\begin{aligned} &\widehat{E}\left|y_1(t) - y_0(t)\right|^2 \\ &\le 3\widehat{E}\left|\int_0^t (Ay_0(s) + f(s, y_0(s), y_0(s-\tau)))\mathrm{d}s\right|^2 + 3\widehat{E}\left|\int_0^t g(s, y_0(s), y_0(s-\tau))\mathrm{d}\omega(s)\right|^2 + 3\widehat{E}\left|\int_0^t h(s, y_0(s), y_0(s-\tau))d\langle\omega\rangle(s)\right|^2 \\ &\le 3t\widehat{E}\int_0^t \left|Ay_0(s) + f(s, y_0(s), y_0(s-\tau))\right|^2\mathrm{d}s + 12\overline{\sigma}^2\int_0^t \left|g(s, y_0(s), y_0(s-\tau))\right|^2\mathrm{d}s + 3\overline{\sigma}^4 t\int_0^t \left|h(s, y_0(s), y_0(s-\tau))\right|^2\mathrm{d}s \\ &\le \left(6t + 12\overline{\sigma}^2 + 3\overline{\sigma}^4 t\right)\widehat{E}\int_0^t \left(G_1 + \left(2G_1 + |A|^2\right)\left|y_0(s)\right|^2\right)\mathrm{d}s \\ &\le \left(6t + 12\overline{\sigma}^2 + 3\overline{\sigma}^4 t\right)G_1 t + \left(6t + 12\overline{\sigma}^2 + 3\overline{\sigma}^4 t\right)\left(2G_1 + |A|^2\right)t\widehat{E}\left|y_0\right|^2, \end{aligned} \tag{49}$$

that is,

$$\max_{0\le s\le t}\widehat{E}\left|y_1(s) - y_0(s)\right|^2 \le \left(6t + 12\overline{\sigma}^2 + 3\overline{\sigma}^4 t\right)G_1 t + \left(6t + 12\overline{\sigma}^2 + 3\overline{\sigma}^4 t\right)\left(2G_1 + |A|^2\right)t\widehat{E}\left|y_0\right|^2. \tag{50}$$

Taking $t = T$,

$$\max_{0\le s\le t}\widehat{E}\left|y_1(s) - y_0(s)\right|^2 \le \left(6t + 12\overline{\sigma}^2 + 3\overline{\sigma}^4 t\right)G_1 T + \left(6t + 12\overline{\sigma}^2 + 3\overline{\sigma}^4 t\right)\left(2G_1 + |A|^2\right)T\widehat{E}\left|y_0\right|^2 := \overline{C}. \tag{51}$$

By the same ways as earlier, we compute

$$\begin{aligned} \widehat{E}\left|y_2(t) - y_1(t)\right|^2 &\le 6t\widehat{E}\int_0^t \left(\left|Ay_1(s) - Ay_0(s)\right|^2 + \left|f(s, y_1, y_1(s-\tau)) - f(s, y_0(s), y_0(s-\tau))\right|^2\right)\mathrm{d}s \\ &\quad + 12\overline{\sigma}^2\int_0^t \left|g(s, y_1(s), y_1(s-\tau)) - g(s, y_0(s), y_0(s-\tau))\right|^2\mathrm{d}s \\ &\quad + 3\overline{\sigma}^4 t\int_0^t \left|h(s, y_1(s), y_1(s-\tau)) - h(s, y_0(s), y_0(s-\tau))\right|^2\mathrm{d}s, \end{aligned} \tag{52}$$

and thus, we derive that

$$\begin{aligned}
&\widehat{E}\left(\sup_{0\le s\le t}\left|y_2(s)-y_1(s)\right|^2\right)\\
&\le 6t\widehat{E}\int_0^t\left(|A|^2|y_1(s)-y_0(s)|^2+L_1\left(\left|y_1(s)-y_0(s)\right|^2+\left|y_1(s-\tau)-y_0(s-\tau)\right|^2\right)\right)\mathrm{d}s\\
&\quad+12\overline{\sigma}^2\int_0^t L_1\left(|y_1(s)-y_0(s)|^2+\left|y_1(s-\tau)-y_0(s-\tau)\right|^2\right)\mathrm{d}s+3\overline{\sigma}^4 t\int_0^t L_1\left(\left|y_1(s)-y_0(s)\right|^2+\left|y_1(s-\tau)-y_0(s-\tau)\right|^2\right)\mathrm{d}s\\
&\le 3\left(2T+4\overline{\sigma}^2+\overline{\sigma}^4T\right)\widehat{E}\int_0^t\left(|A|^2+2L_1\right)\left(\sup_{0\le r\le s}\left|y_1(r)-y_0(r)\right|^2\right)\mathrm{d}s\\
&\le 3\left(2T+4\overline{\sigma}^2+\overline{\sigma}^4T\right)\left(|A|^2+2L_1\right)t\overline{C}.
\end{aligned}\tag{53}$$

Similarly,

$$\begin{aligned}
&\widehat{E}\left(\sup_{0\le s\le t}\left|y_3(s)-y_2(s)\right|^2\right)\\
&\le 3\left(2T+4\overline{\sigma}^2+\overline{\sigma}^4T\right)\widehat{E}\int_0^t\left(|A|^2+2L_1\right)\left(\sup_{0\le r\le s}\left|y_2(r)-y_1(r)\right|^2\right)\mathrm{d}s\\
&\le 3\left(2T+4\overline{\sigma}^2+\overline{\sigma}^4T\right)\left(|A|^2+2L_1\right)\widehat{E}\int_0^t 3\left(2T+4\overline{\sigma}^2+\overline{\sigma}^4T\right)\left(|A|^2+2L_1\right)s\overline{C}\mathrm{d}s\\
&\le\frac{3^2\left(2T+4\overline{\sigma}^2+\overline{\sigma}^4T\right)^2\left(|A|^2+2L_1\right)^2t^2\overline{C}}{2!},
\end{aligned}\tag{54}$$

continuing this process to find that

$$\begin{aligned}
&\widehat{E}\left(\sup_{0\le s\le t}\left|y_4(s)-y_3(s)\right|^2\right)\\
&\le 3\left(2T+4\overline{\sigma}^2+\overline{\sigma}^4T\right)\widehat{E}\int_0^t\left(|A|^2+2L_1\right)\left(\sup_{0\le s\le t}\left|y_3(r)-y_2(r)\right|^2\right)\mathrm{d}s\\
&\le 3\left(2T+4\overline{\sigma}^2+\overline{\sigma}^4T\right)\left(|A|^2+2L_1\right)\widehat{E}\int_0^t 3^2\left(2T+4\overline{\sigma}^2+\overline{\sigma}^4T\right)^2\left(|A|^2+2L_1\right)^2s^2\overline{C}\mathrm{d}s\\
&\le\frac{3^3\left(2T+4\overline{\sigma}^2+\overline{\sigma}^4T\right)^3\left(|A|^2+2L_1\right)^3t^3\overline{C}}{3!}.
\end{aligned}\tag{55}$$

Now, we claim that, for all $n\ge 0$,

$$\widehat{E}\left(\sup_{0\le s\le t}\left|y_{n+1}(s)-y_n(s)\right|^2\right)\le\frac{\overline{C}(Mt)^n}{n!},\quad 0\le t\le T,\tag{56}$$

where $M=3(2T+4\overline{\sigma}^2+\overline{\sigma}^4T)(|A|^2+2L_1)$.

When $n=0,1,2,3$, inequality (56) holds. We suppose that (56) holds for some n, now to check (56) for $(n+1)$. In fact,

$$\begin{aligned}&\widehat{E}\left(\sup_{0\le s\le t}\left|y_{n+2}(s)-y_{n+1}(s)\right|^2\right)\\&\le 3\left(2T+4\overline{\sigma}^2+\overline{\sigma}^4T\right)\widehat{E}\int_0^t\left(|A|^2+2L_1\right)\left(\sup_{0\le r\le s}\left|y_{n+1}(r)-y_n(r)\right|^2\right)\mathrm{d}s\\&=M\int_0^t\widehat{E}\left(|A|^2+2L_1\right)\left(\sup_{0\le r\le s}\left|y_{n+1}(r)-y_n(r)\right|^2\right)\mathrm{d}s.\end{aligned}\tag{57}$$

By induction and (56),

$$\begin{aligned}\widehat{E}\left(\sup_{0\le s\le t}\left|y_{n+2}(s)-y_{n+1}(s)\right|^2\right)&\le M\int_0^t\frac{\overline{C}[M(s)]^n}{n!}\mathrm{d}s\\&=\frac{\overline{C}(Mt)^{n+1}}{(n+1)!}.\end{aligned}\tag{58}$$

It is easy to see that (56) holds for $n+1$. Therefore, by induction, (56) holds for all $n\ge 0$.

