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# Some Quantum Estimates of Hermite-Hadamard Inequalities for Quasi-Convex Functions

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**Abstract:** In this paper, we develop some quantum estimates of Hermite-Hadamard type inequalities for quasi-convex functions. In some special cases, these quantum estimates reduce to the known results.

**Keywords:** quantum estimates; Hermite-Hadamard type inequalities; quasi-convex

## 1. Introduction

### 1.1. Current State of Hermite-Hadamard Inequalities

Many important inequalities are established for the class of convex functions [1], but one of the most famous is the so-called Hermite-Hadamard inequality, which was first discovered by Hermite in 1881, and is stated as follows: Let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a convex function, where  $a, b \in I$  with  $a < b$ . Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

This famous result can be considered as a necessary and sufficient condition for a function to be convex. Hermite-Hadamard's inequality has raised many scholars' attention, and a variety of refinements and generalizations have been found (see [1–20]).

In [16], Özdemir used the following lemma and established some estimates on it via quasi-convex functions.

**Lemma 1.** ([16], Lemma 1) Let  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be a twice differentiable mapping on  $I^0$ ,  $a, b \in I$  with  $a < b$  and  $f''$  be integrable on  $[a, b]$ . Then the following equality holds:

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx = \frac{(b-a)^2}{2} \int_0^1 s(1-s)f''(sa+(1-s)b)ds. \quad (1)$$

**Theorem 1.** ([16], Theorem 2) Let  $f : I^0 \subset [0, \infty) \rightarrow \mathbb{R}$  be a twice differentiable mapping on  $I^0$ , such that  $f'' \in L[a, b]$ ,  $a, b \in I$  with  $a < b$ . If  $|f''|^r$  is quasi-convex on  $[a, b]$  for  $r \geq 1$ , then the following inequality holds:

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)^2}{4} \left( \frac{2}{(r+1)(r+2)} \right)^{\frac{r-1}{r}} \left( \sup\{|f''(a)|^r, |f''(b)|^r\} \right)^{\frac{1}{r}}. \quad (2)$$

**Theorem 2.** ([16], Theorem 3) Let  $f : I^0 \subset [0, \infty) \rightarrow \mathbb{R}$  be a twice differentiable mapping on  $I^0$ , such that  $f'' \in L[a, b]$ ,  $a, b \in I$  with  $a < b$ . If  $|f''|^r$  is quasi-convex on  $[a, b]$  for  $r > 1$ , then the following inequality holds:

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{2^{1+\frac{1}{r}}} (\beta(2, p+1))^{\frac{1}{p}} \left( \sup\{|f''(a)|^r, |f''(b)|^r\} \right)^{\frac{1}{r}}, \quad (3)$$

where  $\frac{1}{p} + \frac{1}{r} = 1$  and  $\beta(\cdot, \cdot)$  is Euler Beta Function:

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \quad x, y > 0.$$

In [2], Alomari et al. established the following inequalities through Lemma 1.

**Theorem 3.** ([2], Theorem 3) Let  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be a twice differentiable mapping on  $I^0$ ,  $a, b \in I$  with  $a < b$  and  $f''$  be integrable on  $[a, b]$ . If  $|f''|$  is quasi-convex on  $[a, b]$ , then the following inequality holds:

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{12} \sup\{|f''(a)|, |f''(b)|\}. \quad (4)$$

**Theorem 4.** ([2], Theorem 4) Let  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be a twice differentiable mapping on  $I^0$ ,  $a, b \in I$  with  $a < b$  and  $f''$  be integrable on  $[a, b]$ . If  $|f''|^{p/(p-1)}$  is quasi-convex on  $[a, b]$  for  $p > 1$ , then the following inequality holds:

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{8} \left( \frac{\sqrt{\pi}}{2} \right)^{\frac{1}{p}} \left( \frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)} \right)^{\frac{1}{p}} \left( \sup\{|f''(a)|^r, |f''(b)|^r\} \right)^{\frac{1}{r}}, \quad (5)$$

where  $r = p/(p-1)$ .

