Some Properties of String Field Algebra

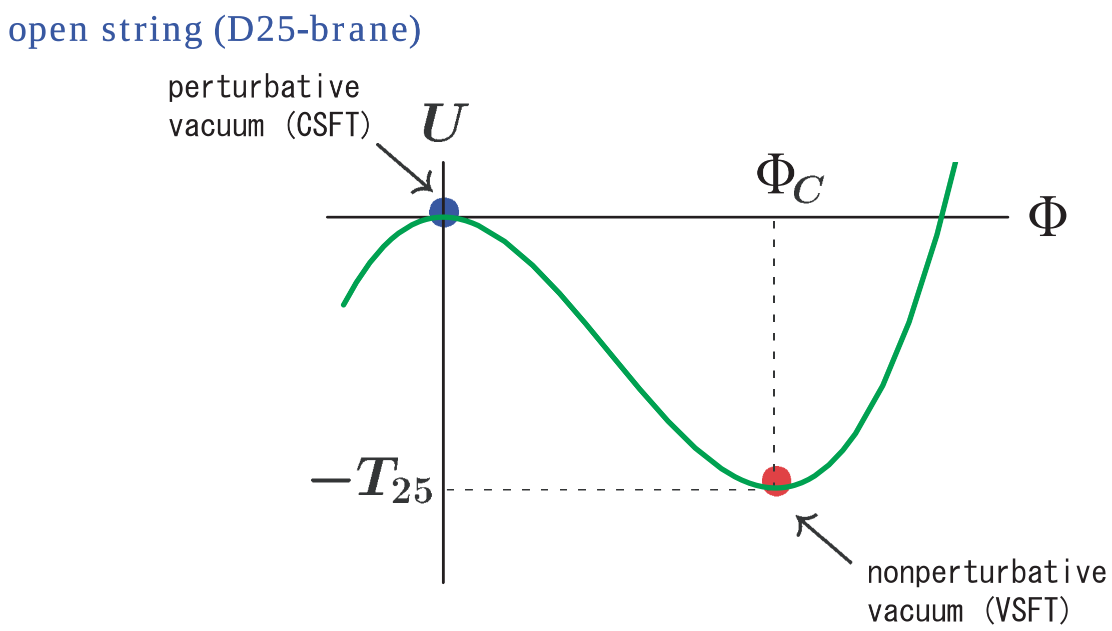
Isao Kishimoto

(YITP, Kyoto Univ.)

based on
I. K., JHEP12(2001)007[hep-th/0110124]
I.K., K.Ohmori, hep-th/0112169

1. Introduction

Sen's conjecture (for bosonic open string field theory)



CSFT(cubic string field theory) Witten

$$S_{\rm CSFT} = -\frac{1}{g_o^2}\left(\frac{1}{2}\langle\Phi, Q_B\Phi\rangle + \frac{1}{3}\langle\Phi, \Phi * \Phi\rangle\right)$$

Sen's conjecture says there is a solution of CSFT Φ_c :
$Q_B\Phi_c + \Phi_c * \Phi_c = 0$ **and** $-S_{\rm CSFT}|_{\Phi_c}/V_{26} = T_{25}$.

VSFT(vacuum string field theory) Rastelli-Sen-Zwiebach (RSZ)

$$S_{\rm VSFT} = -\kappa_0 \left(\frac{1}{2}\langle \Phi, \mathcal{Q}\Phi \rangle + \frac{1}{3}\langle \Phi, \Phi * \Phi \rangle \right)$$

This describes the physics around nonperturbative vacuum (no D25-brane).
$\mathcal{Q}$ should satisfy the following conditions to define a gauge theory

$$\mathcal{Q}^2 = 0, \mathcal{Q}(A * B) = \mathcal{Q}A * B + (-1)^{|A|} A * \mathcal{Q}B, \langle \mathcal{Q}A, B \rangle = -(-1)^{|A|} \langle A, \mathcal{Q}B \rangle$$

and have vanishing cohomology and universality (no matter information).

These requirements are satisfied by

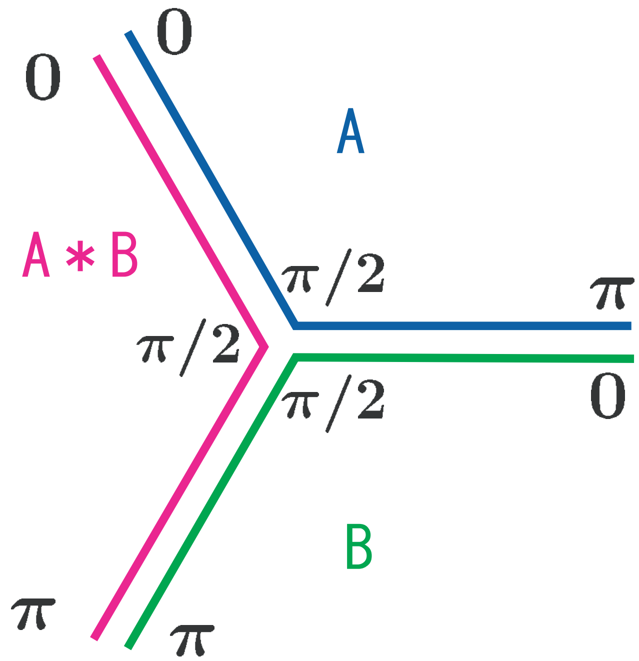
$$\mathcal{Q} = \Sigma_n f_n (c_n + (-1)^n c_{-n}),$$

where f_n is some coefficient. Later, its canonical choice was given by Gaiotto-Rastelli-Sen-Zwiebach (GRSZ):

$$\mathcal{Q} = \frac{1}{2i}(c(i) - c(-i)) = c_0 - (c_2 + c_{-2}) + (c_4 + c_{-4}) + \cdots$$

To realize this scenario, it is necessary to have an analytic solution of CSFT or VSFT which relates them. We investigate Witten's $*$ product for this purpose.

The Witten's $*$ product represents string interaction. This is represented by operator formalism using oscillators or CFT technique.