Next, to verify that $\{y_n(t), n\ge 0\}$ converge to $y(t)$ at the sense of L_G^2 and probability 1 on $M_G^2((0,T];R^d)$, moreover, $y(t)$ is the solution of (7). For (56), taking $t=T$,

$$\widehat{E}\left(\sup_{0\le t\le T}\left|y_{n+1}(t)-y_n(t)\right|^2\right)\le\frac{\overline{C}(\mathrm{MT})^n}{n!}.\tag{59}$$

By Chebyshev's inequality,

$$\widehat{E}\left\{\sup_{0\le t\le T}\left|y_{n+1}(t)-y_n(t)\right|>\frac{1}{2^n}\right\}\le\frac{\overline{C}(2\mathrm{MT})^n}{n!}.\tag{60}$$

Since $\sum_{n=0}^{\infty}\overline{C}(2\mathrm{MT})^n/n!<\infty$, the Borel-Cantelli lemma yields that there exists a positive integer n_0 such that

$$\sup_{0\le t\le T}\left|y_{n+1}(t)-y_n(t)\right|\le\frac{1}{2^n}\text{ whenever } n\ge n_0.\tag{61}$$

It follows that, with probability 1, the partial sums,

$$y_0(t)+\sum_{i=0}^{n-1}\left[y_{i+1}(t)-y_i(t)\right]=y_n(t),\tag{62}$$

are convergent uniformly in $t\in[0,T]$. Denote the limit by $y(t)$. Clearly, $y(t)$ is continuous and $\mathscr{F}_t$-adapted. On the other hand, one sees from (56) that, for every t, $\{y_n(t)\}_{n\ge1}$ is a Cauchy sequence in L_G^2 as well. Hence, we also have $y_n(t)\longrightarrow y(t)$ in L_G^2; that is,

$$\widehat{E}\left|y_n(t)-y(t)\right|^2\longrightarrow 0,\quad n\longrightarrow\infty.\tag{63}$$

Letting $n\longrightarrow\infty$ in (47) then yields that

$$\widehat{E}|y(t)|^2)\le C_3e^{4\left(|A|^2+2G_1\right)\left(2T+4\overline{\sigma}^2+\overline{\sigma}^4T\right)},\quad\text{for all } 0\le t\le T.\tag{64}$$

Therefore, $y(\cdot)\in M^2([0,T];R^d)$. It remains to show that $y(t)$ satisfies the G-SLSDDE in integral form. Note that

$$\begin{aligned}&\widehat{E}\left|\int_0^t\left(Ay_n(s)+f(s,y_n(s),y_n(s-\tau))\right)\mathrm{d}s-\int_0^t\left(Ay(s)+f(s,y(s),y(s-\tau))\right)\mathrm{d}s\right|^2\\&\quad+\widehat{E}\left|\int_0^t g(s,y_n(s),y_n(s-\tau))\mathrm{d}\omega(s)-\int_0^t g(s,y(s),y(s-\tau))\mathrm{d}\omega(s)\right|^2\\&\quad+\widehat{E}\left|\int_0^t h(s,y_n(s),y_n(s-\tau))\mathrm{d}\langle\omega\rangle(s)-\int_0^t h(s,y(s),y(s-\tau))\mathrm{d}\langle\omega\rangle(s)\right|^2\\&\le t\widehat{E}\int_0^t\left|Ay_n(s)+f(s,y_n(s),y_n(s-\tau))-Ay(s)-f(s,y(s),y(s-\tau))\right|^2\mathrm{d}s\\&\quad+4\overline{\sigma}^2\widehat{E}\int_0^t\left|g(s,y_n(s),y_n(s-\tau))-g(s,y(s),y(s-\tau))\right|^2\mathrm{d}s+\overline{\sigma}^4t\widehat{E}\int_0^t\left|h(s,y_n(s),y_n(s-\tau))-h(s,y(s),y(s-\tau))\right|^2\mathrm{d}s\\&\le L_1\left[\left(1+\overline{\sigma}^4\right)T+4\overline{\sigma}^2\right]\left(\frac{|A|^2}{L_1}+2\right)\int_0^t\widehat{E}\sup_{0\le r\le s}\left|y_n(r)-y(r)\right|^2\mathrm{d}s\\&\le L_1\left[\left(1+\overline{\sigma}^4\right)T+4\overline{\sigma}^2\right]\left(\frac{|A|^2}{L_1}+2\right)\int_0^t\widehat{E}\left|y_n(s)-y(s)\right|^2\mathrm{d}s.\end{aligned}\tag{65}$$

Noting that sequence $\{y_n(t), n \geq 0\}$ uniformly converges on $(0, T]$, it means that, for any given $\varepsilon > 0$, there exists a positive integer n_0 such that as $n \geq n_0$, for any $(0, T]$, one then deduces that $\widehat{E}|y_n(t) - y(t)|^2 < \varepsilon$. Furthermore, we obtain

$$\begin{aligned}
&\widehat{E}\left|\int_0^t (Ay_n(s) + f(s, y_n(s), y_n(s-\tau)))\mathrm{d}s - \int_0^t (Ay(s) + f(s, y(s), y(s-\tau)))\mathrm{d}s\right|^2 \\
&\quad + \widehat{E}\left|\int_0^t g(s, y_n(s), y_n(s-\tau))\mathrm{d}\omega(s) - \int_0^t g(s, y(s), y(s-\tau))\mathrm{d}\omega(s)\right|^2 \\
&\quad + \widehat{E}\left|\int_0^t h(s, y_n(s), y_n(s-\tau))\mathrm{d}\langle\omega\rangle(s) - \int_0^t h(s, y(s), y(s-\tau))\mathrm{d}\langle\omega\rangle(s)\right|^2 \\
&\leq L_1\left[\left(1 + \overline{\sigma}^4\right)T + 4\overline{\sigma}^2\right]\left(\frac{|A|^2}{L_1} + 2\right)\int_0^t \widehat{E}|y_n(s) - y(s)|^2\mathrm{d}s \leq L_1\left[\left(1 + \overline{\sigma}^4\right)T + 4\overline{\sigma}^2\right]\left(\frac{|A|^2}{L_1} + 2\right)T\varepsilon.
\end{aligned} \tag{66}$$

In other words, for $t \in [0, T]$, we have

$$\begin{aligned}
\int_0^t (Ay_n(s) + f(s, y_n(s), y_n(s-\tau)))\mathrm{d}s &\longrightarrow \int_0^t (Ay(s) + f(s, y(s), y(s-\tau)))\mathrm{d}s \text{ in } L_G^2, \\
\int_0^t g(s, y_n(s), y_n(s-\tau))\mathrm{d}s &\longrightarrow \int_0^t g(s, y(s), y(s-\tau))\mathrm{d}\omega(s) \text{ in } L_G^2, \\
\int_0^t h(s, y_n(s), y_n(s-\tau))\mathrm{d}s &\longrightarrow \int_0^t h(s, y(s), y(s-\tau))\mathrm{d}\langle\omega\rangle(s) \text{ in } L_G^2.
\end{aligned} \tag{67}$$

For $0 \leq t \leq T$, taking limits on both sides of (47), we obtain

$$\begin{aligned}
\lim_{n\longrightarrow\infty} y_n(t) = y(0) &+ \lim_{n\longrightarrow\infty}\int_0^t (Ay_n(s) + f(s, y_n(s), y_n(s-\tau)))\mathrm{d}s + \lim_{n\longrightarrow\infty}\int_0^t g(s, y_n(s), y_n(s-\tau))\mathrm{d}\omega(s) \\
&+ \lim_{n\longrightarrow\infty}\int_0^t h(s, y_n(s), y_n(s-\tau))\mathrm{d}\langle\omega\rangle(s),
\end{aligned} \tag{68}$$

that is,

$$y(t) = y(0) + \int_0^t (Ay(s) + f(s, y(s), y(s-\tau)))\mathrm{d}s + \int_0^t g(s, y(s), y(s-\tau))\mathrm{d}\omega(s) + \int_0^t h(s, y(s), y(s-\tau))\mathrm{d}\langle\omega\rangle(s), \quad 0 \leq t \leq T. \tag{69}$$

The aformentioned expression demonstrates that $y(t)$ is the solution of (7). So far, the existence of Theorem 1 is completed.

4. Exponential Euler Method and the Numerical Analysis

We can now give the numerical solution for the G-SLSDDE (7). In order to avoid the storage problem and improve the convergence order, we introduce the exponential Euler scheme.

To formulate the grid, let t_0 be an arbitrary but fixed positive number, define $t_n = t_0 + nh$, $n = 0, 1, 2, \ldots, h > 0$ is the step size which satisfies $h = \tau/m$, m is a positive integer, and the solution of (19) at $t_{n+1} = t_n + h$ has the form

$$
\begin{aligned}
y(t_{n+1}) = {} & e^{Ah_n} y(t_n) + \int_{t_n}^{t_{n+1}} e^{As} f(t_n, y(s), y(s-\tau))\mathrm{d}s + \int_{t_n}^{t_{n+1}} e^{As} g(t_n, y(s), y(s-\tau))\mathrm{d}\omega(s) \\
& + \int_{t_n}^{t_{n+1}} e^{As} h(t_n, y(s), y(s-\tau))\mathrm{d}\langle\omega\rangle(s).
\end{aligned} \tag{70}
$$

Approximating the functions f, g, and h with the integral by left rectangle formula at the known values $f(t_n, y(t_n), y(t_n - \tau))$, $g(t_n, y(t_n), y(t_n - \tau))$, and $h(t_n, y(t_n), y(t_n - \tau))$ only leads to the exponential Euler method:

$$
y_{n+1} = e^{Ah_n} y_n + e^{Ah_n} f(t_n, y_n, y_{n-m}) h_n + e^{Ah_n} g(t_n, y_n, y_{n-m})\Delta\omega_n + e^{Ah_n} h(t_n, y_n, y_{n-m})\Delta\langle\omega\rangle_n. \tag{71}
$$

where y_n is an approximation to $y(t_n)$ and y_{n-m} is an approximation to $y(t_n - \tau)$. The increments $\Delta\omega_n = \omega(t_n) - \omega(t_{n-1})$ and $\Delta\langle\omega\rangle_n = \langle\omega\rangle(t_n) - \langle\omega\rangle(t_{n-1})$. We assume that y_n is $\mathcal{F}_{t_n}$–measurable at the mesh point t_n.

