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SOME PROPERTIES OF THE JOINT NUMERICAL RANGE OF THE ALUTHGE TRANSFORM

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Abstract: The study of the Aluthge transform $\widetilde{T}$ was introduced and studied by Aluthge in his study of p-hyponormal operators in 1990. Several researchers have since studied various properties of the transform for a single operator T. For instance, quite a lot has been researched on the numerical range of $\widetilde{T}$ of an operator T. In contrast to this, nothing is known about the joint numerical range of Aluthge transform $\widetilde{T}$ of an $m-$tuple operator $T = (T_1, ..., T_m)$. The main reason for this limitation is that the notion of Aluthge transform is still a new area of study. The focus of this paper is on the study of the properties of the joint numerical range of Aluthge transform for an $m-$tuple operator $T = (T_1, ..., T_m)$. Among other results, we show that the joint approximate point spectrum of $\widetilde{T}$ is contained in the closure of the joint numerical range of of $\widetilde{T}$. This study is therefore helpful in the development of the research on numerical ranges and Aluthge transform.

AMS Subject Classification: 47LXX
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1. Introduction

In this paper, $B(X)$ shall denote the algebra of all bounded linear operators acting on a complex Hilbert space X. The Aluthge transform $\widetilde{T}$ of T was first defined by Aluthge [1] in 1990 as the operator $T = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. Note here

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that $T = U|T|$ is any polar decomposition of T with U a partial isometry and $|T| = (T^*T)^{\frac{1}{2}}$. Here, a linear operator $T^* \in B(X)$ defined by the relation $\langle Tx, y\rangle = \langle x, T^*y\rangle \ \forall \ y, x \in X$ will denote the adjoint of an operator $T \in B(X)$. Note that the adjoint of the operator $T^* \in B(X)$ is not the same as the adjoint of matrix A denoted by $Adj(A)$ which is obtained as the transpose of the cofactor matrix and is used in the determination of the inverse of the matrix. This paper studies the joint numerical range of $\widetilde{T}$ of an $m-$tuple operator $T = (T_1, ..., T_m) \in B(X)$ and establishes some of its properties.

The following section is brief survey of the theory of the joint numerical range and a related topic of joint numerical radius of an $m-$tuple operator $T = (T_1, ..., T_m) \in B(X)$.

2. Joint Numerical Range

The concept of numerical range $W(T)$, also known as the classical field of values on a Hilbert space, was introduced in 1918 by Toeplitz [11] for matrices. Since then, a vast amount of research has been pursued for this notion which has resulted to many proofs of the convexity result and other properties of $W(T)$. For instance, Gustafson and Rao [7] used the following theorem to show that $W(T)$ is convex. It is known that a set S is convex if a line segment joining any two points in S is contained in S.

Theorem 1. *(Toeplitz-Hausdorff). The numerical range of an operator is convex.*

Gustafson and Rao [7] proved this by showing that the segment containing any two points in $W(T)$ is contained in $W(T)$.

Dekker [6] extended the notion of numerical range to joint numerical range in 1969. The joint numerical range has since been used by several researchers as a tool to understand the joint behaviour of several operators. The joint numerical range of $T = (T_1, ..., T_m) \in S(X)^m$ is denoted and defined as, $W_m(T) = \{(\langle T_1x, x\rangle, ..., \langle T_mx, x\rangle) : x \in X, \langle x, x\rangle = 1\}$. Here, $S(X)$ is the set of self adjoint operators in $B(X)$.

The joint numerical range has also been studied by researchers such as Dash [5] and Halmos [8] to establish its properties. It is worth noting that the joint numerical range is generally not convex for $m-$tuple of operators (see [3]) though there are cases in which it is convex. Researchers studied the closure of the joint numerical range, $\overline{W_m(T)}$, and concluded that is usually non-convex. See [2] and [4] for this and more.

The following theorems were used to highlight cases where the joint numerical range is convex.

Theorem 2. *If $T = (T_1, ..., T_m)$ is an $m-$tuple of commuting normal operators, then $W_m(T)$ is a convex subset of $\mathbb{C}^m$.*

See Dekker [6] for the proof.

Theorem 3. *Let $\varphi = (\varphi_1, ..., \varphi_m)$ be an $m-$tuple of functions in L^∞. Then $W_m(T)$ of commuting $m-$tuple $T = (T_\varphi, ..., T_\varphi)$ of Toeplitz operators on a classical Hardy space H^2 is convex.*

See Dash [5] for the proof.