**Theorem 5.** ([2], Theorem 5) Let  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be a twice differentiable mapping on  $I^0$ ,  $a, b \in I$  with  $a < b$  and  $f''$  be integrable on  $[a, b]$ . If  $|f''|^r$  is quasi-convex on  $[a, b]$  for  $q \geq 1$ , then the following inequality holds:

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{12} \left( \sup\{|f''(a)|^r, |f''(b)|^r\} \right)^{\frac{1}{r}}. \quad (6)$$

### 1.2. Motivation of Quantum Estimates

In recent years, many researchers have shown their interest in studying and investigating quantum calculus. Quantum analysis has large applications in many mathematical areas such as number theory ([21]), special functions ([22]), quantum mechanics ([23]) and mathematical inequalities. At present,  $q$ -analogues of many identities and inequalities have been established ([13–15,19,20,24]).

The Hermite-Hadamard inequality has been extended by considering its quantum estimates. For example, in [13], Noor et al. established the following lemma and developed some quantum estimates for it.

**Lemma 2.** ([13], Lemma 3.1) Let  $f : I = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$  be a  $q$ -differentiable function on  $I^0$  (the interior of  $I$ ) with  ${}_a D_q$  be continuous and integrable on  $I$  where  $0 < q < 1$ , then

$$\frac{1}{b-a} \int_a^b f(x) {}_a d_q x - \frac{qf(a) + f(b)}{1+q} = \frac{q(b-a)}{1+q} \int_0^1 (1 - (1+q)t) {}_a D_q f((1-t)a + tb) {}_0 d_q t.$$

**Theorem 6.** ([13], Theorem 3.2) Let  $f : I = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$  be a  $q$ -differentiable function on  $I^0$  (the interior of  $I$ ) with  ${}_aD_q$  be continuous and integrable on  $I$  where  $0 < q < 1$ . If  $|{}_aD_q f|^r, r \geq 1$  is a convex function, then

$$\left| \frac{1}{b-a} \int_a^b f(x) {}_a d_q x - \frac{qf(a) + f(b)}{1+q} \right| \leq \frac{q(b-a)}{1+q} \left( \frac{2q}{(1+q)^2} \right)^{1-\frac{1}{r}} \left( \frac{q(1+3q^2+2q^3)}{(1+q+q^2)(1+q)^3} |{}_aD_q f(a)|^r + \frac{q(1+4q+q^2)}{(1+q+q^2)(1+q)^3} |{}_aD_q f(b)|^r \right)^{\frac{1}{r}}.$$

**Theorem 7.** ([13], Theorem 3.3) Let  $f : I = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$  be a  $q$ -differentiable function on  $I^0$  (the interior of  $I$ ) with  ${}_aD_q$  be continuous and integrable on  $I$  where  $0 < q < 1$ . If  $|{}_aD_q f|^r$  is a convex function where  $p, r > 1, \frac{1}{p} + \frac{1}{r} = 1$ , then

$$\left| \frac{1}{b-a} \int_a^b f(x) {}_a d_q x - \frac{qf(a) + f(b)}{1+q} \right| \leq \frac{q(b-a)}{1+q} \left( \frac{2q}{(1+q)^2} \right)^{\frac{1}{p}} \left( \frac{q(1+3q^2+2q^3)}{(1+q+q^2)(1+q)^3} |{}_aD_q f(a)|^r + \frac{q(1+4q+q^2)}{(1+q+q^2)(1+q)^3} |{}_aD_q f(b)|^r \right)^{\frac{1}{r}}.$$

The main purpose of this paper is to use a new quantum integral identity established in [11] to develop some quantum estimates of Hermite-Hadamard type inequalities for quasi-convex functions (Section 3). These quantum estimates of Hermite-Hadamard type inequalities reduces to Theorems 1–5 as  $q \rightarrow 1$ .

### 1.3. Possible Applications of the Estimates

Quantum calculus has large applications in many mathematical areas. We expect these new quantum estimates for Hermite-Hadamard type inequalities to have potential applications in the fields of integral inequalities, approximation theory, special means theory, optimization theory, information theory and numerical analysis.

## 2. Preliminaries

In this section, we first recall some previously known concepts on  $q$ -calculus which will be used in this paper.