Let $\underline{u} = \lfloor u/h \rfloor h$ with $\lfloor y \rfloor$ denoting the largest integer which is smaller than y, and $z(t) = \sum_{k=0}^{\infty} y_k I_{[t_k, t_{k+1})}(t)$ with $I_{[t_k, t_{k+1})}(t)$ denoting the indicator function on $[t_k, t_{k+1})$; namely, $I_{[t_k, t_{k+1})} = 1$ if $t \in [t_k, t_{k+1}]$ or 0 otherwise. Then, we can extend the discrete exponential Euler method (71) to the continuous one as follows:

$$
\begin{aligned}
\widehat{y(t)} = {} & e^{At} y_0 + \int_0^t e^{A(t-\underline{u})} f(\underline{u}, z(u), (u-\tau))\mathrm{d}u + \int_0^t e^{A(t-\underline{u})} g(\underline{u}, z(u), z(u-\tau))\mathrm{d}\omega(u) \\
& + \int_0^t e^{A(t-\underline{u})} h(\underline{u}, z(u), z(u-\tau))\mathrm{d}\langle\omega\rangle(u).
\end{aligned} \tag{72}
$$

It is not difficult to see that $\widehat{y(t_n)} = y_n$ for $n = 0, 1, 2, \cdots$, that is, the continuous extension $\widehat{y(t)}$ coincides with the discrete numerical solutions at the mesh points.

4.1. Convergence of the Exponential Euler Method. Now, we show the convergence of the exponential Euler approximate solution to the exact solution for system (7). To obtain this result, we first need to show the following several definitions.

Definition 6.

(1) The global error for exponential Euler method is defined as follows:

$$
\varepsilon_n = y(t_n) - y_n. \tag{73}
$$

(2) For fixed $T < \infty$, the approximation y_n is convergent in the mean-square sense on mesh points, with strong order p if

$$
\max_{1 \le n \le N} \left(\widehat{E}|\varepsilon_n|^2\right)^{1/2} \le Ch^p, \text{ as } h \longrightarrow 0, \tag{74}
$$

where C is a positive constant.

We will use the following lemma to analyze the convergence of the exponential Euler method.

Lemma 3. *Under the Lipschitz condition (20), there exists a constant $\widetilde{C}_1 > 0$, such that the analytic solution $y(t)$ of*

Equation (7) and the numerical solution $\widehat{y(t)}$ of continuous exponential Euler approximate (72) satisfy the following inequality:

$$\widehat{E}\sup_{0\le t\le T}|y(t)|^2 \vee \widehat{E}\sup_{0\le t\le T}|\widehat{y(t)}|^2 \le \widetilde{C}_1, \tag{75}$$

where $\widetilde{C}_1$ is independent of h, and

$$\widetilde{C}_1 = \max\left\{4e^{2|A|T}\left(\widehat{E}|y_0|^2 + \left(T + 4\overline{\sigma}^2 + \overline{\sigma}T\right)TG_1\right)e^{8e^{2|A|T}\left(T+4\overline{\sigma}^2+\overline{\sigma}T\right)TG_1}, C_3 e^{4\left(|A|^2+2G_1\right)\left(2T+4\overline{\sigma}^2+\overline{\sigma}T\right)}\right\}. \tag{76}$$

Proof. The detailed proofs are similar to the proofs in [17] which are omitted here. □

Theorem 2. *Under the Lipschitz condition (20), there exists a nonnegative constant G_2, such that, for any $s, t \in [0, T]$ and $t > s$,*

$$|f(t,x,y) - f(s,x,y)|^2 \vee |g(t,x,y) - g(s,x,y)|^2 \vee |h(t,x,y) - g(s,x,y)|^2 \le G_2\left(1 + |x|^2 + |y|^2\right)|s-t|. \tag{77}$$

The numerical solution produced by the continuous exponential Euler method (72) converges to the analytical solution of equation (7) in MS sense with the strong order 1/2; that is, there exists a positive constant C, such that

$$E\sup_{0\le t\le T}|y(t) - \widehat{y(t)}|^2 \le Ch, \text{ as } h \longrightarrow 0. \tag{78}$$

Proof. Taking the difference between (19) and (72) and squaring its both sides, we can get

$$\begin{aligned}
&|y(t) - \widehat{y(t)}|^2 \\
&= \bigg|\int_0^t e^{A(t-u)} f(u, y(u), y(u-\tau))\mathrm{d}u - \int_0^t e^{A(t-\underline{u})} f(\underline{u}, z(u), z(u-\tau))\mathrm{d}u + \int_0^t e^{A(t-u)} g(u, y(u), y(u-\tau))\mathrm{d}\omega(u) \\
&\quad - \int_0^t e^{A(t-\underline{u})} g(\underline{u}, z(u), z(u-\tau))\mathrm{d}\omega(u-\tau)\bigg|^2 \\
&\quad + \int_0^t e^{A(t-u)} h(u, y(u), y(u-\tau))\mathrm{d}\omega(u) - \int_0^t e^{A(t-\underline{u})} h(\underline{u}, z(u), z(u-\tau))\mathrm{d}\langle\omega\rangle(u-\tau)|^2.
\end{aligned} \tag{79}$$

With the help of the elementary inequality and Propositions 2 and 3, we can get

$$\begin{aligned}
&|y(t)-\widehat{y(t)}|^2 \\
&\le 3\left|\int_0^t e^{A(t-u)} f(u, y(u), y(u-\tau))\mathrm{d}u - \int_0^t e^{A(t-\underline{u})} f(\underline{u}, z(u), z(u-\tau))\mathrm{d}u\right|^2 \\
&\quad + 3\left|\int_0^t e^{A(t-u)} g(u, y(u), y(u-\tau))\mathrm{d}\omega(u) - \int_0^t e^{A(t-\underline{u})} g(\underline{u}, z(u), z(u-\tau))\mathrm{d}\omega(u)\right|^2 \\
&\quad + 3\left|\int_0^t e^{A(t-u)} h(u, y(u), y(u-\tau))\mathrm{d}\langle\omega\rangle(u) - \int_0^t e^{A(t-\underline{u})} h(\underline{u}, z(u), z(u-\tau))\mathrm{d}\langle\omega\rangle(u)\right|^2 \\
&\le 9T\int_0^t |e^{A(t-u)} f(u, y(u), y(u-\tau)) - e^{A(t-\underline{u})} f(u, y(u), y(u-\tau))|^2\mathrm{d}u \\
&\quad + 9T\int_0^t \left|e^{A(t-\underline{u})} f(u, y(u), y(u-\tau)) - e^{A(t-\underline{u})} f(\underline{u}, y(u), y(u-\tau))\right|^2\mathrm{d}u \\
&\quad + 9T\int_0^t \left|e^{A(t-\underline{u})} f(\underline{u}, y(u), y(u-\tau)) - e^{A(t-\underline{u})} f(\underline{u}, z(u), z(u-\tau))\right|^2\mathrm{d}u \\
&\quad + 36\overline{\sigma}^2\int_0^t \left|e^{A(t-u)} g(u, y(u), y(u-\tau)) - e^{A(t-\underline{u})} g(u, y(u), y(u-\tau))\right|^2\mathrm{d}u \\
&\quad + 36\overline{\sigma}^2\int_0^t \left|e^{A(t-\underline{u})} g(u, y(u), y(u-\tau)) - e^{A(t-\underline{u})} g(\underline{u}, y(u), y(u-\tau))\right|^2\mathrm{d}u \\
&\quad + 36\overline{\sigma}^2\int_0^t \left|e^{A(t-\underline{u})} g(\underline{u}, y(u), y(u-\tau)) - e^{A(t-\underline{u})} g(\underline{u}, z(u), z(u-\tau))\right|^2\mathrm{d}u \\
&\quad + 9\overline{\sigma}^4T\int_0^t \left|e^{A(t-u)} h(u, y(u), y(u-\tau)) - e^{A(t-\underline{u})} h(u, y(u), y(u-\tau))\right|^2\mathrm{d}u \\
&\quad + 9\overline{\sigma}^4T\int_0^t \left|e^{A(t-\underline{u})} h(u, y(u), y(u-\tau)) - e^{A(t-\underline{u})} h(\underline{u}, y(u), y(u-\tau))\right|^2\mathrm{d}u \\
&\quad + 9\overline{\sigma}^4T\int_0^t \left|e^{A(t-\underline{u})} h(\underline{u}, y(u), y(u-\tau)) - e^{A(t-\underline{u})} h(\underline{u}, z(u), z(u-\tau))\right|^2\mathrm{d}u.
\end{aligned} \tag{80}$$

