Some Properties of Polynomials Orthogonal over the Set $\langle 1, 2, \ldots, N \rangle$ (*) (**).

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Summary. – *Using identities being discrete counterparts of those which are satisfied by the Legendre polynomials, the author proves that if the polynomials* $\langle \Psi_m^{(N)}(t) \rangle$ $(m = 0, 1, \ldots, N-1)$ *form an orthogonal set over the set* $\langle 1, 2, \ldots, N \rangle$ *with equal weight attached to its elements, then*

$$|\Psi_m^{(N)}(t)| < |\Psi_m^{(N)}(1)| = |\Psi_m^{(N)}(N)| \qquad (t = 2, 3, \ldots, N-1)$$

when $m(m+1) \leqslant N-1$. *This result is then extended to a wide class of Hahn polynomials.*

1. – Introduction.

Although polynomials orthogonal over discrete sets were considered as early as the middle of the nineteenth century by Chebyshev, comparatively little attention had been paid to them until recently. We assume throughout that the discrete set in question is the set of consecutive integers from 1 or from 0 to N. The present paper is primarily concerned with the case when these integers are given equal weights, so that we obtain a discrete counterpart of the Legendre polynomials; this case has important applications in statistics, and more particularly applications to the fitting of nonlinear regressions and to trend estimations in time-series analysis (see, for instance, [3], [4], [5]). However, owing to a very recent result obtained by GEORGE GASPER [2], the main theorem can be extended without any difficulty to a wide class of Hahn polynomials (see, for instance, [6]), which are the discrete counterpart of the Jacobi polynomials, and have also found applications to probability theory and statistics.

For our present purpose, the most convenient form of the polynomials being the main object of our study is the following one:

$$\Psi_m^{(N)}(t) = \sum_{k=0}^{m} \frac{(-1)^{m-k}(m+k)!}{(k!)^2(m-k)!} \cdot \frac{(t-1)^{[k]}}{(N-1)^{[k]}} \qquad (m = 0, 1, \ldots, N-1), \tag{1.1}$$

where, as usual, for all x, $x^{[0]} = 1$, and

$$x^{[k]} = x(x-1) \ldots (x-k+1) \qquad (k > 0)$$

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There is an obvious analogy between these polynomials and the polynomials

$$\tilde{P}_m(x) = \sum_{k=0}^{m} \frac{(-1)^{m-k}(m+k)!}{(k!)^2(m-k)!} x^k = (-1)^m P_m(1-2x) \qquad (m = 0, 1, 2, \ldots),$$

where $\langle P_m \rangle$ are the Legendre polynomials. These polynomials satisfy

$$\int_0^1 \tilde{P}_m(x)\tilde{P}_n(x)\,dx = \begin{cases} 0 & \text{when } m \neq n; \\ 1/(2m+1) & \text{when } m = n. \end{cases} \tag{1.3}$$

One of the easiest ways of obtaining the orthogonal properties of the polynomials $\langle \Psi_m^{(N)} \rangle$ is [10] by means of combinatorial identities being immediate consequences of (1.3).

It is easily seen that

$$\lim_{N\to\infty} \{\Psi_m^{(N)}(t) - \tilde{P}_m(t/N)\} = 0 \qquad (t = 1, 2, \ldots, N) \tag{1.4}$$

uniformly in t, but *not* in m. Interesting results have been obtained concerning the order of magnitude of this difference as $N \to \infty$, and of the corresponding difference for more general orthogonal polynomials (see for instance, [8], [9]). Unfortunately, these results are not sufficient for some statistical, and possibly other, applications, notably for those in which the degree m of the polynomial is allowed to increase with N; in such cases the uniform convergence in m within some prescribed, but variable bounds would have to be examined. However, from the view point of such applications a direct study of the polynomials $\langle \Psi_m^{(N)} \rangle$ seems to be more promising.