In the context of VSFT, some techniques using oscillator representation have been developed in *matter part* **especially to construct projectors which satisfy** *reduced* **equation of motion of VSFT :**
$\Phi_M \star \Phi_M = \Phi_M$.
We extend them to *ghost part* **and solve** *full* **equation of motion of VSFT :** $\mathcal{Q}\Phi + \Phi \star \Phi = 0$.

Because $\mathcal{Q}$ is linear in c-ghost, one can take the ansatz $|\Phi_c\rangle = |\Phi_M\rangle|\Phi_G\rangle$ and the e.o.m is reduced to $\Phi_M \star \Phi_M = \Phi_M$ in matter part by assuming the existence of a solution Φ_G in ghost part which satisfies $\mathcal{Q}\Phi_G + \Phi_G \star \Phi_G = 0$. Many authors discussed D-brane solutions of VSFT with this strategy before.

In the context of purely CSFT, Horowitz et.al. discussed (formal) solutions. Using CFT technique, we reexamine them to construct a solution of CSFT which derives GRSZ's proposed VSFT action.

Plan of the talk

§1. **Introduction**

§2. **Oscillator Approach**

Neumann coefficient matrices,
reduced star product,
some formulas for wedge-like states,
application to VSFT,
subtlety of the identity state

§3. **CFT Approach**

Generalized Gluing and Resmoothing Theorem (GGRT),
some formulas for wedge states,
a derivation of VSFT from CSFT

§4. **Summary and Discussion**

2. Oscillator Approach

For two string fields A, B, which are represented by some oscillators on a particular Fock vacuum, we define the Witten's $\star$ product as

$$|A\star B\rangle_1 := {}_2\langle A|_3\langle B|1,2,3\rangle = \langle 2,4|A\rangle_4\langle 3,5|B\rangle_5|1,2,3\rangle,$$

where 3-string vertex $|1,2,3\rangle$ and reflector $\langle 1,2|$ are represented by

$$\begin{aligned}
|V_3\rangle &= |1,2,3\rangle = \tilde{\mu}_3\int d^dp^{(1)}d^dp^{(2)}d^dp^{(3)}(2\pi)^d\delta^d(p^{(1)}+p^{(2)}+p^{(3)})e^{E_3}|0,p\rangle,\\
E_3 &= -\frac{1}{2}\sum_{r,s=1}^{3}\sum_{n.m\geq 1}a_n^{(r)\dagger}V_{nm}^{rs}a_m^{(s)\dagger}-\sum_{r,s=1}^{3}\sum_{n\geq 1}p^{(r)}V_{0n}^{rs}a_n^{(r)\dagger}-\frac{1}{2}\sum_{r,s=1}^{3}p^{(r)}V_{00}^{rs}p^{(s)}-\sum_{r,s=1}^{3}\sum_{n\geq 1,m\geq 0}c_{-n}^{(r)}X_{nm}^{rs}b_{-m}^{(s)},\\
|0,p\rangle &= |0,p^{(1)}\rangle|0,p^{(2)}\rangle|0,p^{(3)}\rangle,\quad b_n^{(i)}|0,p^{(i)}\rangle=0,\quad n\geq 1,\quad c_m^{(i)}|0,p^{(i)}\rangle=0,\quad m\geq 0,\\
\langle V_2| &= \langle 1,2| = \int d^dp^{(1)}d^dp^{(2)}\langle 0,p|e^{E_2}\delta^d(p^{(1)}+p^{(2)})\delta(c_0^{(1)}+c_0^{(2)})\\
E_2 &= -\sum_{n,m\geq 1}a_n^{(1)}C_{nm}a_m^{(2)}-\sum_{n,m\geq 1}(c_n^{(1)}C_{nm}b_m^{(2)}+c_n^{(2)}C_{nm}b_m^{(1)}),\quad \langle 0,p| = {}_1\langle 0,p^{(1)}|_2\langle 0,p^{(2)}|,\quad C_{nm}:=(-1)^n\delta_{n,m}.
\end{aligned}$$

This 3-string vertex is a solution of the connection condition:

$$\left(X^{(r)}(\sigma)-X^{(r-1)}(\pi-\sigma)\right)|V_3\rangle=0,\ \left(P^{(r)}(\sigma)+P^{(r-1)}(\pi-\sigma)\right)|V_3\rangle=0,\quad 0\leq\sigma\leq\frac{\pi}{2},$$
$$\left(c^{\pm(r)}(\sigma)+c^{\pm(r-1)}(\pi-\sigma)\right)|V_3\rangle=0,\quad \left(b^{\pm(r)}(\sigma)-b^{\pm(r-1)}(\pi-\sigma)\right)|V_3\rangle=0,\ r=1,2,3.$$

We can prove useful relations among Neumann coefficients V^{rs}_{nm}, X^{rs}_{nm} :

For the matrices [Gross-Jevicki,Kosteleckey-Potting,RSZ]

$$M_0 := CV^{rr}, \quad M_\pm := CV^{rr\pm1}, \quad \tilde{M}_0 := -CX^{rr}, \quad \tilde{M}_\pm := -CX^{rr\pm1}$$

whose indices run from 1 to ∞, there are some relations

$$\begin{array}{cc} CM_0 = M_0C,\ CM_+ = M_-C, & C\tilde{M}_0 = \tilde{M}_0C,\ C\tilde{M}_+ = \tilde{M}_-C, \\ [M_0, M_\pm] = [M_+, M_-] = 0, & [\tilde{M}_0, \tilde{M}_\pm] = [\tilde{M}_+, \tilde{M}_-] = 0, \\ M_0 + M_+ + M_- = 1, & \tilde{M}_0 + \tilde{M}_+ + \tilde{M}_- = 1, \\ M_+M_- = M_0^2 - M_0, & \tilde{M}_+\tilde{M}_- = \tilde{M}_0^2 - \tilde{M}_0, \\ M_0^2 + M_+^2 + M_-^2 = 1, & \tilde{M}_0^2 + \tilde{M}_+^2 + \tilde{M}_-^2 = 1. \end{array}$$

Neumann coefficient matrices of ghost *nonzero* mode part satisfy the same relations as matter part.