Taking G-expectation on both sides of (80), for arbitrary $0 \le t_1 \le T$, we arrive at

$$\begin{aligned}
&\widehat{E}\left(\sup_{0\le t\le t_1} |y(t) - \widehat{y(t)}|^2\right)\\
&\le 9T\widehat{E}\left(\sup_{0\le t\le t_1}\int_0^t \left|e^{A(t-u)} f(u, y(u), y(u-\tau)) - e^{A(t-\underline{u})} f(u, y(u), y(u-\tau))\right|^2 \mathrm{d}u\right)\\
&\quad + 9T\widehat{E}\left(\sup_{0\le t\le t_1}\int_0^t \left|e^{A(t-\underline{u})} f(u, y(u), y(u-\tau)) - e^{A(t-\underline{u})} f(\underline{u}, y(u), y(u-\tau))\right|^2 \mathrm{d}u\right)\\
&\quad + 9T\widehat{E}\left(\sup_{0\le t\le t_1}\int_0^t \left|e^{A(t-\underline{u})} f(\underline{u}, y(u), y(u-\tau)) - e^{A(t-\underline{u})} f(\underline{u}, z(u), z(u-\tau))\right|^2 \mathrm{d}u\right)\\
&\quad + 36\overline{\sigma}^2\widehat{E}\left(\sup_{0\le t\le t_1}\int_0^t \left|e^{A(t-u)} g(u, y(u), y(u-\tau)) - e^{A(t-\underline{u})} g(u, y(u), y(u-\tau))\right|^2 \mathrm{d}u\right)\\
&\quad + 36\overline{\sigma}^2\widehat{E}\left(\sup_{0\le t\le t_1}\int_0^t \left|e^{A(t-\underline{u})} g(u, y(u), y(u-\tau)) - e^{A(t-\underline{u})} g(\underline{u}, y(u), y(u-\tau))\right|^2 \mathrm{d}u\right)\\
&\quad + 36\overline{\sigma}^2\widehat{E}\left(\sup_{0\le t\le t_1}\int_0^t \left|e^{A(t-\underline{u})} g(\underline{u}, y(u), y(u-\tau)) - e^{A(t-\underline{u})} g(\underline{u}, z(u), z(u-\tau))\right|^2 \mathrm{d}u\right)\\
&\quad + 9\overline{\sigma}^4T\widehat{E}\left(\sup_{0\le t\le t_1}\int_0^t \left|e^{A(t-u)} h(u, y(u), y(u-\tau)) - e^{A(t-\underline{u})} h(u, y(u), y(u-\tau))\right|^2 \mathrm{d}u\right)\\
&\quad + 9\overline{\sigma}^4T\widehat{E}\left(\sup_{0\le t\le t_1}\int_0^t \left|e^{A(t-\underline{u})} h(u, y(u), y(u-\tau)) - e^{A(t-\underline{u})} h(\underline{u}, y(u), y(u-\tau))\right|^2 \mathrm{d}u\right)\\
&\quad + 9\overline{\sigma}^4T\widehat{E}\left(\sup_{0\le t\le t_1}\int_0^t \left|e^{A(t-\underline{u})} h(\underline{u}, y(u), y(u-\tau)) - e^{A(t-\underline{u})} h(\underline{u}, z(u), z(u-\tau))\right|^2 \mathrm{d}u\right)\\
&\le 9TJ_1(t) + 9TJ_2(t) + 9TJ_3(t) + 36\overline{\sigma}^2 J_4(t) + 36\overline{\sigma}^2 J_5(t)\\
&\quad + 36\overline{\sigma}^2 J_6(t) + 9\overline{\sigma}^4 TJ_7(t) + 9\overline{\sigma}^4 TJ_8(t) + 9\overline{\sigma}^4 TJ_9(t),
\end{aligned} \tag{81}$$

where

$$\begin{aligned}
J_1(t) &= \widehat{E}\left(\sup_{0\le t\le t_1}\int_0^t \left|e^{A(t-u)} f(u, y(u), y(u-\tau)) - e^{A(t-\underline{u})} f(u, y(u), y(u-\tau))\right|^2 \mathrm{d}u\right),\\
J_2(t) &= \widehat{E}\left(\sup_{0\le t\le t_1}\int_0^t \left|e^{A(t-\underline{u})} f(u, y(u), y(u-\tau)) - e^{A(t-\underline{u})} f(\underline{u}, y(u), y(u-\tau))\right|^2 \mathrm{d}u\right),\\
J_3(t) &= \widehat{E}\left(\sup_{0\le t\le t_1}\int_0^t \left|e^{A(t-\underline{u})} f(\underline{u}, y(u), y(u-\tau)) - e^{A(t-\underline{u})} f(\underline{u}, z(u), z(u-\tau))\right|^2 \mathrm{d}u\right),\\
J_4(t) &= \widehat{E}\left(\sup_{0\le t\le t_1}\int_0^t \left|e^{A(t-u)} g(u, y(u), y(u-\tau)) - e^{A(t-\underline{u})} g(u, y(u), y(u-\tau))\right|^2 \mathrm{d}u\right),\\
J_5(t) &= \widehat{E}\left(\sup_{0\le t\le t_1}\int_0^t \left|e^{A(t-\underline{u})} g(u, y(u), y(u-\tau)) - e^{A(t-\underline{u})} g(\underline{u}, y(u), y(u-\tau))\right|^2 \mathrm{d}u\right),\\
J_6(t) &= \widehat{E}\left(\sup_{0\le t\le t_1}\int_0^t \left|e^{A(t-\underline{u})} g(\underline{u}, y(u), y(u-\tau)) - e^{A(t-\underline{u})} g(\underline{u}, z(u), z(u-\tau))\right|^2 \mathrm{d}u\right),\\
J_7(t) &= \widehat{E}\left(\sup_{0\le t\le t_1}\int_0^t \left|e^{A(t-u)} h(u, y(u), y(u-\tau)) - e^{A(t-\underline{u})} h(u, y(u), y(u-\tau))\right|^2 \mathrm{d}u\right),\\
J_8(t) &= \widehat{E}\left(\sup_{0\le t\le t_1}\int_0^t \left|e^{A(t-\underline{u})} h(u, y(u), y(u-\tau)) - e^{A(t-\underline{u})} h(\underline{u}, y(u), y(u-\tau))\right|^2 \mathrm{d}u\right),\\
J_9(t) &= \widehat{E}\left(\sup_{0\le t\le t_1}\int_0^t \left|e^{A(t-\underline{u})} h(\underline{u}, y(u), y(u-\tau)) - e^{A(t-\underline{u})} h(\underline{u}, z(u), z(u-\tau))\right|^2 \mathrm{d}u\right).
\end{aligned} \tag{82}$$

To estimate (81), we need to estimate $J_i(t)$ $(i = 1, 2, 3, 4, 5, 7, 8, 9)$.

For $J_1(t)$, we have that

$$J_1(t) \le \widehat{E}\left(\sup_{0\le t\le t_1}\int_0^t \left|e^{A(t-u)} - e^{A(t-\underline{u})}\right|^2 |f(u, y(u), y(u-\tau))|^2 \mathrm{d}u\right)$$
$$= \widehat{E}\left(\sup_{0\le t\le t_1}\int_0^t \left|e^{A(t-u)}\right|^2 \left|I - e^{A(u-\underline{u})}\right|^2 |f(u, y(u), y(u-\tau))|^2 \mathrm{d}u\right). \tag{83}$$

Due to the linear growth condition (21) and $|e^{Ah} - I| \le |A| h e^{|A|T}$, inequality (83) gives

$$\widehat{E}\left(\sup_{0\le t\le t_1}\int_0^t \left|e^{A(t-u)} - e^{A(t-\underline{u})}\right|^2 |f(u, y(u), y(u-\tau))|^2 \mathrm{d}u\right)$$
$$\le G_1 \widehat{E}\left(\sup_{0\le t\le t_1}\int_0^t \left|e^{2A(t-u)}\right|\left|I - e^{A(u-\underline{u})}\right|^2 \left(1 + |y(u)|^2 + |y(u-\tau)|^2\right)\mathrm{d}u\right)$$
$$\le G_1 e^{2|A|T} T |A|^2 h^2 e^{2|A|T}\left(1 + 2\widetilde{C}_1\right). \tag{84}$$

Using (78) and Lipschitz condition (20), it follows from $J_2(t)$ and $J_3(t)$ that

$$J_2(t) \le G_2 e^{2|A|T} \widehat{E}\left(\sup_{0\le t\le t_1}\int_0^t \left(1 + |y(u)|^2 + |y(u-\tau)|^2\right)|u - \underline{u}| \mathrm{d}u\right)$$
$$\le G_2 e^{2|A|T} T h \left(1 + 2\widetilde{C}_1\right), \tag{85}$$

$$J_3(t) \le L_1 e^{2|A|T} \widehat{E}\left(\sup_{0\le t\le t_1}\int_0^t \left(|y(u) - z(u)|^2 + |y(u-\tau) - z(u-\tau)|^2\right)\mathrm{d}u\right)$$
$$\le 2L_1 e^{2|A|T} T \int_0^{t_1} \widehat{E}\left(\sup_{0\le u\le t} |y(u) - \widehat{y(u)}|^2\right)\mathrm{d}t. \tag{86}$$