Let  $J = [a, b] \subseteq \mathbb{R}$  be an interval and  $0 < q < 1$  be a constant.

**Definition 1.** [19] Assume  $f : J \rightarrow \mathbb{R}$  is a continuous function and let  $x \in J$ . Then  $q$ -derivative on  $J$  of function  $f$  at  $x$  is defined as

$${}_aD_q f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, x \neq a, \quad {}_aD_q f(a) = \lim_{x \rightarrow a} {}_aD_q f(x). \tag{7}$$

We say that  $f$  is  $q$ -differentiable on  $J$  provided  ${}_aD_q f(x)$  exists for all  $x \in J$ . Note that if  $a = 0$  in (2.1), then  ${}_0D_q f = D_q f$ , where  $D_q$  is the well-known  $q$ -derivative of the function  $f(x)$  defined by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}. \tag{8}$$

**Definition 2.** [19] Let  $f : J \rightarrow \mathbb{R}$  be a continuous function. We define the second-order  $q$ -derivative on interval  $J$ , which denoted as  ${}_aD_q^2 f$ , provided  ${}_aD_q f$  is  $q$ -differentiable on  $J$  with  ${}_aD_q^2 f = {}_aD_q({}_aD_q f) : J \rightarrow \mathbb{R}$ . Similarly, we define higher order  $q$ -derivative on  $J, {}_aD_q^n : J_k \rightarrow \mathbb{R}$ .

**Definition 3.** [19] Let  $f : J \subset \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function. Then  $q$ -integral on  $J$  is defined by

$$\int_a^x f(t) {}_a d_q t = (1 - q)(x - a) \sum_{n=0}^{\infty} q^n f(q^n x + (1 - q^n)a) \tag{9}$$

for  $x \in J$ . Moreover, if  $c \in (a, x)$  then the definite  $q$ -integral on  $J$  is defined by

$$\begin{aligned} \int_c^x f(t) {}_a d_q t &= \int_a^x f(t) {}_a d_q t - \int_a^c f(t) {}_a d_q t \\ &= (1 - q)(x - a) \sum_{n=0}^{\infty} q^n f(q^n x + (1 - q^n)a) - (1 - q)(c - a) \sum_{n=0}^{\infty} q^n f(q^n c + (1 - q^n)a). \end{aligned}$$

Note that if  $a = 0$ , then we have the classical  $q$ -integral, which is defined by

$$\int_0^x f(t) {}_0 d_q t = (1 - q)x \sum_{n=0}^{\infty} q^n f(q^n x) \tag{10}$$

for  $x \in [0, +\infty)$ .

**Theorem 8.** [19] Assume that  $f, g : J \rightarrow \mathbb{R}$  are continuous functions,  $\alpha \in \mathbb{R}$ . Then, for  $x \in J$ ,

$$\begin{aligned} \int_a^x [f(t) + g(t)] {}_a d_q t &= \int_a^x f(t) {}_a d_q t + \int_a^x g(t) {}_a d_q t; \\ \int_a^x (\alpha f)(t) {}_a d_q t &= \alpha \int_a^x f(t) {}_a d_q t. \end{aligned}$$

In addition, we introduce the  $q$ -analogues of  $a$  and  $(x - a)^n$  and the definition of  $q$ -Beta function.

**Definition 4.** [22] For any real number  $a$ ,

$$[a]_q = \frac{q^a - 1}{q - 1} \tag{11}$$

is called the  $q$ -analogue of  $a$ . In particular, if  $n \in \mathbb{Z}^+$ , we denote

$$[n] = \frac{q^n - 1}{q - 1} = q^{n-1} + \dots + q + 1.$$

**Definition 5.** [22] If  $n$  is an integer, the  $q$ -analogue of  $(x - a)^n$  is the polynomial

$$(x - a)_q^n = \begin{cases} 1, & \text{if } n = 0, \\ (x - a)(x - qa) \dots (x - q^{n-1}a), & \text{if } n \geq 1. \end{cases} \tag{12}$$

**Definition 6.** [22] For any  $t, s > 0$ ,

$$\beta_q(t, s) = \int_0^1 x^{t-1} (1 - qx)_q^{s-1} {}_0 d_q x \tag{13}$$

is called the  $q$ -Beta function. Note that

$$\beta_q(t, 1) = \int_0^1 x^{t-1} {}_0 d_q x = \frac{1}{[t]}, \tag{14}$$

where  $[t]$  is the  $q$ -analogue of  $t$ .