The properties of $\langle \Psi_m^{(N)} \rangle$ are in many ways similar to those of $\langle \tilde{P}_m \rangle$, and, therefore, of the Legendre polynomials, but there are substantial differences, quite apart from the fact that the sequence $\langle \Psi_m^{(N)} \rangle$ is finite for any N. One of those differences is that while Legendre, and $\tilde{P}_m$ polynomials always take their biggest absolute values for the interval over which they are orthogonal at the ends of this interval, this is the case for $\Psi_m^{(N)}$ only when m is sufficiently small in relation to N. Indeed, it will be seen that always

$$|\Psi_m^{(N)}(1)| = |\Psi_m^{(N)}(N)| = 1,$$

but, for instance, according to (1.1),

$$\Psi_2^{(3)}(2) = -2, \qquad \Psi_3^{(6)}(2) = 7/5, \qquad \text{and} \qquad \Psi_3^{(4)}(2) = 3.$$

The main purpose of the present paper is to find conditions on m and N under which

$$|\Psi_m^{(N)}(t)| < 1 \qquad \text{for } t = 2, 3, \ldots, N-1.$$

Various identies, which may also be of some intrinsic interest, will be used to prove the relevant theorem.

2. – Some identities and inequalities.

It is well known that

$$\Psi_m^{(N)}(N-t+1)=(-1)^m\Psi_m^{(N)}(t) \qquad (t=1,\dots,N;\ m=0,\dots,N-1). \tag{2.1}$$

This is easily verified noting that $\Psi_m^{(N)}(N-t+1)$ is orthogonal to all polynomials of degree lower than m, and that its degree in t is precisely m. Since this condition uniquely determines the polynomial apart from an arbitrary constant factor (see, for instance, [5] or [6]), a comparison of the leading coefficients concludes the proof.

According to (1.1),

$$\Psi_m^{(N)}(1)=(-1)^m \qquad (m=0,\dots,N-1)\,. \tag{2.2}$$

Hence also, by (2.1),

$$\Psi_m^{(N)}(N)=1 \qquad (m=0,\dots,N-1)\,. \tag{2.3}$$

In what follows, $f(t)$ being any function, $\Delta f(t)$ will denote the difference $f(t+1)-$ $-f(t)$. Since

$$\Delta(t-1)^{[k]}=k(t-1)^{[k-1]} \qquad (k=1,2,\dots),$$

we have for all t

$$\Delta\Psi_m^{(N)}(t)=\sum_{k=1}^{m}\frac{(-1)^{m-k}(m+k)!\,k}{(k!)^2(m-k)!}\cdot\frac{(t-1)^{[k-1]}}{(N-1)^{[k]}}\,. \tag{2.4}$$

Together with (1.1), this allows us to verify by substitution the following relation, which corresponds to the fourth recurrence relation for Legendre polynomials, as given by WHITTAKER and WATSON [7]:

$$\Delta\Psi_m^{(N)}(t)-\Delta\Psi_{m-2}^{(N)}(t)=\frac{2(2m-1)}{N-1}\,\Psi_{m-1}^{(N-1)}(t)\,, \tag{2.5}$$

valid for all t, $m=2,\dots,N-1$, and $N=3,4,\dots$.

LEMMA 2.1. – The polynomial

$$\Delta\left\{\frac{(t-1)(N-t+1)}{m(m+1)}\,\Delta\Psi_m^{(N)}(t-1)\right\}$$

is orthogonal to every polynomial of degree lower than m over the set $\langle 1,2,\dots,N\rangle$.

PROOF. – Let $\varrho(t)$ be any polynomial of degree lower than m. Since $(t-1)(N-t+1)$ vanishes both for $t=1$ and for $t=N+1$, we find, by the Abel transformation,

$$\sum_{t=1}^{N} \varrho(t)\,\Delta\left\{\frac{(t-1)(N-t+1)}{m(m+1)}\,\Delta\Psi_m^{(N)}(t-1)\right\} = -\sum_{t=1}^{N-1}\frac{t(N-t)}{m(m+1)}\,\Delta\Psi_m^{(N)}(t)\,\Delta\varrho(t) = -\sum_{t=0}^{N}\frac{t(N-t)}{m(m+1)}\,\Delta\varrho(t)\,\Delta\Psi_m^{(N)}(t)\,.$$

