

Some Properties of Quarter-Symmetric Non-Metric Connection in a Kähler Manifold

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Abstract

In the present paper, we studied the properties of a quarter-symmetric non-metric connection in a Kähler manifold.

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1 Introduction

Golab [1] defined and studied the properties of quarter-symmetric connection on a differentiable manifold. In this series Mishra and Pandey [2] defined quarter-symmetric metric F -connections and studied some of its properties. In 2003, Sengupta and Biswas defined a quarter-symmetric non-metric connection on a Sasakian manifold [3]. Recently, Chaubey and Ojha [4] defined a quarter-symmetric non-metric connection on an almost Hermite manifold. They also have been studied the properties of a quarter-symmetric metric connection in an Einstein manifold [5]. The present paper deals with the different geometrical properties of a Kähler manifold equipped with the quarter-symmetric non-metric connection [4].

2 Preliminaries

If on an even dimensional differentiable manifold $V_n, n = 2m$, of differentiability class C^{r+1} , there exists a vector valued function F of differentiability class C^r , satisfying

$$(a) \quad \overline{X} + X = 0 \quad \text{and} \quad (b) \quad \overline{X} \stackrel{\text{def}}{=} FX, \quad (1)$$

for arbitrary vector field X , then V_n is said to be an almost complex manifold and $\{F\}$ is said to give an almost complex structure to V_n [Ehresmann, 1947,1950], [6]. If g denotes the Hermitian metric such that

$$g(\overline{X}, \overline{Y}) = g(X, Y), \quad (2)$$

for arbitrary vector fields X and Y , then an almost complex manifold endowed with an almost complex structure F , know as an almost Hermite manifold [6].

If we define

$$'F(X, Y) \stackrel{\text{def}}{=} g(\overline{X}, Y), \quad (3)$$

then we have

$$'F(X, Y) + 'F(Y, X) = 0. \quad (4)$$

An almost Hermite manifold satisfying

$$(D_X 'F)(Y, Z) = 0, \quad (5)$$

$$(D_X 'F)(Y, Z) + (D_Y 'F)(Z, X) + (D_Z 'F)(X, Y) = 0, \quad (6)$$

for arbitrary vector fields X, Y, Z ; are respectively called Kähler, almost Kähler manifolds [6].

3 Quarter-symmetric non-metric connection

A linear connection B on (V_n, g) defined as

$$B_X Y = D_X Y + u(Y)FX, \quad (7)$$

for arbitrary vector fields X and Y , is said to be a quarter-symmetric non-metric connection [3], [4] if the torsion tensor S of the connection B and the metric tensor g are given by

$$S(X, Y) = u(Y)FX - u(X)FY \quad (8)$$

and

$$(B_X g)(Y, Z) = -u(Y)g(FX, Z) - u(Z)g(FX, Y), \quad (9)$$

for arbitrary vector fields X, Y, Z ; where u is 1-form on V_n with T as associated vector fields, i.e.,

$$u(X) \stackrel{\text{def}}{=} g(X, T) \tag{10}$$

and D being the Riemannian connection. If in addition

$$B_X F = 0, \tag{11}$$

then B is known as a quarter-symmetric non-metric F -connection.

Theorem 3.1 *An almost Hermite manifold equipped with a quarter-symmetric non-metric F -connection B is an almost Kähler manifold.*

Proof We have,

$$\begin{aligned} X('F(Y, Z)) &= (B_X'F)(Y, Z) +'F(B_X Y, Z) +'F(Y, B_X Z) \\ &= (D_X'F)(Y, Z) +'F(D_X Y, Z) +'F(Y, D_X Z), \end{aligned}$$

then in view of (7), above becomes

$$(B_X'F)(Y, Z) = (D_X'F)(Y, Z) + u(Y)g(X, Z) - u(Z)g(X, Y) \tag{12}$$

In view of (11), (12) becomes

$$(D_X'F)(Y, Z) = u(Z)g(X, Y) - u(Y)g(X, Z) \tag{13}$$

Taking cyclic sum of X, Y, Z in (13), we obtain

$$(D_X'F)(Y, Z) + (D_Y'F)(Z, X) + (D_Z'F)(X, Y) = 0$$

Hence the theorem.