At last, we present four simple calculations that will be used in this paper.

**Lemma 3.** Let  $f(x) = 1$ , then we have

$$\int_0^1 {}_0d_q x = (1 - q) \sum_{n=0}^{\infty} q^n = 1.$$

**Lemma 4.** Let  $f(x) = x$  for  $x \in [a, b]$ , then we have

$$\int_0^1 x {}_0d_q x = (1 - q) \sum_{n=0}^{\infty} q^{2n} = \frac{1}{1 + q}.$$

**Lemma 5.** Let  $f(x) = 1 - qx$  for  $x \in [0, 1]$  where  $0 < q < 1$  be a constant, then we have

$$\int_0^1 (1 - qx) {}_0d_q x = \int_0^1 {}_0d_q x - q \int_0^1 x {}_0d_q x = \frac{1}{1 + q}.$$

**Lemma 6.** Let  $f(x) = x(1 - qx)$  for  $x \in [0, 1]$  where  $0 < q < 1$  be a constant, then we have

$$\begin{aligned} \int_0^1 x(1 - qx) {}_0d_q x &= \int_0^1 (x - qx^2) {}_0d_q x = \int_0^1 x {}_0d_q x - q \int_0^1 x^2 {}_0d_q x \\ &= \frac{1}{1 + q} - q(1 - q) \sum_{n=0}^{\infty} q^{3n} = \frac{1}{1 + q} - q \frac{1}{1 + q + q^2} \\ &= \frac{1}{(1 + q)(1 + q + q^2)}. \end{aligned}$$

In [6], we can find the notion of quasi-convex functions generalizes the notion of convex functions. More exactly, a function  $f : [a, b] \rightarrow \mathbb{R}$  is said to be quasi-convex on  $[a, b]$  if

$$f((1 - \lambda)x + \lambda y) \leq \sup\{f(x), f(y)\} \tag{15}$$

holds for any  $x, y \in [a, b]$  and  $\lambda \in [0, 1]$ . It's obviously that any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex.

In [11], we have established the following  $q$ -integral identity and used it to prove some quantum estimates of Hermite-Hadamard type inequalities for convex functions.

**Lemma 7.** ([11], Lemma 4.1) Let  $f : I = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$  be a twice  $q$ -differentiable function on  $I^0$  with  ${}_aD_q^2 f$  be continuous and integrable on  $I$  where  $0 < q < 1$ . Then the following identity holds:

$$\frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x) {}_a d_q x = \frac{q^2(b - a)^2}{1 + q} \int_0^1 t(1 - qt) {}_a D_q^2 f((1 - t)a + tb) {}_0 d_q t. \tag{16}$$

**Remark 1.** If  $q \rightarrow 1$  and substitute  $(1 - t)a + tb$  for  $sa + (1 - s)b$ , then (16) reduces to identity (1) in Lemma 1.

### 3. Hermite-Hadamard Inequalities for Quasi-Convex Functions

In this section, we will give some estimates for the left-hand side of the result of (16) through quasi-convex functions.

**Theorem 9.** Let  $f : I = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$  be a twice  $q$ -differentiable function on  $I^0$  with  ${}_aD_q^2 f$  be continuous and integrable on  $I$  where  $0 < q < 1$ . If  $|{}_aD_q^2 f|^r$  is quasi-convex on  $[a, b]$  for  $r \geq 1$ , then the following inequality holds:

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x) {}_a d_q x \right| \\ & \leq \frac{q^2(b - a)^2}{1 + q} \left( \frac{1}{1 + q} \right)^{1 - \frac{1}{r}} \left( h_1 \sup \{ |{}_aD_q^2 f(a)|^r, |{}_aD_q^2 f(b)|^r \} \right)^{\frac{1}{r}}, \end{aligned} \tag{17}$$

where

$$h_1 = (1 - q) \sum_{n=0}^{\infty} q^{2n} (1 - q^{n+1})^r.$$

**Proof.** Using Lemma 7, Hölder’s inequality and the fact that  $|{}_aD_q^2 f|^r$  is a quasi-convex function, we have

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x) {}_a d_q x \right| \\ & = \left| \frac{q^2(b - a)^2}{1 + q} \int_0^1 t(1 - qt) {}_a D_q^2 f((1 - t)a + tb) {}_0 d_q t \right| \\ & \leq \frac{q^2(b - a)^2}{1 + q} \int_0^1 t(1 - qt) |{}_a D_q^2 f((1 - t)a + tb)| {}_0 d_q t \\ & \leq \frac{q^2(b - a)^2}{1 + q} \left( \int_0^1 t {}_0 d_q t \right)^{1 - \frac{1}{r}} \left( \int_0^1 t(1 - qt)^r |{}_a D_q^2 f((1 - t)a + tb)|^r {}_0 d_q t \right)^{\frac{1}{r}} \\ & \leq \frac{q^2(b - a)^2}{1 + q} \left( \int_0^1 t {}_0 d_q t \right)^{1 - \frac{1}{r}} \left( \sup \{ |{}_a D_q^2 f(a)|^r, |{}_a D_q^2 f(b)|^r \} \int_0^1 t(1 - qt)^r {}_0 d_q t \right)^{\frac{1}{r}} \end{aligned}$$

Applying Lemma 4, we have

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x) {}_a d_q x \right| \\ & \leq \frac{q^2(b - a)^2}{1 + q} \left( \frac{1}{1 + q} \right)^{1 - \frac{1}{r}} \left( \sup \{ |{}_a D_q^2 f(a)|^r, |{}_a D_q^2 f(b)|^r \} \int_0^1 t(1 - qt)^r {}_0 d_q t \right)^{\frac{1}{r}} \\ & = \frac{q^2(b - a)^2}{1 + q} \left( \frac{1}{1 + q} \right)^{1 - \frac{1}{r}} \left( h_1 \sup \{ |{}_a D_q^2 f(a)|^r, |{}_a D_q^2 f(b)|^r \} \right)^{\frac{1}{r}}. \end{aligned}$$

It is easy to check that

$$h_1 = \int_0^1 t(1 - qt)^r {}_0 d_q t = (1 - q) \sum_{n=0}^{\infty} q^{2n} (1 - q^{n+1})^r,$$

thus, we get (17).  $\square$

**Remark 2.** If  $q \rightarrow 1$ , then

$$h_1 = \int_0^1 t(1 - t)^r dt = \frac{1}{(r + 1)(r + 2)}.$$

Inequality (17) reduces to inequality (2) in Theorem 1 due to the fact that

$$\begin{aligned} & \frac{(b-a)^2}{2} \left(\frac{1}{2}\right)^{1-\frac{1}{r}} \left(\frac{1}{(r+1)(r+2)}\right)^{\frac{1}{r}} \left(\sup\{|f''(a)|^r, |f''(b)|^r\}\right)^{\frac{1}{r}} \\ &= \frac{(b-a)^2}{4} \left(\frac{2}{(r+1)(r+2)}\right)^{\frac{1}{r}} \left(\sup\{|f''(a)|^r, |f''(b)|^r\}\right)^{\frac{1}{r}}. \end{aligned}$$

**Corollary 1.** In Theorem 9, if  $r$  is a positive integer, then

$$(1-qt)^r \leq (1-qt)_q^r,$$

and (17) reduces to

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x)_a d_q x \right| \\ & \leq \frac{q^2(b-a)^2}{1+q} \left(\frac{1}{1+q}\right)^{1-\frac{1}{r}} \left(\beta_q(2, r+1) \sup\{|{}_a D_q^2 f(a)|^r, |{}_a D_q^2 f(b)|^r\}\right)^{\frac{1}{r}}. \end{aligned}$$