Applying again the Abel transformation to the last expression and noting the obvious vanishing of some terms, we find eventually

$$\sum_{t=1}^{N} \varrho(t)\,\Delta\left\{\frac{(t-1)(N-t+1)}{m(m+1)}\,\Delta\Psi_m^{(N)}(t-1)\right\} = \sum_{t=0}^{N-1}\Psi_m^{(N)}(t+1)\,\Delta\left\{\frac{t(N-t)}{m(m+1)}\,\Delta\varrho(t)\right\} = \sum_{s=1}^{N}\Psi_m^{(N)}(s)\,\Delta\left\{\frac{(s-1)(N-s+1)}{m(m+1)}\,\Delta\varrho(s-1)\right\}.$$

The degree of

$$\Delta\left\{\frac{(s-1)(N-s+1)}{m(m+1)}\,\Delta\varrho(s-1)\right\}$$

being that of $\varrho(t)$, which is smaller than m, by the orthogonality property of $\langle\Psi_m^{(N)}\rangle$, the last expression is equal to 0, which proves the Lemma.

COROLLARY 2.2. – We have for all t

$$\Delta\left\{\frac{(t-1)(N-t+1)}{m(m+1)}\,\Delta\Psi_m^{(N)}(t-1)\right\} + \Psi_m^{(N)}(t) = 0\,. \tag{2.6}$$

PROOF. – By the unicity, up to constant factors, of the orthogonal polynomials, it follows from the preceding lemma that

$$\Delta\left\{\frac{(t-1)(N-t+1)}{m(m+1)}\,\Delta\Psi_m^{(N)}(t-1)\right\} \tag{2.7}$$

differs from $\Psi_m^{(N)}(t)$ by a constant factor, which is found to be -1 by comparing the leading coefficients of the two polynomials. Hence (2.6).

Working out the first in term (2.6), and then substituting $t+1$ for t, we find

$$\frac{(t+1)(N-t-1)}{m(m+1)}\,\Delta^2\Psi_m^{(N)}(t) + \frac{N-2t-1}{m(m+1)}\,\Delta\Psi_m^{(N)}(t) + \Psi_m^{(N)}(t+1) = 0\,. \tag{2.8}$$

This is the discrete counterpart of the second-order differential equation satisfied by Legendre polynomials and, more generally, by Legendre functions (see, for instance,

[6] or [7]). It can be re-written thus:

$$(2.9)\qquad \frac{(t+1)(N-t-1)}{m(m+1)}\,\Delta^2\Psi_m^{(N)}(t)+\left\{1+\frac{N-2t-1}{m(m+1)}\right\}\Delta\Psi_m^{(N)}(t)+\Psi_m^{(N)}(t)=0\,;$$

both forms of this difference equation will be useful in what follows.

Differencing (2.8) yields

$$\frac{(t+2)(N-t-2)}{m(m+1)}\,\Delta^3\Psi_m^{(N)}(t)+2\,\frac{N-2t-3}{m(m+1)}\,\Delta^2\Psi_m^{(N)}(t)-\frac{2}{m(m+1)}\,\Delta\Psi_m^{(N)}(t)+\Delta\Psi_m^{(N)}(t+1)=0\,,$$

and, substituting in the third term $\Delta\Psi_m^{(N)}(t+1)-\Delta^2\Psi_m^{(N)}(t)$ for $\Delta\Psi_m^{(N)}(t)$, we find

$$(2.10)\qquad \frac{(t+2)(N-t-2)}{m(m+1)}\,\Delta^3\Psi_m^{(N)}(t)+2\,\frac{N-2t-2}{m(m+1)}\,\Delta^2\Psi_m^{(N)}(t)+\left\{1-\frac{2}{m(m+1)}\right\}\Delta\Psi_m^{(N)}(t+1)=0\,.$$

Hence

$$(2.11)\qquad \frac{(t+2)(N-t-2)}{m(m+1)}\,\{\Delta^3\Psi_m^{(N)}(t)+\Delta\Psi_m^{(N)}(t+1)\}=2\,\frac{2t+2-N}{m(m+1)}\,\Delta^2\Psi_m^{(N)}(t)+\frac{(t+2)(N-t-2)+2-m(m+1)}{m(m+1)}\,\Delta\Psi_m^{(N)}(t+1)\,.$$

Note that the coefficient on the left-hand side of the last identity is positive whenever $0<t<N-2$. When $t\geqslant N/2-1$, the coefficient of $\Delta^2\Psi_m^{(N)}(t)$ is nonnegative. The coefficient of $\Delta\Psi_m^{(N)}(t+1)$ decreases when $t\geqslant N/2-2$. However, if $N-3$ is substituted for t, the numerator becomes $N+1-m(m+1)$. It is, therefore, positive whenever $m(m+1)\leqslant N-1$.