For Neumann coefficients which have zero mode indices, using vector notation, we have found

$$CV^{rs}_{\ \ 0} = V^{sr}_{\ \ 0},\ \sum_{t=1}^{3} V^{ts}_{\ \ 0} = \sum_{t=1}^{3} V^{rt}_{\ \ 0} = 0, \qquad CX^{rs}_{\ \ 0} = X^{sr}_{\ \ 0},\ \sum_{t=1}^{3} X^{ts}_{\ \ 0} = \sum_{t=1}^{3} X^{rt}_{\ \ 0} = 0,$$

$$V^{21}_{\ \ 0} = \frac{3M_+ - 2}{1 + 3M_0} V^{11}_{\ \ 0},\ V^{31}_{\ \ 0} = \frac{3M_- - 2}{1 + 3M_0} V^{11}_{\ \ 0}, \qquad X^{21}_{\ \ 0} = -\frac{\tilde{M}_+}{1 - \tilde{M}_0} X^{11}_{\ \ 0},\ X^{31}_{\ \ 0} = -\frac{\tilde{M}_-}{1 - \tilde{M}_0} X^{11}_{\ \ 0}.$$

Matter part

We consider particular squeezed states : 'wedge-like' state [Furuuchi-Okuyama]

$$|n_\beta\rangle := e^{\beta a^\dagger}|n\rangle = \mu_n \exp\left(\beta a^\dagger - \frac{1}{2}a^\dagger C T_n a^\dagger\right)|0\rangle$$

where $|n\rangle$ is given by the state which is obtained by taking $\star$ product $n-1$ times with a particular squeezed states $|2\rangle$:

$$|n\rangle := (|2\rangle)_\star^{n-1}, \quad |2\rangle = \mu_2 e^{-\frac{1}{2}a^\dagger C T_2 a^\dagger}|0\rangle, \quad CT_2 = T_2 C, \ T_2^T = T_2, \quad [M_0, T_2] = 0, \quad T_2 \neq 1.$$

Here T_n, μ_n are given by

$$T_n = \frac{T(1-T_2T)^{n-1} + (T_2-T)^{n-1}}{(1-T_2T)^{n-1} + T(T_2-T)^{n-1}}, \quad M_0 T^2 - (M_0+1)T + M_0 = 0,$$

$$\mu_n = \mu_2 \left(\mu_2 \mu_3^M \det{}^{-\frac{d}{2}}\left(\frac{1-T}{1-T+T^2}\right)\right)^{n-2} \det{}^{\frac{d}{2}}\left(\frac{1-T^2}{(1-T_2T)^{n-1} + T(T_2-T)^{n-1}}\right),$$

We have $\star$ product formula between them [RSZ]:

$$|n_{\beta_1} \star m_{\beta_2}\rangle = \exp\left(-\mathcal{C}_{n_{\beta_1}, m_{\beta_2}}\right)\left|(n+m-1)_{\beta_1 \rho_{1(n,m)} + \beta_2 \rho_{2(n,m)}}\right\rangle,$$

where

$$\mathcal{C}_{n_{\beta_1}, m_{\beta_2}} = \frac{1}{2}(\beta_1, \beta_2)\frac{C}{T_{n,m}}\begin{pmatrix} M_0(1-T_m) & M_- \\ M_+ & M_0(1-T_n)\end{pmatrix}\begin{pmatrix}\beta_1^T \\ \beta_2^T\end{pmatrix} = \mathcal{C}_{m_{\beta_2 C}, n_{\beta_1 C}},$$

$$\rho_{1(n,m)} = \frac{M_- + M_+ T_m}{T_{n,m}}, \quad \rho_{2(n,m)} = \frac{M_+ + M_- T_n}{T_{n,m}}, \quad C\rho_{1(n,m)} = \rho_{2(m,n)}C, \quad T_{n,m} = 1 + M_0(T_nT_m - T_n - T_m).$$

One can calculate $\star$ product between states of the form $a_k^\dagger \cdots a_l^\dagger |n\rangle$ by differentiating it with parameter β and setting $\beta = 0$ appropriately.

Ghost part

Noting similarity of relations among Neumann coefficients matrices for matter and ghost nonzero modes, we define *reduced* product (denoted as $\star^r$):

$$|A\star^r B\rangle := {}_2\langle A^r|_3\langle B^r|V_3^r\rangle_{123}, \quad \langle A^r| := \langle V_2^r|A\rangle,$$

where we restrict string fields $|A\rangle, |B\rangle$ such that they have no b_0, c_0 modes on the Fock vacuum $|+\rangle_G$. $(c_0|+\rangle_G = 0, b_0|+\rangle_G \neq 0)$
Here we introduced reduced reflector $\langle V_2^r|$ and reduced 3-string vertex $|V_3^r\rangle$ which contain no b_0, c_0 modes on the vacuum ${}_G\langle\tilde{+}|$, $|+\rangle_G$, i.e. they are related with usual reflector and 3-string vertex by

$${}_{12}\langle V_2| = {}_{12}\langle V_2^r|(c_0^{(1)} + c_0^{(2)}), \quad |V_3\rangle_{123} = e^{-\sum_{r,s=1}^3 c^{\dagger(r)} X^{rs}_{0} b_0^{(s)}}|V_3^r\rangle_{123}.$$

Under this $\star^r$ product in ghost part, one can obtain similar formulas to those of matter part as follows.
We define ghost squeezed state $|n_{\xi,\eta}\rangle$ with Grassmann odd parameters ξ, η which corresponds to $|n_\beta\rangle$ in matter part :

$$|n_{\xi,\eta}\rangle := e^{\xi b^\dagger + \eta c^\dagger}|n\rangle_G = \tilde{\mu}_n \exp\left(\xi b^\dagger + \eta c^\dagger + c^\dagger C\tilde{T}_n b^\dagger\right)|+\rangle_G.$$