**Theorem 10.** Let  $f : I = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$  be a twice  $q$ -differentiable function on  $I^0$  with  ${}_a D_q^2 f$  be continuous and integrable on  $I$  where  $0 < q < 1$ . If  $|{}_a D_q^2 f|^r$  is quasi-convex on  $[a, b]$  where  $p, r > 1, \frac{1}{p} + \frac{1}{r} = 1$ , then

$$\left| \frac{qf(a) + f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x)_a d_q x \right| \leq \frac{q^2(b-a)^2}{1+q} (l_1)^{\frac{1}{p}} \left( \frac{\sup\{|{}_a D_q^2 f(a)|^r, |{}_a D_q^2 f(b)|^r\}}{1+q} \right)^{\frac{1}{r}}, \quad (18)$$

where

$$l_1 = (1-q) \sum_{n=0}^{\infty} q^{2n} (1-q^{n+1})^p.$$

**Proof.** Using Lemma 7, Hölder’s inequality and the fact that  $|{}_a D_q^2 f|^r$  is a quasi-convex function, we have

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x)_a d_q x \right| \\ &= \left| \frac{q^2(b-a)^2}{1+q} \int_0^1 t(1-qt)_a D_q^2 f((1-t)a + tb)_0 d_q t \right| \\ & \leq \frac{q^2(b-a)^2}{1+q} \int_0^1 t(1-qt) |{}_a D_q^2 f((1-t)a + tb)|_0 d_q t \\ & \leq \frac{q^2(b-a)^2}{1+q} \left( \int_0^1 t(1-qt)^p_0 d_q t \right)^{\frac{1}{p}} \left( \int_0^1 t |{}_a D_q^2 f((1-t)a + tb)|^r_0 d_q t \right)^{\frac{1}{r}} \\ & \leq \frac{q^2(b-a)^2}{1+q} \left( \int_0^1 t(1-qt)^p_0 d_q t \right)^{\frac{1}{p}} \left( \sup\{|{}_a D_q^2 f(a)|^r, |{}_a D_q^2 f(b)|^r\} \int_0^1 t_0 d_q t \right)^{\frac{1}{r}} \end{aligned}$$

Applying Lemma 4, we have

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x) {}_a d_q x \right| \\ & \leq \frac{q^2(b - a)^2}{1 + q} \left( \int_0^1 t(1 - qt)^p {}_0 d_q t \right)^{\frac{1}{p}} \left( \frac{\sup\{|{}_a D_q^2 f(a)|^r, |{}_a D_q^2 f(b)|^r\}}{1 + q} \right)^{\frac{1}{r}} \\ & = \frac{q^2(b - a)^2}{1 + q} (l_1)^{\frac{1}{p}} \left( \frac{\sup\{|{}_a D_q^2 f(a)|^r, |{}_a D_q^2 f(b)|^r\}}{1 + q} \right)^{\frac{1}{r}}. \end{aligned}$$

It is easy to check that

$$l_1 = \int_0^1 t(1 - qt)^p {}_0 d_q t = (1 - q) \sum_{n=0}^{\infty} q^{2n} (1 - q^{n+1})^p,$$

thus, we get (18).  $\square$

**Remark 3.** If  $q \rightarrow 1$ , then

$$l_1 = \int_0^1 t(1 - t)^p dt = \beta(2, p + 1).$$

Inequality (18) reduces to inequality (3) in Theorem 2.

**Corollary 2.** In Theorem 10, if  $p$  is a positive integer and  $p > 1$ , then

$$(1 - qt)^p \leq (1 - qt)_q^p,$$

and (18) reduces to

$$\left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x) {}_a d_q x \right| \leq \frac{q^2(b - a)^2}{1 + q} (\beta_q(2, p + 1))^{\frac{1}{p}} \left( \frac{\sup\{|{}_a D_q^2 f(a)|^r, |{}_a D_q^2 f(b)|^r\}}{1 + q} \right)^{\frac{1}{r}}.$$