Summing up, we see, assuming $m(m+1)\leqslant N-1$, that the coefficient of the left-hand side of (2.11) is positive and that of the coefficients of the right-hand side one is nonnegative and the other is positive whenever

$$(2.12)\qquad N/2-1\leqslant t\leqslant N-3\,.$$

This proves the following lemma:

LEMMA 2.3. – Let $m(m+1)\leqslant N-1$. Then, if (2.12) is satisfied, and if $\Delta^2\Psi_m^{(N)}(t)$ and $\Delta\Psi_m^{(N)}(t+1)$ have the same sign,

$$\Delta^3\Psi_m^{(N)}(t)+\Delta\Psi_m^{(N)}(t+1)$$

also has the same sign.

COROLLARY 2.4. – Under the assumptions of the preceding lemma, a fortiori

$$\Delta^3\Psi_m^{(N)}(t)+2\Delta\Psi_m^{(N)}(t+1)$$

has the same sign as $\Delta^2\Psi_m^{(N)}(t)$ and $\Delta\Psi_m^{(N)}(t+1)$.

3. – An auxiliary function.

The polynomial in t

$$F_m^{(N)}(t)=\Psi_m^{(N)}(t)\Psi_m^{(N)}(t+1)+\frac{t(N-t)}{m(m+1)}\{\Delta\Psi_m^{(N)}(t)\}^2 \tag{3.1}$$

is a discrete counterpart of a similar auxiliary function introduced in order to investigate the local extrema of the Legendre polynomials (see, for instance, [6], p. 163). Clearly,

$$F_m^{(N)}(N-t)=F_m^{(N)}(t)\qquad(\forall t)\,. \tag{3.2}$$

Differencing (3.1), we find

$$\begin{aligned}\Delta F_m^{(N)}(t)&=\{\Psi_m^{(N)}(t)+\Delta\Psi_m^{(N)}(t)\}\{2\Delta\Psi_m^{(N)}(t)+\Delta^2\Psi_m^{(N)}(t)\}+\\&\quad+\frac{N-2t-1}{m(m+1)}\{\Delta\Psi_m^{(N)}(t)\}^2+\frac{(t+1)(N-t-1)}{m(m+1)}\{2\Delta\Psi_m^{(N)}(t)+\Delta^2\Psi_m^{(N)}(t)\}\Delta^2\Psi_m^{(N)}(t)\\&=2\Delta\Psi_m^{(N)}(t)\left\{\frac{(t+1)(N-t-1)}{m(m+1)}\Delta^2\Psi_m^{(N)}(t)+\left(1+\frac{N-2t-1}{2m(m+1)}\right)\Delta\Psi_m^{(N)}(t)+\Psi_m^{(N)}(t)\right\}+\\&\quad+\Delta^2\Psi_m^{(N)}(t)\left\{\frac{(t+1)(N-t-1)}{m(m+1)}\Delta^2\Psi_m^{(N)}(t)+\Delta\Psi_m^{(N)}(t)+\Psi_m^{(N)}(t)\right\};\end{aligned}$$

subtracting from the contents of each pair of curly brackets the left-hand side of (2.9), we find eventually

$$\Delta F_m^{(N)}(t)=\frac{2t-N+1}{m(m+1)}\Delta\Psi_m^{(N)}(t)\,\Delta\Psi_m^{(N)}(t+1)\,. \tag{3.3}$$

Thus, if $t\geqslant(N-1)/2$, then $\Delta F_m^{(N)}(t)\geqslant 0$ unless $\Delta\Psi_m^{(N)}(t)\Delta\Psi_m^{(N)}(t+1)<0$. However, we have the following:

LEMMA 3.1. – Assuming $m(m+1)\leqslant N-1$, if $\Delta\Psi_m^{(N)}(t)\Delta\Psi_m^{(N)}(t+1)<0$, and $(N-1)/2\leqslant t\leqslant N-2$, then

$$\Delta F_m^{(N)}(t)+\Delta F_m^{(N)}(t+1)>0\,. \tag{3.4}$$

PROOF. – Let $\Delta\Psi_m^{(N)}(t) > 0$, the case of $\Delta\Psi_m^{(N)}(t) < 0$ being exactly similar. Thus in our case $\Delta\Psi_m^{(N)}(t+1) < 0$, which also entails

$$\Delta^2\Psi_m^{(N)}(t) < 0 \, . \tag{3.5}$$

Now

$$\Delta\Psi_m^{(N)}(t+2) = \Delta\Psi_m^{(N)}(t+1) + \Delta^3\Psi_m^{(N)}(t) + \Delta^2\Psi_m^{(N)}(t) \, .$$

Consequently, by (3.5) and Lemma 2.3,

$$\Delta\Psi_m^{(N)}(t+2) < 0 \, ,$$

which entails $\Delta F(t+1) > 0$. Hence, by (3.3),

$$\Delta F_m^{(N)}(t) + \Delta F_m^{(N)}(t+1) > \frac{2t-N+1}{m(m+1)} \Delta\Psi_m^{(N)}(t+1) \{\Delta\Psi_m^{(N)}(t) + \Delta\Psi_m^{(N)}(t+2)\} \, .$$

But

$$\Delta\Psi_m^{(N)}(t) + \Delta\Psi_m^{(N)}(t+2) = 2\Delta\Psi_m^{(N)}(t+1) + \Delta^3\Psi_m^{(N)}(t) < 0$$

according to Corollary 2.4. Since $\Delta\Psi_m^{(N)}(t+1) < 0$, (3.4) follows owing to (3.3).

Some particular values of $\Psi_m^{(N)}(t)$, $\Delta\Psi_m^{(N)}(t)$ and $F_m^{(N)}(t)$ are easily computed. By (1.1) and (2.1),

$$\begin{cases} (-1)^m\Psi_m^{(N)}(1) = \Psi_m^{(N)}(N) = 1 \, ; \\ (-1)^m\Psi_m^{(N)}(2) = \Psi_m^{(N)}(N-1) = 1 - \dfrac{m(m+1)}{N-1} \, ; \\ (-1)^m\Psi_m^{(N)}(3) = \Psi_m^{(N)}(N-2) = 1 - \dfrac{2m(m+1)}{N-1} + \dfrac{(m-1)m(m+1)(m+2)}{2(N-1)(N-2)} \, . \end{cases} \tag{3.6}$$

But it follows from (2.1) that

$$\Delta\Psi_m^{(N)}(N-t) = (-1)^{m-1}\Delta\Psi_m^{(N)}(t) \, , \tag{3.7}$$

and either from (3.6) or directly from (2.4) we obtain

$$\begin{cases} (-1)^{m-1}\Delta\Psi_m^{(N)}(1) = \Delta\Psi_m^{(N)}(N-1) = \dfrac{m(m+1)}{N-1} \, ; \\ (-1)^{m-1}\Delta\Psi_m^{(N)}(2) = \Delta\Psi_m^{(N)}(N-2) = \dfrac{m(m+1)}{N-1} - \dfrac{(m-1)m(m+1)(m+2)}{2(N-1)(N-2)} \, . \end{cases} \tag{3.8}$$

Furthermore, by (2.4) and (3.7), if $6 \leqslant m(m+1) \leqslant N-1$,

$$\Delta \Psi_m^{(N)}(N-3) = (-1)^{m-1} \Delta \Psi_m^{(N)}(3) =$$
$$= \frac{m(m+1)}{N-1} - \frac{(m-1)m(m+1)(m+2)}{(N-1)(N-2)} + \frac{2(m+3)!}{3!\,2!\,(m-3)!\,(N-1)^{[3]}} >$$
$$> \frac{m(m+1)}{N-1} - \frac{(m-1)m(m+1)(m+2)}{(N-1)(N-2)},$$

and since $(m-1)(m+2) = m(m+1) - 2 < N-2$ under our assumption,

$$\Delta \Psi_m^{(N)}(N-3) > 0\,, \tag{3.9}$$

and all the more

$$\Delta \Psi_m^{(N)}(N-2) > 0\,. \tag{3.10}$$

According to (3.1), (3.6), and (3.8),

$$F_m^{(N)}(N-1) = 1\,. \tag{3.11}$$

Owing to (3.3) and (3.8), it follows that

$$F_m^{(N)}(N-2) = 1 - \frac{N-3}{N-1}\left\{\frac{m(m+1)}{N-1} - \frac{(m-1)m(m+1)(m+2)}{2(N-1)(N-2)}\right\}. \tag{3.12}$$