Here we defined $|n\rangle_G$ as the state which is obtained by taking the $\star^r$ product $n-1$ times with a particular ghost squeezed state $|2\rangle_G$:

$$|n\rangle_G=(|2\rangle_G)^{n-1}_{\star^r},\quad |2\rangle_G=\exp\left(c^\dagger C\tilde{T}_2 b^\dagger\right)|+\rangle_G,\quad C\tilde{T}_2=\tilde{T}_2C,\quad [\tilde{M}_0,\tilde{T}_2]=0,\quad \tilde{T}_2\neq 1,$$

and then we have obtained formulas for $\tilde{T}_n$, $\tilde{\mu}_n$,

$$\tilde{T}_n=\frac{\tilde{T}(1-\tilde{T}_2\tilde{T})^{n-1}+(\tilde{T}_2-\tilde{T})^{n-1}}{(1-\tilde{T}_2\tilde{T})^{n-1}+\tilde{T}(\tilde{T}_2-\tilde{T})^{n-1}},\quad \tilde{M}_0\tilde{T}^2-(\tilde{M}_0+1)\tilde{T}+\tilde{M}_0=0,$$

$$\tilde{\mu}_n=\tilde{\mu}_2\left(\tilde{\mu}_2\tilde{\mu}_3^r\det\left(\frac{1-\tilde{T}}{1-\tilde{T}+\tilde{T}^2}\right)\right)^{n-2}\det\left(\frac{(1-\tilde{T}_2\tilde{T})^{n-1}+\tilde{T}(\tilde{T}_2-\tilde{T})^{n-1}}{1-\tilde{T}^2}\right),$$

by solving the same recurrence equation as that in matter part.

For these ghost squeezed states, we have the $\star^r$ product formula:

$$|n_{\xi,\eta}\star^r m_{\xi',\eta'}\rangle=\exp\left(-\mathcal{C}_{n_{\xi,\eta},m_{\xi',\eta'}}\right)\Big|(n+m-1)_{\xi\tilde{\rho}_{1(n,m)}+\xi'\tilde{\rho}_{2(n,m)},\eta\tilde{\rho}^T_{1(n,m)}+\eta'\tilde{\rho}^T_{2(n,m)}}\Big\rangle,$$

where

$$\mathcal{C}_{n_{\xi,\eta},m_{\xi',\eta'}}=(\xi,\xi')\frac{C}{\tilde{T}_{n,m}}\begin{pmatrix}\tilde{M}_0(1-\tilde{T}_m) & \tilde{M}_-\\ \tilde{M}_+ & \tilde{M}_0(1-\tilde{T}_n)\end{pmatrix}\begin{pmatrix}\eta^T\\ \eta'^T\end{pmatrix}=\mathcal{C}_{m_{\xi'C,\eta'C},n_{\xi C,\eta C}},$$

$$\tilde{\rho}_{1(n,m)}=\frac{\tilde{M}_-+\tilde{M}_+\tilde{T}_m}{\tilde{T}_{n,m}},\quad \tilde{\rho}_{2(n,m)}=\frac{\tilde{M}_++\tilde{M}_-\tilde{T}_n}{\tilde{T}_{n,m}},\quad C\tilde{\rho}_{1(n,m)}=\tilde{\rho}_{2(m,n)}C,\quad \tilde{T}_{n,m}=1+\tilde{M}_0(\tilde{T}_n\tilde{T}_m-\tilde{T}_n-\tilde{T}_m).$$

Using the $\star^r$ product, we get the $\star$ product formula between string fields $|\Phi\rangle = b_0|\phi\rangle,\ |\Psi\rangle = b_0|\psi\rangle$ in the Siegel gauge:

$$
\begin{aligned}
|\Phi\star\Psi\rangle &= |\phi\star^r\psi\rangle + b_0\left({}_2\langle\phi^r|{}_3\langle\psi^r|\sum_{s=1}^{3}c^{(s)\dagger}X^{s1}_{0}|V_3^r\rangle_{123}\right)\\
&= (1+b_0c^\dagger X^{11}_{0})|\phi\star^r\psi\rangle + b_0\sum_{s=2,3}{}_2\langle\phi^r|{}_3\langle\psi^r|c^{(s)\dagger}X^{s1}_{0}|V_3^r\rangle_{123}.
\end{aligned}
$$

Especially, we have obtained $\star$ product formula between squeezed states in ghost part in the Siegel gauge:

$$
\begin{aligned}
&|(b_0n_{\xi,\eta})\star(b_0m_{\xi',\eta'})\rangle\\
&=\left(1+b_0\left(c^\dagger X^{11}_{0}+\left(\xi C+\frac{\partial}{\partial\eta}\tilde{T}_n\right)X^{21}_{0}+\left(\xi' C+\frac{\partial}{\partial\eta'}\tilde{T}_m\right)X^{31}_{0}\right)\right)|n_{\xi,\eta}\star^r m_{\xi',\eta'}\rangle\\
&=\left(1+b_0c^\dagger\frac{1-\tilde{T}_n\tilde{T}_m}{\tilde{T}_{n,m}}X^{11}_{0}-b_0(\xi\tilde{\rho}_{1(n,m)}+\xi'\tilde{\rho}_{2(n,m)})\frac{1}{1-\tilde{M}_0}X^{11}_{0}\right)|n_{\xi,\eta}\star^r m_{\xi',\eta'}\rangle.
\end{aligned}
$$

We can obtain $\star$ product between the states of the form $b_0b_k^\dagger\cdots c_l^\dagger|n\rangle_G$ by differentiating it with respect to parameters ξ,η and setting them zero appropriately.

Later, Okuyama further investigated and rearranged these algebra in the Siegel gauge elegantly. Especially, our $\star^r$ corresponds to his $\star_{b_0}$: $|\Phi\star_{b_0}\Psi\rangle = b_0|\phi\star^r\psi\rangle$.