Similar to estimates $J_1(t)$, $J_2(t)$, and $J_3(t)$, one obtains

$$J_4(t) \le G_1 e^{2|A|T} T |A|^2 h^2 e^{2|A|T}\left(1 + 2\widetilde{C}_1\right), \tag{87}$$

$$J_5(t) \le G_2 e^{2|A|T} T h \left(1 + 2\widetilde{C}_1\right), \tag{88}$$

$$J_6(t) \le 2L_1 e^{2|A|T} T \int_0^{t_1} \widehat{E} \sup_{0\le u\le t} y(u) - \widehat{y(u)}|^2 \mathrm{d}t, \tag{89}$$

$$J_7(t) \le G_1 e^{2|A|T} T |A|^2 h^2 e^{2|A|T}\left(1 + 2\widetilde{C}_1\right), \tag{90}$$

$$J_8(t) \le G_2 e^{2|A|T} T h \left(1 + 2\widetilde{C}_1\right), \tag{91}$$

$$J_9(t) \le 2L_1 e^{2|A|T} T \int_0^{t_1} \widehat{E} \sup_{0\le u\le t} |y(u) - \widehat{y(u)}|^2 \mathrm{d}t. \tag{92}$$

Substituting (84)–(92) into (81) and rearranging (81), we obtain

$$\begin{aligned}&\widehat{E}\left(\sup_{0\le t\le t_1}|y(t)-\widehat{y(t)}|^2\right)\\&\le\left(9T+36\overline{\sigma}^2+9T\overline{\sigma}^4\right)G_1e^{2|A|T}T|A|^2h^2e^{2|A|T}\left(1+2\widetilde{C}_1\right),\end{aligned}\tag{93}$$

$$\begin{aligned}&+\left(9T+36\overline{\sigma}^2+9T\overline{\sigma}^4\right)G_2e^{2|A|T}Th\left(1+2\widetilde{C}_1\right)\\&+2\left(9T+36\overline{\sigma}^2+9T\overline{\sigma}^4\right)L_1e^{2|A|T}T\int_0^{t_1}\widehat{E}\left(\sup_{0\le u\le t}y(u)-\widehat{y(u)}|^2\right)dt\\&\le\left(9T+36\overline{\sigma}^2+9T\overline{\sigma}^4\right)e^{2|A|T}Th\left(G_1|A|^2he^{2|A|T}+G_2\right)\left(1+2\widetilde{C}_1\right)\\&+2\left(9T+36\overline{\sigma}^2+9T\overline{\sigma}^4\right)L_1e^{2|A|T}T\int_0^{t_1}\widehat{E}\left(\sup_{0\le u\le t}|y(u)-\widehat{y(u)}|^2\right)dt.\end{aligned}\tag{94}$$

By Gronwall's inequality, it gives

$$\begin{aligned}&\widehat{E}\left(\sup_{0\le t\le T}|y(t)-\widehat{y(t)}|^2\right)\\&\le\left(9T+36\overline{\sigma}^2+9T\overline{\sigma}^4\right)e^{2|A|T}h(T+4)\left(G_1|A|^2he^{2|A|T}+G_2\right)\left(1+2\widetilde{C}_1\right)\cdot e^{2\left(9T+36\overline{\sigma}^2+9T\overline{\sigma}^4\right)L_1e^{2|A|T}T},\end{aligned}\tag{95}$$

and then, letting $h\longrightarrow 0$, one can draw the conclusion. The proof is completed.

Theorem 2 shows that if the coefficients of f, g, and h obey the Lipschitz condition, in addition to the conditions imposed in Lemma 3, then the exponential Euler approximate solution converges to the exact solution for system (7). □

4.2. The Mean-Square Stability. In this section, we are in a position to explore the exponential stability of the exact solution and the numerical approximation. For this purpose, we further assume that $f(t,0,0)=0$, $g(t,0,0)=0$, and $h(t,0,0)=0$. Therefore, system (7) admits a trivial solution. Moreover, from condition (20), it is easy to get

$$|f(t,x,y)|^2\vee|g(t,x,y)|^2\vee|h(t,x,y)|^2\le L_1\left(|x|^2+|y|^2\right),\quad\forall x,y\in R^d.\tag{96}$$

In the following, we first give some necessary assumptions and definitions for the mean-square exponential stability.

Definition 7. Equation (7) is said to be mean-square exponentially stable if there exist positive constants v and C, which is dependent on the initial data $y_0(t)\in C^b_{\mathscr{F}_0}([0,T];R^d)$ and independent of t, such that

$$\widehat{E}|y(t)|^2\le Ce^{-vt},\quad t\ge 0.\tag{97}$$

For convenience, we introduce the logarithmic norm $\mu[A]$ proposed in [38].

Definition 8. The logarithmic norm $\mu[A]$ of A is defined by

$$\mu[A]=\lim_{\Delta\longrightarrow 0^+}\frac{|I+\Delta A|-1}{\Delta}.\tag{98}$$

In particular, if $|\cdot|$ is an inner product norm, $\mu[A]$ can also be written as

$$\mu[A]=\max_{y\ne 0}\frac{\langle Ay,y\rangle}{|y|^2}.\tag{99}$$

Theorem 3. *Assume that f, g, and h satisfy the Lipschitz condition (20). If there exist positive constants λ_1 and λ_2, such that*

$$2x^Tf(t,x,y)\le-\lambda_1|x|^2+\lambda_2|y|^2,\quad x,y\in R^d,\tag{100}$$

holds, then the analytical solution of equation (7) is mean-square exponentially stable under the condition

$$\lambda_1-2\mu[A]-\lambda_2-6\overline{\sigma}^4L_1-4\overline{\sigma}^4\sqrt{L_1}>0.\tag{101}$$

Proof. Let $V(t,y(t))=1+|y(t)|^2=1+y^T(t)y(t)$. By the matrix derivative rule and the Itô formula, we can derive that

$$\begin{aligned}\mathrm{d}V(t,y(t)) &= \left[V_t(t,y(t)) + V_y(t,y(t))f(t,y(t),y(t-\tau))\right]\mathrm{d}t \\ &\quad + V_y(t,y(t))g(t,y(t),y(t-\tau))\mathrm{d}\omega(t) + \Big[V_y(t,y(t))h(t,y(t),y(t-\tau)) \\ &\quad + \frac{1}{2}\mathrm{trace}\, g^T V_{yy}(t,y(t),y(t-\tau))g\Big]\mathrm{d}\langle\omega\rangle(t).\end{aligned} \tag{102}$$

The aforementioned equation is equivalent to the following stochastic integral equation:

$$\begin{aligned}V(t,y(t)) &= V(0,y(0)) + \int_0^t \left[V_s(s,y(s)) + V_y(s,y(s))f(s,y(s),y(s-\tau))\right]\mathrm{d}s + \int_0^t V_y(s,y(s))g(s,y(s),y(s-\tau))\mathrm{d}\omega(s) \\ &\quad + \int_0^t \left[V_y(s,y(s))h(s,y(s),y(s-\tau)) + \frac{1}{2}\mathrm{trace}\, g^T V_{yy}(s,y(s),y(s-\tau))g\right]\mathrm{d}\langle\omega\rangle(s),\end{aligned} \tag{103}$$

that is,

$$\begin{aligned}1 + |y(t)|^2 &= 1 + |y(0)|^2 + 2\int_0^t \langle y(s), Ay(s) + f(s,y(s),y(s-\tau))\rangle \mathrm{d}s + 2\int_0^t \langle y(s), g(s,y(s),y(s-\tau))\rangle \mathrm{d}\omega(s) \\ &\quad + 2\int_0^t \langle y(s), h(s,y(s),y(s-\tau))\rangle \mathrm{d}\langle\omega\rangle(s) + \int_0^t |g(s,y(s),y(s-\tau))|^2 d\langle\omega\rangle(s).\end{aligned} \tag{104}$$

Taking G-expectation on both sides of (101), we get

$$\begin{aligned}\widehat{E}\left(1 + |y(t)|^2\right) &= 1 + |y(0)|^2 + 2\widehat{E}\left(\int_0^t \langle y(s), Ay(s) + f(s,y(s),y(s-\tau))\rangle \mathrm{d}s\right) \\ &\quad + 2\widehat{E}\left(\int_0^t \langle y(s), g(s,y(s),y(s-\tau))\rangle \mathrm{d}\omega(s)\right) \\ &\quad + 2\widehat{E}\left(\int_0^t \langle y(s), h(s,y(s),y(s-\tau))\rangle \mathrm{d}\langle\omega\rangle(s)\right) + \widehat{E}\left(\int_0^t |g(s,y(s),y(s-\tau))|^2 \mathrm{d}\langle\omega\rangle(s)\right).\end{aligned} \tag{105}$$