But if $6 \leqslant m(m+1) \leqslant N-1$, the first factor in the right-hand side of (3.3) with $t = N-3$ is positive. Consequently, by (3.9) and (3.10),

$$\Delta F_m^{(N)}(N-3) > 0\,.$$

Hence, by (3.12), whenever $6 \leqslant m(m+1) \leqslant N-1$,

$$F_m^{(N)}(N-3) < 1 - \frac{N-3}{N-1}\left\{\frac{m(m+1)}{N-1} - \frac{(m-1)m(m+1)(m+2)}{2(N-1)(N-2)}\right\}. \tag{3.13}$$

In view of Lemma 3.1, it follows from (3.12) and (3.13) that if $6 \leqslant m(m+1) \leqslant N-1$, then

$$F_m^{(N)}(t) \leqslant 1 - \frac{N-3}{N-1}\left\{\frac{m(m+1)}{N-1} - \frac{(m-1)m(m+1)(m+2)}{2(N-1)(N-2)}\right\} \tag{3.14}$$

whenever $(N-1)/2 \leqslant t \leqslant N-2$. Furthermore, owing to (3.2), we have, more generally:

Corollary 3.2. – If $6 \leqslant m(m+1) \leqslant N-1$, then (3.14) holds whenever $2 \leqslant t \leqslant N-2$.

4. – The main theorem.

LEMMA 4.1. – If $6\leqslant m(m+1)\leqslant N-1$ and $2\leqslant t\leqslant N-2$, then

$$\frac{1}{2}\,|\Psi_m^{(N)}(t)+\Psi_m^{(N)}(t+1)|\leqslant 1-\frac{N-3}{2(N-1)}\left\{\frac{m(m+1)}{N-1}-\frac{(m-1)m(m+1)(m+2)}{2(N-1)(N-2)}\right\}. \tag{4.1}$$

PROOF. – According to (3.1),

$$\Psi_m^{(N)}(t)\Psi_m^{(N)}(t+1)=F_m^{(N)}(t)-\frac{t(N-t)}{m(m+1)}\,\{\Delta\Psi_m^{(N)}(t)\}^2\,,$$

or

$$\left\{\frac{\Psi_m^{(N)}(t)+\Psi_m^{(N)}(t+1)}{2}\right\}^2=F_m^{(N)}(t)+\left\{\frac{1}{4}-\frac{t(N-1)}{m(m+1)}\right\}\{\Delta\Psi_m^{(N)}(t)\}^2\,.$$

But, under our assumptions,

$$\frac{1}{4}-\frac{t(N-t)}{m(m+1)}<0\,.$$

Hence

$$\tfrac{1}{2}\,|\Psi_m^{(N)}(t)+\Psi_m^{(N)}(t+1)|<\sqrt{F_m^{(N)}(t)}\,,$$

while it follows from Corollary 3.2 that

$$\sqrt{F_m^{(N)}(t)}\leqslant 1-\frac{N-3}{2(N-1)}\left\{\frac{m(m+1)}{N-1}-\frac{(m-1)m(m+1)(m+2)}{2(N-1)(N-2)}\right\},$$

and (4.1) is an immediate consequence of the last two inequalities.

COROLLARY 4.2. – If $6\leqslant m(m+1)\leqslant N-1$, $2\leqslant t\leqslant N-2$, and

$$\Delta\Psi_m^{(N)}(t)\,\Delta\Psi_m^{(N)}(t+1)\leqslant 0\,, \tag{4.2}$$

then

$$|\Psi_m^{(N)}(t+1)|<\\ 1-\frac{N-3}{2(N-1)}\left\{\frac{m(m+1)}{N-1}-\frac{(m-1)\,m(m+1)(m+2)}{2(N-1)(N-2)}\right\}+\frac{1}{2}\,|\Delta^2\Psi_m^{(N)}(t)|\,. \tag{4.3}$$

PROOF. – Since (4.2) implies $|\Delta\Psi_m^{(N)}(t)|\leqslant|\Delta^2\Psi_m^{(N)}(t)|$, (4.3) is an immediate consequence of (4.1).