Application to VSFT

Equation of motion of VSFT:

$$\mathcal{Q}|\Psi\rangle + |\Psi\star\Psi\rangle = 0, \quad \mathcal{Q} = c_0 + \sum_{n=1}^{\infty} f_n(c_n + (-1)^n c_n^\dagger) = c_0 + f\cdot(c + Cc^\dagger).$$

To solve it we set the ansatz in the Siegel gauge :

$$|\Psi\rangle = b_0|P\rangle_M \left(\sum_{n=1}^{\infty} g_n|n\rangle_G\right), \quad |P\star P\rangle_M = |P\rangle_M.$$

As usual, the matter part is factorized and solved by a projector $|P\rangle_M$ which was well investigated earlier. [Gross-Taylor,RSZ,Kawano-Okuyama]

We have obtained some solutions by using previous formulas in ghost part:

1. **identity-like solution**

$$\mathcal{Q} = c_0, \quad |\Psi\rangle = -b_0|P\rangle_M|I^r\rangle_G.$$

2. **sliver-like solution**

$$\mathcal{Q} = c_0 - (c + c^\dagger)\frac{1}{1-\tilde{M}_0}X^{11}_{\ 0}, \quad |\Psi\rangle = -b_0|P\rangle_M|\Xi^r\rangle_G.$$

   This solution was constructed by Hata-Kawano (HK). **(Our formula is simpler than HK's.)**

3. **another solution**

$$\mathcal{Q} = c_0 - (c + c^\dagger)\frac{1}{1-\tilde{M}_0}X^{11}_{\ 0}, \quad |\Psi\rangle = -b_0|P\rangle_M(|I^r\rangle_G - |\Xi^r\rangle_G).$$

Here we denoted as

$$|n=1\rangle_G =: |I^r\rangle_G, \quad |n=\infty\rangle_G =: |\Xi^r\rangle_G,$$

which are analogies of identity and sliver states with respect to the $\star^r$ product:

$$|I^r \star^r A\rangle = |A \star^r I^r\rangle = |A\rangle, \quad |\Xi^r \star^r \Xi^r\rangle = |\Xi^r\rangle.$$

These $|I^r\rangle_G, |\Xi^r\rangle_G$ are *not* the ghost part of identity or sliver state which are defined as surface states.

Later, GRSZ proposed their canonical choice of the kinetic term of VSFT : $\mathcal{Q} = \frac{1}{2i}(c(i) - c(-i))$, and observed that it would coincide with that of HK solution numerically, and then Okuyama proved

$$\frac{1}{2i}(c(i) - c(-i)) = c_0 - (c + c^\dagger)\frac{1}{1-\tilde{M}_0} X^{11}_{\;0}$$

analytically.

GRSZ also observed numerically $|\Xi^r\rangle_G$ would coincide with their sliver state $|\Xi'\rangle_G$ with respect to the $*'$ product on twisted bc-ghost system.

Subtlety of the identity state

The identity state $|\mathcal{I}\rangle$ is defined by

$$(X(\sigma) - X(\pi - \sigma))|\mathcal{I}\rangle = 0, \quad 0 \leq \sigma \leq \pi/2,$$

in matter part and corresponding connection condition in bc-ghost, but there is subtlety which comes from midpoint singularity especially in ghost part.
The identity state $|\mathcal{I}\rangle$ is expected to be the *identity* with respect to the $\star$ product at least naively.

The identity state $|\mathcal{I}\rangle$ in oscillator representation is given as [LPP]

$$\begin{aligned}
\langle \mathcal{I}| &= \mu_{1M}\langle 0|_G\langle \Omega|c_{-1}c_0c_1 \\
&\quad \cdot \int_{\zeta_1\zeta_0\zeta_{-1}} \exp\Big(\frac{1}{2}\sum_{n,m\geq 1}\alpha_n N_{nm}\alpha_m + \sum_{n\geq 2, m\geq -1} c_n \tilde{N}_{nm} b_m - \sum_{i=\pm 1,0, m\geq 1}\zeta_i M_{im} b_m\Big), \\
N_{nm} &= \frac{1}{nm}\oint \frac{dz}{2\pi i} z^{-n} f'(z) \oint \frac{dw}{2\pi i} w^{-m} f'(w) \frac{1}{(f(z)-f(w))^2}, \\
\tilde{N}_{nm} &= \oint \frac{dz}{2\pi i} z^{-n+1}(f'(z))^2 \oint \frac{dw}{2\pi i} w^{-m-2}(f'(w))^{-1} \frac{-1}{f(z)-f(w)}, \\
M_{im} &= \oint \frac{dz}{2\pi i} z^{-m-2}(f'(z))^{-1}(f(z))^{i+1}
\end{aligned}$$

where the map $f(z)$ is defined by $f(z) = \frac{2z}{1-z^2}$.

This formula gives the oscillator representation of the identity state $|\mathcal{I}\rangle$ which is the same as that in Gross-Jevicki(II):

$$|\mathcal{I}\rangle = \frac{1}{4i} b^+\left(\frac{\pi}{2}\right) b^-\left(\frac{\pi}{2}\right)|I\rangle_M |I^r\rangle_G = [b^\dagger]_{\mathcal{O}}\left(b_0 + 2[b^\dagger]_{\mathcal{E}}\right)|I\rangle_M |I^r\rangle_G,$$

$$[\ ]_{\mathcal{E}} := \sum_{n=1}^{\infty}(-1)^n[\ ]_{2n}, \quad [\ ]_{\mathcal{O}} := \sum_{n=0}^{\infty}(-1)^n[\ ]_{2n+1}.$$

By pure oscillator calculation, we can show the following equations :

$$\left(a_n - (-1)^n a_n^\dagger\right)|\mathcal{I}\rangle = 0, \quad \left(b_n - (-1)^n b_n^\dagger\right)|\mathcal{I}\rangle = 0,$$

$$\left(c_{2k} + c_{2k}^\dagger\right)|\mathcal{I}\rangle = (-1)^k 2c_0|\mathcal{I}\rangle, \quad \left(c_{2k+1} - c_{2k+1}^\dagger\right)|\mathcal{I}\rangle = (-1)^k(c_1 - c_{-1})|\mathcal{I}\rangle,$$

$$Q_B|\mathcal{I}\rangle = -\frac{d-26}{2}\sum_{l=1}^{\infty} l c_{2l}^\dagger|\mathcal{I}\rangle + (1-a_0)c_0|\mathcal{I}\rangle = 0. \quad (d = 26, a_0 = 1)$$

Note $|\mathcal{I}\rangle$ is BRST invariant, but $(c_k + (-1)^k c_k^\dagger)|\mathcal{I}\rangle \neq 0$, i.e., there is anomaly for c-ghost in oscillator representation.