Theorem 3.2 *An almost Hermite manifold equipped with a quarter-symmetric non-metric F -connection is a Hermite manifold.*

Proof Differentiating $FY = \bar{Y}$ with respect to X and using (1) (a), (1) (b) and (7), we get

$$(B_X F)(Y) = (D_X F)(Y) - u(Y)\bar{X} - u(\bar{Y})\bar{X} \tag{14}$$

In consequence of (11), (14) becomes

$$(D_X F)(Y) = u(Y)\bar{X} + u(\bar{Y})\bar{X} \tag{15}$$

The Nijenhuis tensor in an almost Hermite manifold [6] is

$$N(X, Y) = (D_{\bar{X}} F)(Y) - (D_{\bar{Y}} F)(X) - \overline{(D_X F)(Y)} + \overline{(D_Y F)(X)} \tag{16}$$

In view of (1) (a) and (15), (16) becomes

$$N(X, Y) = 0$$

An almost Hermite manifold for which the Nijenhuis tensor vanishes, is called a Hermite manifold [6, pp- 70]. Hence the statement of the theorem.

4 Curvature tensor with respect to the quarter-symmetric non-metric connection

The curvature tensor R with respect to the quarter-symmetric non-metric connection B is defined

$$R(X, Y, Z) = B_X B_Y Z - B_Y B_X Z - B_{[X, Y]} Z$$

which satisfy [3]

$$\begin{aligned} R(X, Y, Z) = & K(X, Y, Z) + (D_X u)(Z)(\bar{Y}) - (D_Y u)(Z)(\bar{X}) \\ & + u(Z) \{ (D_X F)(Y) - (D_Y F)(X) \}, \end{aligned} \quad (17)$$

where

$$K(X, Y, Z) = D_X D_Y Z - D_Y D_X Z - D_{[X, Y]} Z \quad (18)$$

is the curvature tensor of the Riemannian connection D .

If we define

$$'R(X, Y, Z, U) \stackrel{\text{def}}{=} g(R(X, Y, Z), U)$$

and

$$'K(X, Y, Z, U) \stackrel{\text{def}}{=} g(K(X, Y, Z), U),$$

then in a Kähler manifold (17) becomes

$$'R(X, Y, Z, U) = 'K(X, Y, Z, U) + (D_X u)(Z)g(\bar{Y}, U) - (D_Y u)(Z)g(\bar{X}, U) \quad (19)$$

Theorem 4.1 *If a Kähler manifold admitting a quarter-symmetric non-metric connection B , then $(D_X u)(Z)$ is hybrid if and only if any one of the following holds*

- (a) $'R(\bar{X}, \bar{Y}, Z, U) = 'R(X, Y, \bar{Z}, \bar{U})$
- (b) $'R(\bar{X}, \bar{Y}, \bar{Z}, \bar{U}) = 'R(X, Y, Z, U)$
- (c) $'R(X, \bar{Y}, \bar{Z}, U) = 'R(\bar{X}, Y, Z, \bar{U})$.

Proof Barring Z and U in (19) and using (2), we get

$$'R(X, Y, \bar{Z}, \bar{U}) = 'K(X, Y, \bar{Z}, \bar{U}) + (D_X u)(\bar{Z})g(Y, U) - (D_Y u)(\bar{Z})g(X, U) \quad (20)$$

Again barring X and Y in (19) and using (1) (a), we find

$$'R(\bar{X}, \bar{Y}, Z, U) = 'K(\bar{X}, \bar{Y}, Z, U) - (D_{\bar{X}} u)(Z)g(Y, U) - (D_{\bar{Y}} u)(Z)g(X, U) \quad (21)$$

Subtracting (21) from (20) and using

$$'K(\bar{X}, \bar{Y}, Z, U) = 'K(X, Y, \bar{Z}, \bar{U}),$$

we obtain

$$\begin{aligned} {}'R(X, Y, \bar{Z}, \bar{U}) - {}'R(\bar{X}, \bar{Y}, Z, U) &= [(D_X u)(\bar{Z}) + (D_{\bar{X}} u)(Z)]g(Y, U) \\ &\quad - [(D_Y u)(\bar{Z}) + (D_{\bar{Y}} u)(Z)]g(X, U) \end{aligned} \quad (22)$$

We have from (22)

$${}'R(\bar{X}, \bar{Y}, Z, U) = {}'R(X, Y, \bar{Z}, \bar{U})$$

if and only if

$$(D_X u)(\bar{Z}) + (D_{\bar{X}} u)(Z) = 0$$

$$\Leftrightarrow (D_X u)(Z) \text{ is hybrid.}$$

Again barring X, Y, Z and U in (19) and using (1) (a), we obtain

$$\begin{aligned} {}'R(\bar{X}, \bar{Y}, \bar{Z}, \bar{U}) &= {}'K(\bar{X}, \bar{Y}, \bar{Z}, \bar{U}) \\ &\quad - (D_{\bar{X}} u)(\bar{Z})g(Y, \bar{U}) + (D_{\bar{Y}} u)(\bar{Z})g(X, \bar{U}) \end{aligned} \quad (23)$$