THEOREM 4.3. – For any $m \geqslant 0$ and N satisfying $m(m+1) \leqslant N-1$, we have

$$|\Psi_m^{(N)}(t)| \leqslant 1 \qquad (t=1, \ldots, N), \tag{4.4}$$

equality occurring only when $m=0$ or $t=1$ or $t=N$;

$$|\Delta \Psi_m^{(N)}(t)| \leqslant \frac{m(m+1)}{N-1} \qquad (t=1, \ldots, N-1;\ N \geqslant 2), \tag{4.5}$$

equality occuring only when $m=0$, or $t=1$, or $t=N-1$; and

$$|\Delta^2 \Psi_m^{(N)}(t)| \leqslant \frac{(m-1)\,m(m+1)(m+2)}{2(N-1)(N-2)} \qquad (t=1, \ldots, N-2;\ N \geqslant 3), \tag{4.6}$$

equality occuring only when $m \leqslant 1$, or $t=1$, or $t=N-2$.

PROOF. – The proposition is trivial when $m=0$ or $m=1$. When t is equal to $1, 2, N-1$, or N, (4.4) follows from (3.6), because $m(m+1) \leqslant N-1$ is equivalent to

$$(m-1)(m+2) \leqslant N-3\,. \tag{4.7}$$

For the same reason, (4.5) with t equal to $1, 2$, or $N-1$, is a consequence of (3.8), and so is (4.6) with t equal to 1 or $N-2$.

Now we proceed by induction. Given $m \geqslant 2$ and N satisfying $m(m+1) \leqslant N-1$, assume that the statement of the theorem holds with μ and ν substituted for m and N respectively whenever $\mu < m$ and $\mu(\mu+1) \leqslant \nu - 1$. We note that the last inequality holds in particular when $\mu \leqslant m-1$ and $\nu = N-1$.

According to the inductive assumption and owing to (2.5),

$$|\Delta \Psi_m^{(N)}(t)| \leqslant \frac{(m-2)(m-1)}{N-1} + \frac{2(2m-1)}{N-1} = \frac{m(m+1)}{N-1} \qquad (t=1, \ldots, N-1),$$

equality occurring only when $t=1$ or $t=N-1$.

Differencing (2.5) yields

$$\Delta^2 \Psi_m^{(N)}(t) - \Delta^2 \Psi_{m-2}^{(N)}(t) = \frac{2(2m-1)}{N-1}\, \Delta \Psi_{m-1}^{(N-1)}(t)\,.$$

Hence, under the inductive assumption,

$$|\Delta^2 \Psi_m^{(N)}(t)| \leqslant \frac{(m-3)(m-2)(m-1)m}{2(N-1)(N-2)} + \frac{2(2m-1)}{N-1} \cdot \frac{(m-1)m}{N-2} \qquad (t=1, \ldots, N-2)$$

which is nothing else but (4.6), and it is clear that equality occurs only when $t=1$ or $t=N-2$.

What remains to be proved now is only

$$|\Psi_m^{(N)}(t+1)|<1 \qquad (t=2,\dots,N-2),$$

but obviously we can confine ourselves to extreme values of $\Psi_m^{(N)}$ over this set, which enables us to apply Corollary 4.2. Substituting (4.6) in (4.3), we find

$$|\Psi_m^{(N)}(t+1)|<$$
$$1-\frac{N-3}{2(N-1)}\left\{\frac{m(m+1)}{N-1}-\frac{(m-1)m(m+1)(m+2)}{2(N-1)(N-2)}\right\}+\frac{(m-1)m(m+1)(m+2)}{4(N-1)(N-2)}\,.$$

But if $m(m+1)\leqslant N-1$, the right-hand side of this inequality is smaller than, or equal to, 1. Indeed, the last statement is equivalent to

$$\frac{m(m+1)}{N-1}\geqslant\left(\frac{N-1}{N-3}+1\right)\frac{(m-1)m(m+1)(m+2)}{2(N-1)(N-2)}=\frac{(m-1)m(m+1)(m+2)}{(N-1)(N-3)}\,,$$

which in turn is equivalent to (4.7). Hence the proof is complete.