Related to the study of the joint numerical range is the notion of the joint numerical radius of an operator $T = (T_1, ..., T_m) \in B(X)$. The joint numerical radius of T is defined as

$$w_m(T) = \sup\{|\langle T_m x, x\rangle| : x \in X, \|x\| = 1\} = \sup\{|\lambda_k| : \lambda_k \in W_m(T)\}, \quad 1 \le k \le m.$$

A lot has been done on the concept of joint numerical radius. For instance, it is known that the joint numerical radius of a self adjoint and normal operator $T = (T_1, ..., T_m) \in B(X)$ is the norm of the operator i.e $w_m(T) = \|T_k\|$.

Theorem 4. *Suppose $T \in B(X)$. Then, $w_m(\mathcal{R}e(T)) \le w_m(T)$ and $w_m(Im(T)) \le w_m(T)$. Here, $\mathcal{R}e$ stands for "real part of " and Im stands for "imaginary part of ".*

Proof. Recall that $\mathcal{R}e(T) = \frac{1}{2}(T+T^*)$ and $Im(T) = \frac{1}{2i}(T-T^*)$. Now, from the definition,

$$\begin{aligned} w_m(\mathcal{R}e(T)) &= \sup\left\{\left|\left\langle\left(\frac{T_k + T_k^*}{2}\right)x, x\right\rangle\right| : x \in X, \|x\| = 1\right\} \\ &\le \sup\left\{\frac{1}{2}\left|\langle T_k x, x\rangle\right| + \frac{1}{2}\left|\langle x, T_k x\rangle\right| : x \in X, \|x\| = 1\right\} \\ &= w_m(T). \end{aligned}$$

Similarly,

$$\begin{aligned} w_m(Im(T)) &= \sup\left\{\left|\left\langle\left(\frac{T_k - T_k^*}{2i}\right)x, x\right\rangle\right| : x \in X, \|x\| = 1\right\} \\ &\le \sup\left\{\frac{1}{2i}\left|\langle T_k x, x\rangle\right| - \frac{1}{2i}\left|\langle x, T_k x\rangle\right| : x \in X, \|x\| = 1\right\} \\ &= w_m(T). \end{aligned}$$

□

The following theorem demonstrates that the joint numerical radius is invariant under unitary equivalence.

Theorem 5. *Suppose $T=(T_1,T_2,...,T_m)\in B(X)$. Then, for every unitary operator $U\in B(X)$, $w_m(UTU^*)=w_m(T)$.*

Proof. Let U be a unitary operator. Recall that $x\in X$ is a unit vector of X if and only if U^*x is a unit vector. Since $\langle UTU^*x,x\rangle=\langle TU^*x,U^*x\rangle=\langle Tx,x\rangle$, the proof follows from the definition of joint numerical radius. □

It is known that $w_m(T)$ is a norm equivalent to the operator norm $\|T_k\|$ which satisfies $\frac{1}{2}\|T_k\|\le w_m(T)\le\|T_k\|$.

3. Joint Numerical Range of Aluthge Transform

While studying the properties of the Aluthge transform, many authors have investigated the relation between numerical range of T and $\widetilde{T}$. For instance, Jung, Ko and Pearcy in [9] showed that $W(\widetilde{T})\subseteq W(T)$ for any T on a two-dimensional space. In [13], Yamazaki showed that $\overline{W(\widetilde{T})}\subseteq\overline{W(T)}$ for an operator T with $\dim\ker T\le\dim\ker T^*$. Wu showed in [12] that the containment $\overline{W(\widetilde{T})}\subseteq\overline{W(T)}$ holds for any operator T on a Hilbert space X. In this section, we focus on the properties of the joint numerical range of $\widetilde{T}$.

Let $T=U|T|$ be a polar decomposition of $T=(T_1,...,T_m)\in B(X)$ and let $r=(r_1,...,r_m)\in\mathbb{C}^m$. The joint numerical range of Aluthge transform is denoted and defined by

$$W_m(\widetilde{T})=\{r_k\in\mathbb{C}^m:(\langle\widetilde{T}_1x,x\rangle,...,\langle\widetilde{T}_mx,x\rangle)=r_k,$$

where $x\in X,\ \|x\|=1$ and $1\le k\le m\}$ and its closure $\overline{W_m(\widetilde{T})}$ defined by

$$\overline{W_m(\widetilde{T})}=\bigcap_{z_k\in\mathbb{C}^m}\{r_k\in\mathbb{C}^m:|r_k-z_k|\le w_m(T_k-z_kI),\ 1\le k\le m\},$$

where $w_m(T_k)$ is the joint numerical radius of an $m-$tuple operator $T=(T_1,...,T_m)\in B(X)$. The joint numerical radius of T_k is defined as

$$w_m(T_k)=\sup\{|r_k|:r_k\in W_m(T),\ 1\le k\le m\}.$$

We define the joint approximate point spectrum $\sigma_\pi(\widetilde{T})$ of Aluthge transform of an operator $T=(T_1,...,T_m)$ as a point $\lambda=(\lambda_i,...,\lambda_m)\in\mathbb{C}^m$ such that for a sequence $\{x_m\}$ of unit vectors in X we have

$$\|(\lambda_i-\widetilde{T}_i)x_m\|\longrightarrow 0\quad(m\longrightarrow\infty),\ \ i=1,...,m.$$