If we use the relations among Neumann coefficients *formally*, we have

$$
\begin{aligned}
{}_3\langle \mathcal{I}|1,2,3\rangle &= \mu_1\mu_3\,(\det(1-M_0))^{-\frac{d}{2}}\det(1-\tilde{M}_0)|1,2\rangle_M|1,2\rangle'_G\ (\neq|1,2\rangle),\\
|1,2\rangle_M &= \exp\left(-\sum_{n,m\geq 0}a_n^{\dagger(1)}C_{nm}a_m^{\dagger(2)}\right)|0\rangle_{M12},\\
|1,2\rangle'_G &= (1-2[(1-\tilde{M}_0)^{-1}X^{11}{}_0]_{\mathcal{E}})\,\cdot\\
&\quad\cdot\left([(1-\tilde{M}_0)^{-1}X^{21}{}_0]_{\mathcal{O}}(b_0^{(1)}-b_0^{(2)})-[(1-\tilde{M}_0)^{-1}(\tilde{M}_+b^{\dagger(1)}+\tilde{M}_-b^{\dagger(2)})]_{\mathcal{O}}\right)\cdot\\
&\quad\cdot\exp\left(\sum_{n,m\geq 1}(c_{-n}^{(1)}C_{nm}b_{-m}^{(2)}+c_{-n}^{(2)}C_{nm}b_{-m}^{(1)})\right)e^{\Delta E}|+\rangle_{G12},\\
\Delta E &= -(c^{\dagger(1)}-c^{\dagger(2)})\frac{1}{1-\tilde{M}_0}X^{11}{}_0(b_0^{(1)}-b_0^{(2)}),
\end{aligned}
$$

and this shows the identity state in oscillator representation is *not* the identity with respect to the $\star$ product because ${}_3\langle\mathcal{I}|1,2,3\rangle=|1,2\rangle$ should be satisfied if $\mathcal{I}\star A=A\star\mathcal{I}=A,\ \forall A$.

This would be caused by c-ghost anomaly in oscillator representation. But the above calculation might be subtle because we treated $\infty\times\infty$ matrices as usual number here.

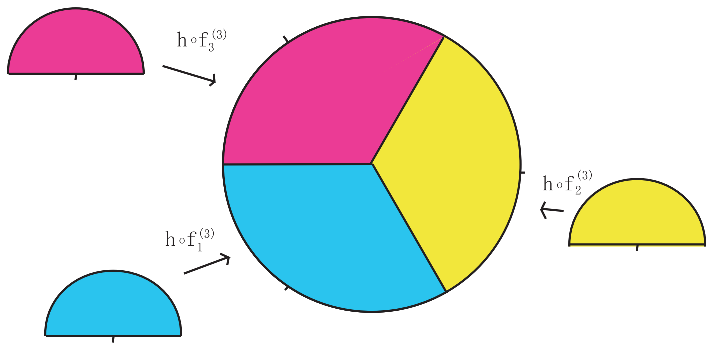
3. CFT Approach

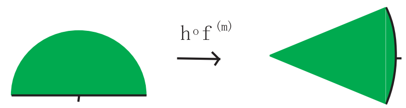
The Witten's $*$ product in CFT language which was developed by LeClair-Peskin-Preitschopf (LPP) is defined as:

$$\langle A, B * C\rangle \;=\; \left\langle f_1^{(3)}\circ A(0)\; f_2^{(3)}\circ B(0)\; f_3^{(3)}\circ C(0)\right\rangle_{\mathrm{UHP}},$$

where conformal maps are given by

$$f_1^{(3)}(z)=h^{-1}\left(e^{-\frac{2}{3}\pi i}h(z)^{\frac{2}{3}}\right),\quad f_2^{(3)}(z)=h^{-1}\left(h(z)^{\frac{2}{3}}\right),\quad f_3^{(3)}(z)=h^{-1}\left(e^{\frac{2}{3}\pi i}h(z)^{\frac{2}{3}}\right),\quad h(z)=\frac{1+iz}{1-iz}.$$





For wedge state $|m\rangle$ which is defined by

$$\langle m, \varphi\rangle = \left\langle f^{(m)}\circ\varphi(0)\right\rangle_{\mathrm{UHP}},\quad f^{(m)}(z)=h^{-1}\left(h(z)^{\frac{2}{m}}\right),$$

we have the $*$ product between them [David]

$$\langle\varphi, m*n\rangle = \langle\varphi, m+n-1\rangle,\quad \forall\varphi.$$

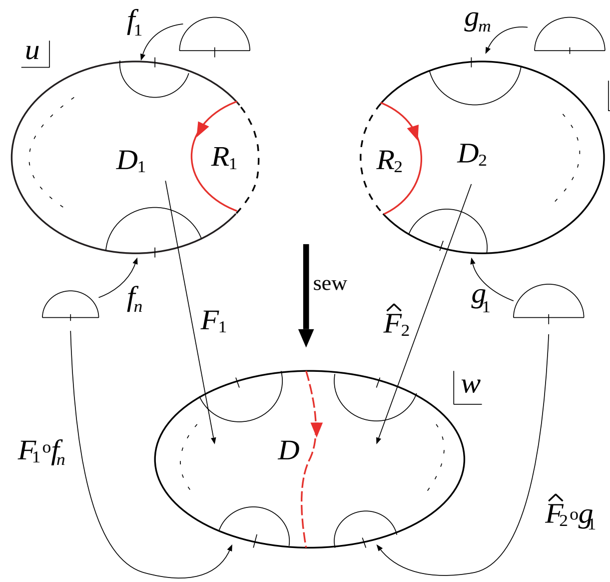
To prove this algebra we followed only the definition of wedge state and generalized gluing and resmoothing theorem (GGRT)[Schwarz-Sen]:

$$\sum_r \langle f_1\circ\Phi_{r_1}(0)\ldots f_n\circ\Phi_{r_n}(0)\; f\circ\Phi_r(0)\rangle_{\mathcal{D}_1}\langle g_1\circ\Phi_{s_1}(0)\ldots g_m\circ\Phi_{s_m}(0)\; g\circ\Phi^c_r(0)\rangle_{\mathcal{D}_2}$$

$$=\left\langle F_1\circ f_1\circ\Phi_{r_1}(0)\ldots F_1\circ f_n\circ\Phi_{r_n}(0)\;\widehat{F}_2\circ g_1\circ\Phi_{s_1}(0)\ldots\widehat{F}_2\circ g_m\circ\Phi_{s_m}(0)\right\rangle_{\mathcal{D}},\qquad F_1\circ f=\hat{F}_2\circ g\circ I.$$

and constructed resmoothing maps $F_1,\hat{F}_2$ concretely.

