SOME PROPERTIES OF THE ZEROS OF BESSEL FUNCTIONS

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1. Let j_{nm} be the mth positive zero of $J_n(x)$ (n not necessarily integral). Then Relton (**1**), p. 59, has conjectured from numerical considerations that

$$j_{1m}^2+j_{1m+1}^2-2j_{2m}^2>0, \qquad (\alpha)$$

$$2j_{1m+1}^2-j_{2m}^2-j_{2m+1}^2>0. \qquad (\beta)$$

It is shown here that certain relations given by Gatteschi (**2**) enable the inequalities to be proved along with the additional relations

$$\lim_{n\to\infty}\{j_{nm}^2+j_{nm+1}^2-2j_{n+1m}^2\}=\frac{\pi^2}{2}+(4n+2), \qquad (\gamma)$$

$$\lim_{n\to\infty}\{2j_{nm+1}^2-j_{n+1m}^2-j_{n+1m+1}^2\}=-\frac{\pi^2}{2}+(4n+2). \qquad (\delta)$$

2. Gatteschi (**2**) has shown that there exists a relation of the form

$$|j_{nm}-x_{nm}+\lambda_n/x_{nm}|<K_n/x_{nm}^3, \qquad (1)$$

where

$$x_{nm}=(m+\tfrac{1}{2}n-\tfrac{1}{4})\pi, \qquad (2)$$

$$\lambda_n=\tfrac{1}{2}(n^2-\tfrac{1}{4}), \qquad (3)$$

and K_n is a function of n only, whose exact form will not be reproduced here. The relation (1) holds provided that

$$\pi j_{nm}>(2n+1)(2n+3). \qquad (4)$$

In particular $\lambda_1=0{\cdot}375$, $\lambda_2=1{\cdot}875$, $K_1=0{\cdot}575$, $K_2=10{\cdot}8$, and the results are applicable if $j_{1m}>4{\cdot}775$, $j_{2m}>11{\cdot}141$. These conditions are certainly satisfied if $m\geqq 3$. If $m=1$ or 2, the relations (α) and (β) can easily be proved numerically. It follows therefore that

$$\begin{aligned}&x_{nm}^2-2\lambda_n+(\lambda_n^2-2\kappa_n)x_{nm}^{-2}+2\lambda_nK_nx_{nm}^{-4}+K_n^2x_{nm}^{-6}\\ &<j_{nm}^2 \qquad (5)\\ &<x_{nm}^2-2\lambda_n+(\lambda_n^2+2\kappa_n)x_{nm}^{-2}-2\lambda_nK_nx_{nm}^{-4}+K_n^2x_{nm}^{-6}.\end{aligned}$$

3. In order to prove (α) and (γ) we apply (5) to j_{nm}, j_{nm+1}, j_{n+1m} and find that

$$j_{nm}^2+j_{nm+1}^2-2j_{n+1m}^2>\frac{\pi^2}{2}+(4n+2)+a_n, \qquad (6)$$

where

$$\begin{aligned}a_n=&(\lambda_n^2-2K_n)(x_{nm}^{-2}+x_{nm+1}^{-2})-2(\lambda_{n+1}^2+2K_{n+1})x_{n+1m}^{-2}\\ &+2\lambda_nK_n(x_{nm}^{-4}+x_{nm+1}^{-4})+4\lambda_{n+1}K_{n+1}x_{n+1m}^{-4}\\ &+K_n^2(x_{nm}^{-6}+x_{nm+1}^{-6})-2K_{n+1}^2x_{n+1m}^{-6}. \qquad (7)\end{aligned}$$

It is easy to see that the left side of (6) is less than $\frac{1}{2}\pi^2+(4n+2)+b_n$, where b_n is a quantity similar to a_n. Both a_n and b_n tend to zero as m tends to infinity and hence (γ) is proved.

Relton's conjecture (α) will be proved if we can show that

$$\tfrac{1}{2}\pi^2+6+a_1>0. \qquad (8)$$

The absolute value μ of the sum of the negative terms in a, is given by

$$\mu=(2K_1-\lambda_1^2)(x_{1m}^{-2}+x_{1m+1}^{-2})+2(2K_2+\lambda_2^2)x_{2m}^{-2}+2K_2^2x_{2m}^{-6}, \qquad (9)$$

and

$$x_{1m+1}^{-1}<x_{1m}^{-1}<x_{13}^{-1}, \qquad (10)$$

$$x_{2m+1}^{-1}<x_{2m}<x_{23}^{-1}, \qquad (11)$$

where $x_{13}=(13\pi)/4\doteqdot 10{\cdot}2$, $x_{23}=(15\pi)/4\doteqdot 11{\cdot}8$. Substituting the appropriate values for the K's and the λ's, we find that

$$\mu<2{\cdot}020x_{13}^{-2}+48{\cdot}2x_{23}^{-2}+106{\cdot}6x_{23}^{-6}, \qquad (12)$$

$$=0{\cdot}02+0{\cdot}35+0{\cdot}0001. \qquad (13)$$

It is clear from this that the negative part of a_1 is less than $\dfrac{\pi^2}{2}+6$, and so the inequality (8) is true and Relton's conjecture (α) is proved. Relton's conjecture (β) can be proved in exactly the same manner, together with the result (δ).

In conclusion, it may be remarked that any number of results similar to (α), (β), (γ), (δ) may be obtained by the methods of this paper.

REFERENCES

(1) F. E. Relton, *Applied Bessel Functions*, Blackie, London (1946).

(2) L. Gatteschi, *Proc. Kon. Ned Acad. Wet.* (A), **55** (1952), 224.

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