With the condition in (100), the inequality in (105) obtains

$$\begin{aligned}\widehat{E}\left(\sup_{0\le s\le t}\left(1 + |y(s)|^2\right)\right) &\le 1 + |y(0)|^2 + \widehat{E}\left(\int_0^t \left((2\mu[A] - \lambda_1)|y(s)|^2 + \lambda_2|y(s-\tau)|^2\right)\mathrm{d}s\right) \\ &\quad + 2\widehat{E}\left(\sup_{0\le s\le t}\int_0^s \langle y(u), g(u,y(u),y(u-\tau))\rangle \mathrm{d}\omega(u)\right) \\ &\quad + 2\widehat{E}\left(\sup_{0\le s\le t}\int_0^s \langle y(u), h(u,y(u),y(u-\tau))\rangle \mathrm{d}\langle\omega\rangle(u)\right) + \widehat{E}\left(\sup_{0\le s\le t}\int_0^s |g(u,y(u),y(u-\tau))|^2 \mathrm{d}\langle\omega\rangle(u)\right).\end{aligned} \tag{106}$$

Combining Proposition 3 with Lemma 1, we have

$$\begin{aligned}
&2\widehat{E}\left(\sup_{0\le s\le t}\int_0^s\langle y(u), g(u, y(u), y(u-\tau))\rangle \mathrm{d}\omega(u)\right)\\
&\le 2\overline{\sigma}^2\widehat{E}\left(\int_0^t |y(s)|^2|g(s, y(s), y(s-\tau))|^2\mathrm{d}s\right)^{1/2}\\
&\le 2\overline{\sigma}^2\widehat{E}\left\{\left(\sup_{0\le s\le t}|y(s)|^2\right)\int_0^t |g(s, y(s), y(s-\tau))|^2\mathrm{d}s\right\}^{1/2}\\
&\le \frac{1}{2}\widehat{E}\left(\sup_{0\le s\le t}\left(1+|y(s)|^2\right)\right)+2\overline{\sigma}^4L_1\widehat{E}\left(\int_0^t\left(|y(s)|^2+|y(s-\tau)|^2\right)\mathrm{d}s\right).
\end{aligned} \tag{107}$$

Similarly, we have

$$\begin{aligned}
&2\widehat{E}\left(\sup_{0\le s\le t}\int_0^s\langle y(u), h(u, y(u), y(u-\tau))\rangle \mathrm{d}\langle\omega\rangle(u)\right)\\
&\le 2\overline{\sigma}^4\widehat{E}\left(\int_0^{st}|y(s)||h(s, y(s), y(s-\tau))|\mathrm{d}s\right)\\
&\le \overline{\sigma}^4\widehat{E}\left(\int_0^t 2\sqrt[4]{L_1}|y(s)|\frac{|h(s, y(s), y(s-\tau))|}{\sqrt[4]{L_1}}\mathrm{d}s\right)\\
&\le \overline{\sigma}^4\widehat{E}\left(\int_0^t\left(\sqrt{L_1}|y(s)|^2+\frac{|h(s, y(s), y(s-\tau))|^2}{\sqrt{L_1}}\right)\mathrm{d}s\right)\\
&\le \overline{\sigma}^4\sqrt{L_1}\widehat{E}\left(\int_0^t\left(|y(s)|^2+|y(s)|^2+|y(s-\tau)|^2\right)\mathrm{d}s\right)\\
&\le 2\overline{\sigma}^4\sqrt{L_1}\widehat{E}\left(\int_0^t\left(|y(s)|^2+|y(s-\tau)|^2\right)\mathrm{d}s\right).
\end{aligned} \tag{108}$$

and

$$\begin{aligned}
&\widehat{E}\left(\sup_{0\le s\le t}\int_0^s|g(u, y(u), y(u-\tau))|^2\mathrm{d}\langle\omega\rangle(u)\right)\\
&\le \overline{\sigma}^4L_1\widehat{E}\left(\int_0^t\left(|y(s)|^2+|y(s-\tau)|^2\right)\mathrm{d}s\right).
\end{aligned} \tag{109}$$

Substituting (107)–(109) into (106), we obtain

$$\begin{aligned}
&\widehat{E}\left(\sup_{0\le s\le t}\left(1+|y(s)|^2\right)\right)\\
&\le 1+|y(0)|^2+\widehat{E}\left(\int_0^t\left((2\mu[A]-\lambda_1)|y(u)|^2+\lambda_2|y(u-\tau))|^2\right)\mathrm{d}u\right)\\
&\quad+\frac{1}{2}\widehat{E}\left(\sup_{0\le s\le t}\left(1+|y(s)|^2\right)\right)+2\overline{\sigma}^4L_1\widehat{E}\left(\int_0^t\left(|y(s)|^2+|y(s-\tau)|^2\right)\mathrm{d}s\right)\\
&\quad+2\overline{\sigma}^4\sqrt{L_1}\widehat{E}\left(\int_0^t\left(|y(s)|^2+|y(s-\tau)|^2\right)\mathrm{d}s\right)+\overline{\sigma}^4L_1\widehat{E}\left(\int_0^t\left(|y(s)|^2+|y(s-\tau)|^2\right)\mathrm{d}s\right),
\end{aligned} \tag{110}$$

that is,

$$\widehat{E}\left(\sup_{0\le s\le t}\left(1+|y(s)|^2\right)\right) \le 2\left(1+|y(0)|^2\right)+2\left(2\mu[A]-\lambda_1+\lambda_2+6\overline{\sigma}^4L_1+4\overline{\sigma}^4\sqrt{L_1}\right)\widehat{E}\left(\int_0^t \sup_{0\le s\le t}|y(s)|^2\mathrm{d}s\right). \tag{111}$$

From the Gronwall inequality, we have

$$\widehat{E}\left(\sup_{0\le s\le t}\left(|y(s)|^2\right)\right)\le 2\left(1+|y(0)|^2\right)e^{2\left(2\mu[A]-\lambda_1+\lambda_2+6\overline{\sigma}^4L_1+4\overline{\sigma}^4\sqrt{L_1}\right)t}\le Ce^{-\nu t}, \tag{112}$$

where $C=2(1+|y(0)|^2)$ and $\nu=2(\lambda_1-2\mu[A]-\lambda_2-6\overline{\sigma}^4L_1-4\overline{\sigma}^4\sqrt{L_1})$. The proof is completed. □

Then the question is, will the exponential Euler method reproduce the stability of analytical solutions of equation (7) under the Lipschitz condition? In order to answer this question, we introduce the following definition of mean-square stability from [39] at first and present the stability theorem of the exponential Euler approximation.

Definition 9. A numerical method is said to be asymptotically mean-square stable (with respect to a given G-SLSDDE) if

$$\lim_{n\longrightarrow\infty}\widehat{E}|y_n|^2=0. \tag{113}$$

Theorem 4. *Let (H1) hold. Assume that there exist positive constants λ_3 and λ_4, such that*

$$2\overline{\sigma}^2x^Th(t,x,y)\le-\lambda_3|x|^2+\lambda_4|y|^2,\quad x,y\in R^d. \tag{114}$$

If the inequalities $L_1\overline{\sigma}^2-\lambda_1-\lambda_3<0$ and $2\mu[A]+\lambda_2+\lambda_4+\lambda_1+\lambda_3<0$ hold, then the exponential Euler method is asymptotically mean-square stable for all

$$h_n<h_0=\frac{1/2\lambda_1+1/2\lambda_3-1/2L_1\overline{\sigma}^2}{L_1+L_1\overline{\sigma}^4+L_1\overline{\sigma}^2}. \tag{115}$$

Proof. Squaring both sides of (71), we have

$$\begin{aligned} y_{n+1}^2 &= e^{2\mu[A]h_n}\left(y_n^2+f^2(t_n,y_n,y_{n-m})h_n^2+g^2(t_n,y_n,y_{n-m})\left|\Delta\omega_n\right|^2+h^2(t_n,y_n,y_{n-m})\left|\Delta\langle\omega\rangle_n\right|^2\right)+2e^{2\mu[A]h_n}y_n^Tf(t_n,y_n,y_{n-m})h_n \\ &\quad +2e^{2\mu[A]h_n}y_n^Tg(t_n,y_n,y_{n-m})\Delta\omega_n+2e^{2\mu[A]h_n}y_n^Th(t_n,y_n,y_{n-m})\Delta\langle\omega\rangle_n+2e^{2\mu[A]h_n}f(t_n,y_n,y_{n-m})^Tg(t_n,y_n,y_{n-m})h_n\Delta\omega_n \\ &\quad +2e^{2\mu[A]h_n}f(t_n,y_n,y_{n-m})^Th(t_n,y_n,y_{n-m})h_n\Delta\langle\omega\rangle_n+2e^{2\mu[A]h_n}g(t_n,y_n,y_{n-m})^Th(t_n,y_n,y_{n-m})\Delta\omega_n\Delta\langle\omega\rangle_n. \end{aligned} \tag{116}$$