Our strategy for computation of the $*$ product including a wedge state $|m\rangle$ is as follows. First insert complete set $\sum_r|\Phi_r\rangle\langle\Phi^c_r|$, and then apply GGRT:



$$\langle\varphi,A*(\mathcal{O}_Bm)\rangle=\sum_r\langle\varphi,A*\Phi_r\rangle\langle\Phi^c_r,\mathcal{O}_Bm\rangle$$

$$=\sum_r\left\langle f_1^{(3)}\circ\varphi\; f_2^{(3)}\circ A\; f_3^{(3)}\circ\Phi_r\right\rangle\left\langle f^{(m)}\circ I\circ\mathcal{O}_B\; f^{(m)}\circ\Phi^c_r\right\rangle$$

$$=\left\langle F_1\circ f_1^{(3)}\circ\varphi\; F_1\circ f_2^{(3)}\circ A\;\hat{F}_2\circ f^{(m)}\circ I\circ\mathcal{O}_B\right\rangle.$$

In this case, $F_1,\hat{F}_2$ are given by

$$F_1(z)=h^{-1}\left(e^{\frac{m+2}{m+1}\pi i}h(z)^{\frac{3}{m+1}}\right),\;\hat{F}_2(z)=h^{-1}\left(e^{\frac{m+2}{m+1}\pi i}h(z)^{\frac{m}{m+1}}\right),\quad F_1\circ f_3^{(3)}=\hat{F}_2\circ f^{(m)}\circ I.$$

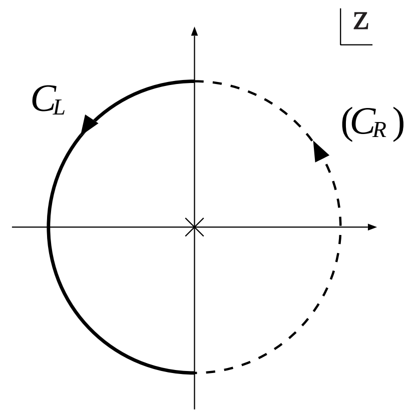
Using this technique, we proved some algebras about the identity state $|\mathcal{I}\rangle = |m=1\rangle$:

$$\langle\varphi, \mathcal{I} * \psi\rangle = \langle\varphi, \psi * \mathcal{I}\rangle = \langle\varphi, \psi\rangle, \quad \langle\varphi, \mathcal{I} * \mathcal{O}\mathcal{I}\rangle = \langle\varphi, \mathcal{O}\mathcal{I} * \mathcal{I}\rangle = \langle\varphi, \mathcal{O}\mathcal{I}\rangle$$

In this sense, we found the identity state $\mathcal{I}$ behaves like the identity with respect to the $*$ product in CFT language.

In the same way, we have checked 'partial integration formula'

$$\langle\varphi, (Q_R A) * B\rangle = -(-1)^{|A|}\langle\varphi, A * (Q_L B)\rangle,$$



even on the wedge state: $|A\rangle = \mathcal{O}_A|m\rangle$ or $|B\rangle = \mathcal{O}_B|m\rangle$.
Here we defined $Q_{L(R)}$ using the primary BRST current j_B as $Q_{L(R)} := \int_{C_{L(R)}} \frac{dz}{2\pi i} j_B(z)$.

From these results we have verified that

$$|\Phi_0\rangle = -Q_L|\mathcal{I}\rangle + \frac{a}{2}\mathcal{Q}^\epsilon|\mathcal{I}\rangle, \quad \left(\mathcal{Q}^\epsilon := \frac{1}{2i}\left(e^{-i\epsilon}c(ie^{i\epsilon}) - e^{i\epsilon}c(-ie^{-i\epsilon})\right)\right)$$

satisfies equation of motion of CSFT :

$$\langle\varphi, Q_B\Phi_0 + \Phi_0 * \Phi_0\rangle = 0, \quad \forall\varphi.$$

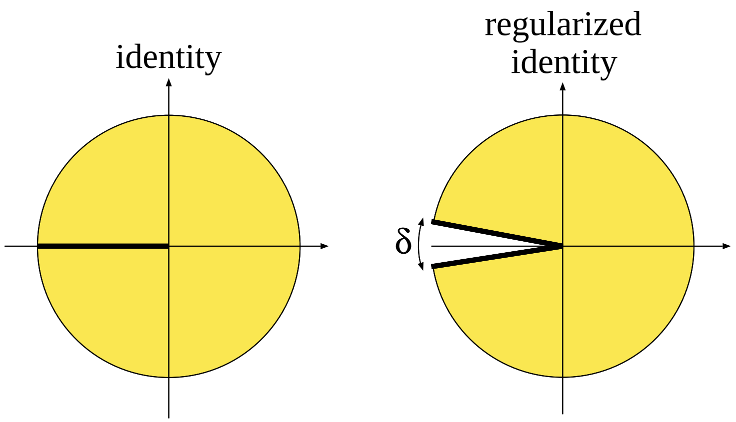
By expanding CSFT action around our solution Φ_0 :

$$S_{\mathrm{CSFT}}|_{\Phi_0+\Psi} = -\frac{1}{g_o^2}\left(\frac{a}{2}\langle\Psi, \mathcal{Q}_\epsilon\Psi\rangle + \frac{1}{3}\langle\Psi, \Psi * \Psi\rangle\right) + S_{\mathrm{CSFT}}|_{\Phi_0},$$

we have derived GRSZ's VSFT action which is regularized by ϵ in the kinetic term:

$$\mathcal{Q}_\epsilon = \frac{1}{4i}\left(e^{-i\epsilon}c(ie^{i\epsilon}) + e^{i\epsilon}c(ie^{-i\epsilon}) - e^{-i\epsilon}c(-ie^{-i\epsilon}) - e^{i\epsilon}c(-ie^{-i\epsilon})\right) \xrightarrow{\epsilon\to 0} \frac{1}{2i}(c(i) - c(-i)).$$

Naively one might think the value of the CSFT action at Φ_0 would be zero, but it may be possible to give a nonzero value for D25-brane tension.