Theorem 6. *The joint approximate point spectrum $\sigma_\pi(\widetilde{T})$ is contained in $\overline{W_m(\widetilde{T})}$*

Proof. Suppose $\lambda = (\lambda_1, ..., \lambda_m) \in \sigma_\pi(\widetilde{T})$. There is a sequence $x_m \in X$ such that $\|(\widetilde{T}_i - \lambda_i)x_m\| \longrightarrow 0 \quad (m \longrightarrow \infty), \;\; i = 1, ..., m.$

Then, by Schwarz inequality,

$$|(\langle\, \widetilde{T}_i x_m, x_m \,\rangle) \; - \lambda_i| \; = \; |(\langle\, (\widetilde{T}_i - \lambda_i)x_m, x_m\rangle)| \leq \; \|(\widetilde{T}_i - \lambda_i)x_m\|$$

Thus $(\langle\, \widetilde{T}_i x_m, x_m \,\rangle) \to \lambda$ as $m \to \infty$.

Therefore, $\lambda \in \overline{W_m(\widetilde{T})}$ and $\sigma_\pi(\widetilde{T}) \subset \overline{W_m(\widetilde{T})}$. □

The immediate consequence of the above theorem is the next corollary which we state without proof.

Corollary 7. *Conv* $\sigma_\pi(\widetilde{T}) \;\subseteq\; \overline{W_m(\widetilde{T})}$.

Here Conv $\sigma_\pi(\widetilde{T})$ denotes the convex hull of the joint approximate point spectrum of the Aluthge transform $\widetilde{T}$.

Theorem 8. *Let $T = V|T|$ and $T = U|T|$ be the polar decompositions of T, where U and V are partial isometries. Then $\widetilde{T} = |T|^{\frac{1}{2}}V|T|^{\frac{1}{2}} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$.*

See [9] for the proof.

It is clear that the joint numerical range of $\widetilde{T}$ is invariant under the unitary equivalence of operators as shown by the following theorem.

Theorem 9. *Let U be a unitary operator on X. Then $W_m(\widetilde{T}) = W_m(U^*\widetilde{T}U)$.*

Proof. If $W_m(\widetilde{T}) = \emptyset$ and $W_m(U^*\widetilde{T}U) = \emptyset$ the result would follow automatically for $r = (r_1, ..., r_m) \in \mathbb{C}^m$.

Let $r = (r_1, ..., r_m) \in W_m(\widetilde{T}) \neq \emptyset$. Then there exists a unit vector $x \in X$ such that

$$(\langle\widetilde{T}_1 x, x\rangle, ..., \langle\widetilde{T}_m x, x\rangle) \to r_k$$

where $x \in X, \; \|x\| = 1$ and $1 \leq k \leq m$.

Note that U is a unitary operator and $x \in X$ is a unit vector if and only if U^*x is a unit vector. Note also that

$$\langle U\widetilde{T}U^*x, x\rangle = \langle\widetilde{T}U^*x, U^*x\rangle = \langle\widetilde{T}x, x\rangle.$$

Also, $\|U^*x\| = 1$ if and only if $\|x\| = 1$.

Then simple computation gives $r = (r_1, ..., r_m) \in W_m(U^*\widetilde{T}U)$. Thus

$$W_m(\widetilde{T}) \subseteq W_m(U^*\widetilde{T}U).$$

Conversely, let $r = (r_1, ..., r_m) \in W_m(U^*\widetilde{T}U) \neq \emptyset$. The prove runs through as above to give $r = (r_1, ..., r_m) \in W_m(\widetilde{T})$.

This implies that $W_m(U^*\widetilde{T}U) \subseteq W_m(\widetilde{T})$ which completes the proof. □

We use the following theorem to show that $W_m(\widetilde{T})$ behaves nicely under the operation of taking the adjoint of an operator.