Taking G-expectation on both sides of (116) and considering that y_k and y_{k-m} are $\mathscr{F}_{t_k}$-measurable, by Proposition 3, one can get

$$\begin{aligned} \widehat{E}\left[g^2(t_k,y_k,y_{k-m})\left(\left|\Delta\omega_k\right|^2-\overline{\sigma}^2h_k\right)\right] &= \widehat{E}\left[\widehat{E}\left(g^2(t_k,y_k,y_{k-m})\left(\left|\Delta\omega_k\right|^2-\overline{\sigma}^2h_k\right)|\mathscr{F}_{t_k}\right)\right] \\ &= \widehat{E}\left[g^2(t_k,y_k,y_{k-m})\widehat{E}\left(\left(\Delta\omega_k\right)^2-\overline{\sigma}^2h_k|\mathscr{F}_{t_k}\right)\right]=0, \\ \widehat{E}\left[y_k^Tg(t_k,y_k,y_{k-m})\Delta\omega_k\right] &= \widehat{E}\left[\widehat{E}\left(y_k^Tg(t_k,y_k,y_{k-m})\Delta\omega_k|\mathscr{F}_{t_k}\right)\right] \\ &= \widehat{E}\left[y_k^Tg(t_k,y_k,y_{k-m})\widehat{E}\left(\Delta\omega_k|\mathscr{F}_{t_k}\right)\right]=0. \end{aligned} \tag{117}$$

By Proposition 2, one obtains

$$\begin{aligned}
\widehat{E}\left[h^2(t_k,y_k,y_{k-m})\left(\left|\Delta\langle\omega\rangle_k\right|^2-\overline{\sigma}^4h_k^2\right)\right] &= \widehat{E}\left[\widehat{E}\left(h^2(t_k,y_k,y_{k-m})\left(\left|\Delta\langle\omega\rangle_k\right|^2-\overline{\sigma}^4h_k^2\right)|\mathscr{F}_{t_k}\right)\right] \\
&= \widehat{E}\left[g^2(t_k,y_k,y_{k-m})\widehat{E}\left(\left(\Delta\langle\omega\rangle_k\right)^2-\overline{\sigma}^4h_k^2|\mathscr{F}_{t_k}\right)\right]\le 0, \\
\widehat{E}\left[y_k^Th(t_k,y_k,y_{k-m})\Delta\langle\omega\rangle_k\right] &= \widehat{E}\left[\widehat{E}\left(y_k^Tg(t_k,y_k,y_{k-m})\Delta\langle\omega\rangle_k|\mathscr{F}_{t_k}\right)\right] \\
&= \widehat{E}\left[y_k^Th(t_k,y_k,y_{k-m})\widehat{E}\left(\Delta\langle\omega\rangle_k|\mathscr{F}_{t_k}\right)\right]\le\overline{\sigma}^2h_k\widehat{E}\left[y_k^Th(t_k,y_k,y_{k-m})\right], \\
\widehat{E}\left[g^T(t_k,y_k,y_{k-m})h(t_k,y_k,y_{k-m})\Delta\omega_k\Delta\langle\omega\rangle_k\right] &= \widehat{E}\left[\widehat{E}\left(g^T(t_k,y_k,y_{k-m})h(t_k,y_k,y_{k-m})\Delta\omega_k\Delta\langle\omega\rangle_k|\mathscr{F}_{t_k}\right)\right] \\
&= \widehat{E}\left[g^T(t_k,y_k,y_{k-m})h(t_k,y_k,y_{k-m})\widehat{E}\left(\Delta\omega_k\Delta\langle\omega\rangle_k|\mathscr{F}_{t_k}\right)\right] \\
&\le\overline{\sigma}^2h_k\widehat{E}\left[g^T(t_k,y_k,y_{k-m})h(t_k,y_k,y_{k-m})\widehat{E}\left(\Delta\omega_k|\mathscr{F}_{t_k}\right)\right]=0, \\
\widehat{E}\left[f^T(t_k,y_k,y_{k-m})h(t_k,y_k,y_{k-m})h_k\Delta\langle\omega\rangle_k\right] &= \widehat{E}\left[\widehat{E}\left(f^T(t_k,y_k,y_{k-m})h(t_k,y_k,y_{k-m})h_k\Delta\langle\omega\rangle_k|\mathscr{F}_{t_k}\right)\right] \\
&= h_k\widehat{E}\left[f^T(t_k,y_k,y_{k-m})h(t_k,y_k,y_{k-m})\widehat{E}\left(\Delta\langle\omega\rangle_k|\mathscr{F}_{t_k}\right)\right] \\
&\le\overline{\sigma}^2h_k^2\widehat{E}\left[f^T(t_k,y_k,y_{k-m})h(t_k,y_k,y_{k-m})\right].
\end{aligned} \tag{118}$$

Then, it follows from (100), (114), and (99) that

$$\begin{aligned}
\widehat{E}\left|y_{n+1}\right|^2 &= e^{2\mu[A]h_n}|\widehat{E}\left|y_n\right|^2+\widehat{E}\left[f^2(t_n,y_n,y_{n-m})\right]h_n^2+\widehat{E}\left[g^2(t_n,y_n,y_{n-m})\right]\overline{\sigma}^2h_n+\widehat{E}\left[h^2(t_n,y_n,y_{n-m})\right]\overline{\sigma}^4h_n^2 \\
&\quad+2y_n^Tf(t_n,y_n,y_{n-m})h_n+2\overline{\sigma}^2h_n\widehat{E}\left[y_n^Th(t_n,y_n,y_{n-m})\right] \\
&\quad+2\overline{\sigma}^2h_n^2\widehat{E}\left[f(t_n,y_n,y_{n-m})^Th(t_n,y_n,y_{n-m})\right]|\le e^{2\mu[A]h_n}|\widehat{E}\left|y_n\right|^2+\widehat{E}\left(L_1\left|y_n\right|^2+L_1\left|y_{n-m}\right|^2\right)h_n^2 \\
&\quad+\widehat{E}\left(L_1\left|y_n\right|^2+L_1\left|y_{n-m}\right|^2\right)\overline{\sigma}^2h_n \\
&\quad+\widehat{E}\left(L_1\left|y_n\right|^2+L_1\left|y_{n-m}\right|^2\right)\overline{\sigma}^4h_n^2+\left(-\lambda_1\widehat{E}\left|y_n\right|^2+\lambda_2\widehat{E}\left|y_{n-m}\right|^2\right)h_n+h_n\widehat{E}\left(-\lambda_3\left|y_n\right|^2+\lambda_4\left|y_{n-m}\right|^2\right) \\
&\quad+2\overline{\sigma}^2h_n^2\widehat{E}\left(L_1\left|y_n\right|^2+L_1\left|y_{n-m}\right|^2\right)| \\
&= e^{2\mu[A]h_n}|\left(1+L_1h_n^2+L_1\overline{\sigma}^2h_n+L_1\overline{\sigma}^4h_n^2-\lambda_1h_n-\lambda_3h_n+L_1\overline{\sigma}^2h_n^2\right)\widehat{E}\left|y_n\right|^2 \\
&\quad+\left(L_1h_n^2+L_1\overline{\sigma}^2h_n+L_1\overline{\sigma}^4h_n^2+\lambda_2h_n+\lambda_4h_n+L_1\overline{\sigma}^2h_n^2\right)\widehat{E}\left|y_{n-m}\right|^2=\left|A_1\widehat{E}\left|y_n\right|^2+A_2\widehat{E}\left|y_{n-m}\right|^2\right|,
\end{aligned} \tag{119}$$

where $A_1=e^{2\mu[A]h_n}(1+(L_1\overline{\sigma}^2-\lambda_1-\lambda_3)h_n+(L_1+L_1\overline{\sigma}^4+L_1\overline{\sigma}^2)h_n^2)$ and $A_2=e^{2\mu[A]h_n}((L_1\overline{\sigma}^2+\lambda_2+\lambda_4)h_n+(L_1+L_1\overline{\sigma}^4+L_1\overline{\sigma}^2)h_n^2)$.

Considering $L_1\overline{\sigma}^2-\lambda_1-\lambda_3<0$, for any $h_n<h_0=1/2\lambda_1+1/2\lambda_3-1/2L_1\overline{\sigma}^2/L_1+L_1\overline{\sigma}^4+L_1\overline{\sigma}^2$, the following inequalities hold:

$$A_1\le e^{2\mu[A]h_n}\left(1+\left(\frac{1}{2}L_1\overline{\sigma}^2-\frac{1}{2}\lambda_1-\frac{1}{2}\lambda_3\right)h_n\right), \tag{120}$$

$$\begin{aligned}
\left|A_1+A_2\right|\le\left|A_1\right|+\left|A_2\right| &= \left|e^{2\mu[A]h_n}\left(1+\left(\frac{1}{2}L_1\overline{\sigma}^2-\frac{1}{2}\lambda_1-\frac{1}{2}\lambda_3\right)h_n\right)\right|+\left|e^{2\mu[A]h_n}\left(\left(L_1\overline{\sigma}^2+\lambda_2+\lambda_4\right)h_n+\left(L_1+L_1\overline{\sigma}^4+L_1\overline{\sigma}^2\right)h_n^2\right)\right| \\
&\le e^{2\mu[A]h_n}\left(1+\left(\frac{1}{2}\lambda_1+\frac{1}{2}\lambda_3-\frac{1}{2}L_1\overline{\sigma}^2\right)h_n\right)+e^{2\mu[A]h_n}\left(L_1\overline{\sigma}^2+\lambda_2+\lambda_4+\frac{1}{2}\lambda_1+\frac{1}{2}\lambda_3-\frac{1}{2}L_1\overline{\sigma}^2\right)h_n \\
&\le e^{2\mu[A]h_n}\left(1+(\lambda_2+\lambda_4+\lambda_1+\lambda_3)h_n\right).
\end{aligned} \tag{121}$$