In fact we have

$$\langle\mathcal{Q}^\epsilon\widetilde{\mathcal{I}}_\delta, Q_B\mathcal{Q}^\epsilon\widetilde{\mathcal{I}}_\delta\rangle$$
$$= -\delta^2\sin^2\epsilon\left[\frac{1}{2}\left\{\left(\tan\frac{\epsilon}{2}\right)^{\frac{2}{\delta}} + \left(\tan\frac{\epsilon}{2}\right)^{-\frac{2}{\delta}}\right\} + 3\right]V_{26},$$

where $\widetilde{\mathcal{I}}_\delta$ is regularized identity state which is necessary to apply GGRT. (At $\delta = 0$ this quantity would vanish if one uses equation of motion naively.)

Remark

The solution of the CSFT such as

$$\Psi_c = -Q_L\mathcal{I} + C_L(f)\mathcal{I}, \quad C_L(f) = \int_{C_L} d\sigma f(\sigma)(c(\sigma) + c(-\sigma)), \quad f(\pi - \sigma) = f(\sigma), f\left(\frac{\pi}{2}\right) = 0$$

was considered earlier by Horowitz et.al. in the context of purely cubic SFT, but they treated identity state rather formally (i.e., they treated $\mathcal{I}$ as a formal object which behaves like the identity).

If one uses the equations which were proved formally

$$Q_B\Psi_c + \Psi_c \star \Psi_c = 0, \quad Q_L\mathcal{I} \star Q_L\mathcal{I} = C_L(f)\mathcal{I} \star C_L(f)\mathcal{I} = 0,$$

the value of the action at this solution vanishes :

$$S|_{\Psi_c} \propto \langle \Psi_c, \Psi_c \star \Psi_c \rangle = 0.$$

Recently Takahashi-Tanimoto constructed a solution of CSFT of the form $-Q_L(f)\mathcal{I} + C_L(g)\mathcal{I}, \quad f \neq 1$.

4. Summary and Discussion

We examined Witten's $*$ product both in oscillator and in CFT language.

We constructed solutions of VSFT in oscillator representation and a solution of CSFT in CFT language. The latter one derives GRSZ's VSFT action from Witten's CSFT, but to confirm Sen's conjecture we should obtain D25-brane tension from potential height.

The identity state $\mathcal{I}$ is rather complicated in ghost part in oscillator representation, and *naive* computation (using relations among Neumann coefficient matrices *formally*) gives some unexpected results: for example $\mathcal{I}\star\mathcal{I}=0$.
This subtlety would be caused not only by c-ghost anomaly but also by regarding $\infty\times\infty$ matrices as usual number. We might have to treat them more carefully using Neumann coefficient matrices spectroscopy [RSZ].
On the other hand, we proved some relations expected of the identity state using GGRT in CFT language. But the evaluation of the action including $\mathcal{I}$ is still rather subtle because an appropriate regularization is required.

Appendix

Gaussian integral formula:

matter part (momentum zero sector)

$$\exp\left(\frac{1}{2}aMa+\lambda a\right)\exp\left(\frac{1}{2}a^\dagger Na^\dagger+\mu a^\dagger\right)|0\rangle$$
$$=\frac{1}{\sqrt{\det(1-MN)}}\exp\left(\frac{1}{2}\lambda N(1-MN)^{-1}\lambda+\frac{1}{2}\mu M(1-NM)^{-1}\mu+\lambda(1-NM)^{-1}\mu\right)$$
$$\cdot\exp\left((\lambda N+\mu)(1-MN)^{-1}a^\dagger+\frac{1}{2}a^\dagger N(1-MN)^{-1}a^\dagger\right)|0\rangle,\quad [a_m,a_n^\dagger]=\delta_{mn},\ a_n|0\rangle=0,n\geq 1.$$

ghost part

$$\exp(cAb+c_0\alpha b+c\mu+\nu b+c_0\gamma)\exp(c^\dagger Bb^\dagger+c^\dagger\beta b_0+c^\dagger\rho+\sigma b^\dagger+\delta b_0)|+\rangle=\det(1+BA)\det\Delta\cdot e^{E_1+E_0}|+\rangle,$$
$$\Delta=1+\alpha(1+BA)^{-1}\beta,$$
$$E_1=c^\dagger(1+BA)^{-1}Bb^\dagger+c^\dagger(1+BA)^{-1}(\rho-B\mu)+(\nu B+\sigma)(1+AB)^{-1}b^\dagger$$
$$+\nu(1+BA)^{-1}(\rho-B\mu)-\sigma(1+AB)^{-1}(A\rho+\mu),$$
$$E_0=-c^\dagger(1+BA)^{-1}\beta\Delta^{-1}(\alpha(1+BA)^{-1}Bb^\dagger-b_0)-c^\dagger(1+BA)^{-1}\beta\Delta^{-1}(\alpha(1+BA)^{-1}(\rho-B\mu)+\gamma)$$
$$-((\nu-\sigma A)(1+BA)^{-1}\beta+\delta)\Delta^{-1}(\alpha(1+BA)^{-1}Bb^\dagger-b_0)$$
$$-((\nu-\sigma A)(1+BA)^{-1}\beta+\delta)\Delta^{-1}(\alpha(1+BA)^{-1}(\rho-B\mu)+\gamma),$$
$$\{c_n,b_m\}=\delta_{n+m,0},\ c_n|+\rangle=0,n\geq 0,\ b_n|+\rangle=0,n\geq 1,\qquad c_n^\dagger:=c_{-n},b_n^\dagger:=b_{-n},n\geq 1.$$