5. – Hahn polynomials.

Following GASPER's [2] notations, for any α and β bigger than -1, and for any n smaller than N, we can write the corresponding Hahn polynomial in the following form

$$Q_n(x;\alpha,\beta,N)=\sum_{k=0}^{n}\frac{(-n)_k(n+\alpha+\beta+1)_k(-x)_k}{k!\,(\alpha+1)_k(-N)_k}\,,$$

where $(a)_0=1$, $(a)_{k+1}=(a)_k(a+k)$ $(k=0,1,\dots)$. These polynomials are orthogonal over the set $\langle 0,1,\dots,N\rangle$ with weights

$$\varrho(x)=\frac{\binom{x+\alpha}{x}\binom{N-x+\beta}{N-x}}{\binom{N+\alpha+\beta+1}{N}} \qquad (x=0,1,\dots,N)$$

and tend to the corresponding Jacobi polynomials when $N\to\infty$.

One verifies easily that identically

(5.1) $$Q_n(0;\alpha,\beta,N)=1$$

and

(5.2) $$Q_n(x;0,0,N)=(-1)^n\Psi_n^{(N+1)}(x+1)\,.$$

Obviously, one can always write a projection formula

$$Q_n(x;\, a,\, b,\, N) = \sum_{k=0}^{n} B(n,\, k)\, Q_k(x;\, \alpha,\, \beta,\, N)\,, \tag{5.3}$$

where the coefficients $B(n, k)$ depend also on α, β, a, and b $(a, b, \alpha, \beta > -1)$.

Using a result by ASKEY and GASPER [1], the latter found some necessary and some sufficient conditions under which the coefficients $B(n, k)$ are all non-negative. In particular, when $-1 < \alpha = \beta \leqslant 0$, these coefficients are all non-negative if, and only if, $a \leqslant b$ and

$$(\alpha + 2)(a^2 + b^2) - 2(\alpha + 1)ab - (\alpha - 2)(a + b) - 4\alpha \geqslant 0\,.$$

When $\alpha = \beta = 0$, the last inequality reduces to

$$a^2 + b^2 - ab + a + b \geqslant 0\,; \tag{5.4}$$

this is the case when $Q_n(x; \alpha, \beta, N)$ is given by (5.2). Then, in view of (5.1),

$$\sum_{k=0}^{n} B(n,\, k) = 1\,,$$

and owing to Theorem 4.3, we arrive, by (5.2) and (5.3), at the following conclusion:

THEOREM 5.1. – If $a \geqslant b > -1$, and if (5.4) is satisfied, then, for every n and N such that $n(n+1) \leqslant N$,

$$|Q_n(x;\; a,\, b,\, N)| \leqslant 1 \qquad (x = 0, 1, \ldots, N)\,,$$

equality being impossible when $0 < x < N$ and $n > 0$.

The shape of the domain defined by $a \geqslant b > -1$ and (5.4) is easily seen, since (5.4) defines the outside of an ellipse which is symmetric with rescpect to the line $y = x$, is centered at the point $(-1, -1)$ and passes through $(0, 0)$ and $(0, -1)$.

Since the Hahn polynomials tend to the corresponding Jacobi polynomials as $N \to \infty$, we must have, for $a \geqslant b$, $a \geqslant -\frac{1}{2}$, and $b > -1$, the inequality

$$|Q_n(x;\, a,\, b,\, N)| \leqslant 1 \qquad (x = 0, 1, \ldots, N)$$

when N is sufficiently large in relation to n. One might. therefore, be tempted to wonder whether the conditions of Theorem 5.1, could not be relaxed by substituting the inequality $a > -\frac{1}{2}$ for (5.4); however, the answer is negative, as can be seen, for instance from

$$Q_n\left(2; -\frac{1}{2}, -\frac{1}{2}, N\right) = 1 - \frac{4n^2}{N} + \frac{4n^2(n^2-1)}{3N(N-1)} = -\frac{5}{3}$$

when $N = n^2$.

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