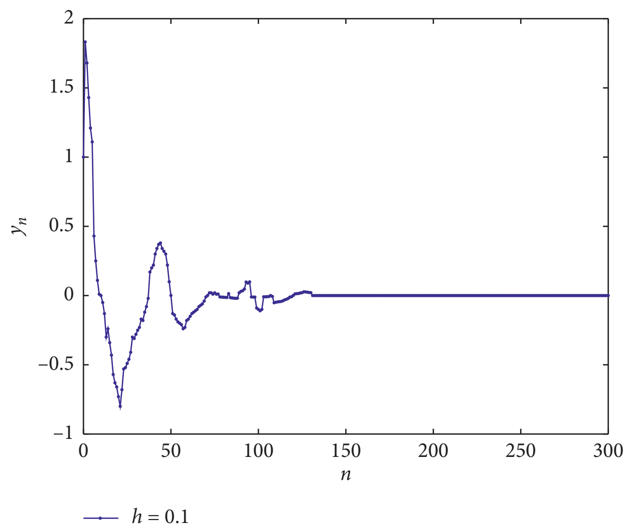

FIGURE 1: The numerical solution with $h = 0.1$ of the exponential Euler method.

Therefore,

$$\begin{aligned}\widehat{E}|y_{n+1}|^2 &\le e^{2\mu[A]h_n}(1+(\lambda_2+\lambda_4+\lambda_1+\lambda_3)h_n)\max\{\widehat{E}|y_n|^2, \widehat{E}|y_{n-m}|^2\}\\ &\le e^{(2\mu[A]+\lambda_2+\lambda_4+\lambda_1+\lambda_3)h_n}\max\{\widehat{E}|y_n|^2, \widehat{E}|y_{n-m}|^2\}.\end{aligned} \tag{122}$$

The condition $2\mu[A]+\lambda_2+\lambda_4+\lambda_1+\lambda_3<0$ gives that $e^{(2\mu[A]+\lambda_2+\lambda_4+\lambda_1+\lambda_3)h_n}<1$; it is not difficult to see that $\widehat{E}|y_n|^2 \le \widehat{E}|y_{n-m}|^2$; therefore,

$$\begin{aligned}\widehat{E}|y_{n+1}|^2 &\le e^{(2\mu[A]+\lambda_2+\lambda_4+\lambda_1+\lambda_3)h_n}\widehat{E}|y_{n-m}|^2\\ &\le e^{2(2\mu[A]+\lambda_2+\lambda_4+\lambda_1+\lambda_3)h_n}\widehat{E}|y_{n-2m}|^2\\ &\vdots\\ &\le e^{\lfloor n-2/m\rfloor(2\mu[A]+\lambda_2+\lambda_4+\lambda_1+\lambda_3)h_n}\widehat{E}|y_0|^2\end{aligned}, \tag{123}$$

and hence

$$\lim_{n\longrightarrow\infty}\widehat{E}|y_n|^2 = 0. \tag{124}$$

The above theorem gives the sufficient conditions for keeping mean-square stability by exponential Euler method for equation (7). On the premise of its stability, it is found that the stability and step size of exponential Euler method depend on the norm of A. □

5. Numerical Examples

In this section, we discuss a numerical example to illustrate the effectiveness of the obtained results.

Example 1. Let $\omega(t)$ be a scalar G-Brownian motion with $\omega(t) \sim N(0,[1/8,1/4])$. Consider the scalar nonlinear G-SLSDDE with the form

$$\begin{aligned} dy(t) &= \left(-\frac{1}{2}y(t)-\frac{1}{8}[y(t)-\sin y(t-1)]\right)dt+\frac{1}{2}y(t-1)d\omega(t)-\frac{1}{4}y(t)d\langle\omega\rangle(t), \quad t\ge 0,\\ y(t) &= t+1, \quad t\in[-1,0].\end{aligned} \tag{125}$$

Then, the numerical solution of this system is convergent and asymptotically mean-square stable.

Obviously, from system (125), one can obtain $\mu[A] = -1/2$ and $f(t,0,0)=g(t,0,0)=h(t,0,0)=0$. It is easy to see that f,g,h satisfy the condition in (H1). Moreover, $|f(t,x,y)|^2 \le 1/32(|x|^2+|y|^2)$, $|g(t,x,y)|^2 = 1/4|y|^2$, and $|h(t,x,y)|^2 = 1/16|x|^2$. Therefore, $|f(t,x,y)|^2 \vee |g(t,x,y)|^2 \vee |h(t,x,y)|^2 \le 1/4(|x|^2+|y|^2)$; that is, $L_1 = 1/4$.

Further, by Young's inequality, one can obtain

$$\begin{aligned}2x^T f(t,x,y) &= 2x^T\left(-\frac{1}{8}[x-\sin y]\right) \le -\frac{3}{16}|x|^2+\frac{1}{4}|y|^2,\\ 2\overline{\sigma}^2x^T h(t,x,y) &= 2\frac{1}{4}x^T\left(-\frac{1}{4}x\right) \le -\frac{1}{8}|x|^2+\frac{1}{8}|y|^2.\end{aligned} \tag{126}$$

Therefore, $\lambda_1 = 3/16$, $\lambda_2 = 1/4$, $\lambda_3 = 1/8$, and $\lambda_4 = 1/8$ satisfy $L_1\overline{\sigma}^2-\lambda_1-\lambda_3<0$, $2\mu[A]+\lambda_2+\lambda_4+\lambda_1+\lambda_3<0$, and $\lambda_1-2\mu[A]-\lambda_2-6\overline{\sigma}^4L_1-4\overline{\sigma}^4\sqrt{L_1}>0$. Then, it follows from

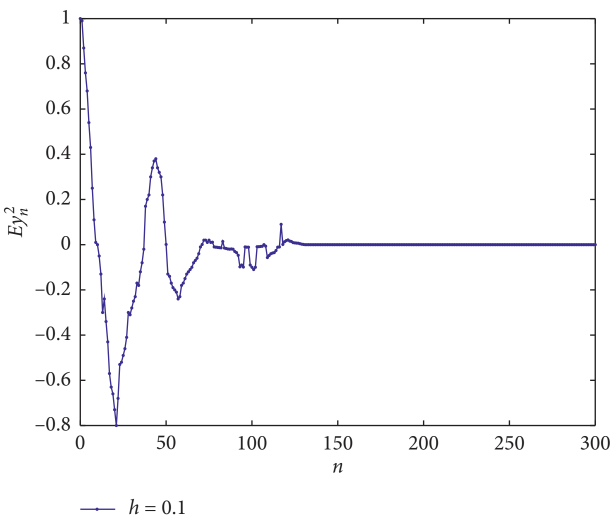

FIGURE 2: The G-expectation of numerical solution with $h = 0.1$ of the exponential Euler method.

Theorem 3 that the trivial solution of the example is mean-square exponentially stable.

Noticing that

$$h_0 = \frac{1/2\lambda_1 + 1/2\lambda_3 - 1/2L_1\overline{\sigma}^2}{L_1 + L_1\overline{\sigma}^4 + L_1\overline{\sigma}^2} = \frac{8}{21}, \tag{127}$$

by Theorem 2 and Theorem 4, we see that the exponential Euler method is convergent and asymptotically mean-square stable.

On the other hand, by the exponential Euler method, choose the step size $h < 8/21$ and the initial value $y(0) = 1$ to simulate the numerical solution y_n and $\widehat{E}y_n^2$ for system (125), which are shown in Figures 1 and 2 , respectively. It follows from Figure 1 that the numerical solution of system (125) converges to zero. Moreover, it follows from Figure 2 that numerical solution of system (125) is asymptotically mean-square stable. In brief, the numerical simulation results are consistent with our theoretical results.

6. Conclusions

This paper is devoted to applying the exponential integrators to the semilinear stochastic delay differential equations driven by G-Brownian motion (G-SLSDDEs) and dealing with the convergence and stability properties of exponential integrators for G-SLSDDEs. It first investigates some suitable conditions for the mean-square stability of the analytic solution and then shows that the exponential integrators numerical solution converges to the analytic solution for the G-SLSDDEs with the strong order 1/2. Furthermore, it proves that the exponential Euler method can keep the mean-square exponential stability of the analytic solution under some restrictions on the step size.

G-framework is a new study area which many scholars begin to pay attention to. Recently, some results related to G-framework have been obtained; however, it is a pity that there are few numerical conclusions. In the future, we will further discover more efficient numerical methods, such as the transferred Legendre pseudospectral method in [40], to solve semilinear stochastic delay differential equations with time-variable delay driven by G-Brownian motion. Influenced by experience gained from solving stochastic fractional differential equations driven by fractional Brownian motion, we will further study stochastic fractional differential equations driven by G-Brownian motion with "$dy(t)$" as fractional order in future research [41].

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares no conflicts of interest.

Acknowledgments

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