Theorem 10. $W_m(\widetilde{T^*}) = (W_m(\widetilde{T}))^* = \{\bar{r} \ : r = (r_1, ..., r_m) \in W_m(\widetilde{T})\}$

Proof. Let $r = (r_1, ..., r_m) \in W_m(\widetilde{T})$. Then, there is a unit vector $x \in X$ such that $(\langle \widetilde{T}_1 x, x\rangle, ..., \langle \widetilde{T}_m x, x\rangle) = r_k$ where $x \in X$, $\|x\| = 1$ and $1 \leq k \leq m$. Then

$$|\langle x, x\rangle - \langle \widetilde{T}_k x, \widetilde{T}_k x\rangle| = |\langle (1 - \widetilde{T}_k^* \widetilde{T}_k)x, x\rangle \,|= 0,$$

implies that $\|\sqrt{(1 - \widetilde{T}_k^*\widetilde{T}_k)}\; x\|^2 = 0$. Thus $\|(1 - \widetilde{T}_k^*\widetilde{T}_k)\; x\| = 0$ and $\|\widetilde{T}_k^*\widetilde{T}_k x\| = 1$. Hence $\|\widetilde{T}_k x\| = 1$. Thus,

$$\begin{aligned} |\langle \widetilde{T}_k^*\widetilde{T}_k x, \widetilde{T}_k x\rangle - \langle x, \widetilde{T}_k x\rangle| =& |\langle (\widetilde{T}_k^*\widetilde{T}_k - 1)x, \widetilde{T}_k x\rangle| \\ \leq& \|(\widetilde{T}_k^*\widetilde{T}_k - 1)x\|\|\widetilde{T}_k x\| \\ =& 0. \end{aligned}$$

This implies $\langle \widetilde{T}_k^*\widetilde{T}_k x, \widetilde{T}_k x\rangle = \langle x, \widetilde{T}_k x\rangle = \bar{r}$. Puting $z = \frac{\widetilde{T}_k x}{\|\widetilde{T}_k x\|}$ where z is a unit vector we obtain $\|\widetilde{T}_k^* z\| = 1$ and $\langle \widetilde{T}_k^* z, z\rangle = \bar{r}$. Thus $\bar{r} \in W_m(\widetilde{T^*})$. Hence $(W_m(\widetilde{T}))^* \subseteq W_m(\widetilde{T^*})$. By symmetry, we obtain $W_m(\widetilde{T^*}) \subseteq (W_m(\widetilde{T}))^*$ meaning $W_m(\widetilde{T^*}) = (W_m(\widetilde{T}))^*$ □

We now generalise the theorem by Wu in [12] on single operator case to come up with the following theorem which we state without proof.

Theorem 11. *Let* $T = (T_1, ..., T_m) \in B(X)$. *Then* $\overline{W_m(\widetilde{T})} \subseteq \overline{W_m(T)}$.

It is known (from [10]) that $\|\widetilde{T}\| \subseteq \|T\|$. This leaves us with very limited relationships between joint numerical range of T and $\widetilde{T}$ as the following theorem demonstrates.

Theorem 12. *Let* $T = (T_{1,...,}T_m) \in B(X)$ *and* $|T| = (T^*T)^{\frac{1}{2}}$. *Then* $W_m(\widetilde{T}) \subseteq W_m(T)$.

Proof. There would be nothing to prove if $W_m(\widetilde{T}) = \emptyset$. Therefore, we let $r = (r_1, ..., r_m) \in W_m(\widetilde{T})$. Then, there is a unit vector $x \in X$ such that $\langle \widetilde{T}_k x, x\rangle = r_k$

where $x \in X$, $\|x\| = 1$ and $1 \leq k \leq m$. It follows that, $\||T_k|^{1/2}x\| = \||T_k|^{1/2}\| = 1$ and $\|(1 - |T_k|)x\| = 0, 1 \leq k \leq m$. Also, $\|(1 - |T_k|^3)x\| = 0, 1 \leq k \leq m$. Thus,

$$\|T_k|T_k|^{1/2}x\| = \langle |T_k|^3 x, x\rangle = 1 = \|T_k\|, \quad 1 \leq k \leq m.$$

Also,

$$\begin{aligned} |\langle \widetilde{T}_k x, x\rangle - \langle T_k|T_k|^{1/2}x, |T_k|^{1/2}x\rangle| =& |\langle U|T_k|^{1/2}x, |T_k|^{1/2}x\rangle - \langle T|T_k|^{1/2}x, |T_k|^{1/2}x\rangle| \\ =& |\langle (U|T_k|^{1/2} - U|T_k||T_k|^{1/2})x, |T_k|^{1/2}x\rangle| \\ =& |\langle (U|T_k|^{1/2})(1 - |T_k|)x, |T_k|^{1/2}x\rangle| \\ \leq& \|U|T_k|^{1/2}\|\|(1 - |T_k|)x\|\||T_k|^{1/2}x\| = 0, \end{aligned}$$

where $1 \leq k \leq m$.

If we let $z = (|T_k|^{1/2}x)(\||T_k|^{1/2}x\|)^{-1}$ then $z \in X$ is a unit vector such that $\langle\ T_k z, z\rangle = r_k$ and $\|z\| = 1, 1 \leq k \leq m$. Thus $r = (r_1, ..., r_m) \in W_m(T)$. Hence $W_m(\widetilde{T}) \subseteq W_m(T)$. □

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