Subtracting (19) from (23) and using

$${}'K(\bar{X}, \bar{Y}, \bar{Z}, \bar{U}) = K(X, Y, Z, U),$$

we find

$$\begin{aligned} {}'R(\bar{X}, \bar{Y}, \bar{Z}, \bar{U}) - {}'R(X, Y, Z, U) &= [(D_X u)(Z) - (D_{\bar{X}} u)(\bar{Z})]g(Y, \bar{U}) \\ &\quad + [(D_Y u)(Z) - (D_{\bar{Y}} u)(\bar{Z})]g(\bar{X}, U) \end{aligned} \quad (24)$$

Since ${}'R(\bar{X}, \bar{Y}, \bar{Z}, \bar{U}) = {}'R(X, Y, Z, U)$ if and only if $(D_{\bar{X}} u)(\bar{Z}) = (D_X u)(Z) \Leftrightarrow (D_X u)(Z)$ is hybrid. Also barring Y and Z in (19) and using (1) (a) and (2), we have

$${}'R(X, \bar{Y}, \bar{Z}, U) = {}'K(X, \bar{Y}, \bar{Z}, U) - (D_X u)(\bar{Z})g(Y, U) - (D_{\bar{Y}} u)(\bar{Z})g(\bar{X}, U) \quad (25)$$

In the last, barring X and U in (19) and using (1) (a) and (2), we get

$${}'R(\bar{X}, Y, Z, \bar{U}) = {}'K(\bar{X}, Y, Z, \bar{U}) + (D_{\bar{X}} u)(Z)g(Y, U) + (D_Y u)(Z)g(X, \bar{U}) \quad (26)$$

Subtracting (25) from (26) and using

$${}'K(X, \bar{Y}, \bar{Z}, U) = {}'K(\bar{X}, Y, Z, \bar{U}),$$

we have

$$\begin{aligned} {}'R(\bar{X}, Y, Z, \bar{U}) - {}'R(X, \bar{Y}, \bar{Z}, U) &= [(D_Y u)(Z) - (D_{\bar{Y}} u)(\bar{Z})]g(\bar{X}, U) \\ &\quad - [(D_X u)(\bar{Z}) + (D_{\bar{X}} u)(Z)]g(Y, U) \end{aligned} \quad (27)$$

From (27), we have

$${}'R(\bar{X}, Y, Z, \bar{U}) = {}'R(X, \bar{Y}, \bar{Z}, U)$$

if and only if

$$(D_{\bar{Y}} u)(\bar{Z}) = (D_Y u)(Z)$$

Hence the theorem.

Theorem 4.2 *Let V_n be a Kähler manifold equipped with a quarter-symmetric non-metric connection B whose curvature tensor vanishes, then the 1-form u is a gradient vector field if and only if the manifold is flat.*

Proof In view of (5), (17) becomes

$$R(X, Y, Z) = K(X, Y, Z) + (D_X u)(Z)(\bar{Y}) - (D_Y u)(Z)(\bar{X}) \quad (28)$$

Since $R(X, Y, Z) = 0$, therefore (28) becomes

$$K(X, Y, Z) = (D_Y u)(Z)(\bar{X}) - (D_X u)(Z)(\bar{Y}) \quad (29)$$

If the manifold is flat, then

$$(D_Y u)(Z)(\bar{X}) = (D_X u)(Z)(\bar{Y}) \quad (30)$$

$\Rightarrow u$ is a gradient vector field.

Converse part is obvious from (29) and (30).

Theorem 4.3 *In a Kähler manifold, endowed with a quarter-symmetric non-metric connection B , if the curvature tensor with respect to B vanishes, then we have*

$$K(X, Y, Z) = Ric(\bar{Y}, Z)\bar{X} - Ric(\bar{X}, Z)\bar{Y}.$$

Proof Contracting (29) with respect to X , we get

$$Ric(Y, Z) = -(D_{\bar{Y}} u)(Z)$$

Barring Y in the above relation and then using (1) (a), we get

$$Ric(\bar{Y}, Z) = (D_Y u)(Z) \quad (31)$$

In consequence of (31), (29) gives the required result.

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