**Theorem 11.** Let  $f : I = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$  be a twice  $q$ -differentiable function on  $I^0$  with  ${}_a D_q^2 f$  be continuous and integrable on  $I$  where  $0 < q < 1$ . If  $|{}_a D_q^2 f|^r$  is quasi-convex on  $[a, b]$  where  $p, r > 1, \frac{1}{p} + \frac{1}{r} = 1$ , then the following inequality holds:

$$\left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x) {}_a d_q x \right| \leq \frac{q^2(b - a)^2}{1 + q} (s_1)^{\frac{1}{p}} \left( \sup\{|{}_a D_q^2 f(a)|^r, |{}_a D_q^2 f(b)|^r\} \right)^{\frac{1}{r}}, \quad (19)$$

where

$$s_1 = (1 - q) \sum_{n=0}^{\infty} (q^n)^{p+1} (1 - q^{n+1})^p.$$



**Proof.** Using Lemma 7, Hölder’s inequality and the fact that  $|{}_aD_q^2 f|^r$  is a quasi-convex function, we have

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x) {}_a d_q x \right| \\ &= \left| \frac{q^2(b - a)^2}{1 + q} \int_0^1 t(1 - qt) {}_a D_q^2 f((1 - t)a + tb) {}_0 d_q t \right| \\ &\leq \frac{q^2(b - a)^2}{1 + q} \int_0^1 t(1 - qt) \left| {}_a D_q^2 f((1 - t)a + tb) \right| {}_0 d_q t \\ &\leq \frac{q^2(b - a)^2}{1 + q} \left( \int_0^1 t^p (1 - qt)^p {}_0 d_q t \right)^{\frac{1}{p}} \left( \int_0^1 \left| {}_a D_q^2 f((1 - t)a + tb) \right|^r {}_0 d_q t \right)^{\frac{1}{r}} \\ &\leq \frac{q^2(b - a)^2}{1 + q} \left( \int_0^1 t^p (1 - qt)^p {}_0 d_q t \right)^{\frac{1}{p}} \left( \sup \{ \left| {}_a D_q^2 f(a) \right|^r, \left| {}_a D_q^2 f(b) \right|^r \} \int_0^1 {}_0 d_q t \right)^{\frac{1}{r}} \end{aligned}$$

Applying Lemma 3, we have

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x) {}_a d_q x \right| \\ &\leq \frac{q^2(b - a)^2}{1 + q} (s_1)^{\frac{1}{p}} \left( \sup \{ \left| {}_a D_q^2 f(a) \right|^r, \left| {}_a D_q^2 f(b) \right|^r \} \right)^{\frac{1}{r}}. \end{aligned}$$

It is easy to check that

$$s_1 = \int_0^1 t^p (1 - qt)^p {}_0 d_q t = (1 - q) \sum_{n=0}^{\infty} (q^n)^{p+1} (1 - q^{n+1})^p,$$

thus, we get (19). □

**Remark 4.** If  $q \rightarrow 1$ , then

$$s_1 = \int_0^1 t^p (1 - t)^p dt = \beta(p + 1, p + 1).$$

Using the properties of Beta function, that is,  $\beta(x, x) = 2^{1-2x} \beta\left(\frac{1}{2}, x\right)$  and  $\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(xy)}$ , we can obtain that

$$\beta(p + 1, p + 1) = 2^{1-2(p+1)} \beta\left(\frac{1}{2}, p + 1\right) = 2^{-2p-1} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma(p + 1)}{\Gamma\left(\frac{3}{2} + p\right)},$$

where  $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$  and  $\Gamma(t)$  is Gamma function:

$$\Gamma(t) = \int_0^{\infty} x^{t-1} e^{-x} dx, \quad t > 0.$$

Inequality (19) reduces to inequality (5) in Theorem 4 due to the fact that

$$\begin{aligned} & \frac{(b - a)^2}{2} \left( 2^{-2p-1} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma(p + 1)}{\Gamma\left(\frac{3}{2} + p\right)} \right)^{\frac{1}{p}} \left( \sup \{ |f''(a)|^r, |f''(b)|^r \} \right)^{\frac{1}{r}} \\ &= \frac{(b - a)^2}{8} \left( \frac{\sqrt{\pi}}{2} \right)^{\frac{1}{p}} \left( \frac{\Gamma(1 + p)}{\Gamma\left(\frac{3}{2} + p\right)} \right)^{\frac{1}{p}} \left( \sup \{ |f''(a)|^r, |f''(b)|^r \} \right)^{\frac{1}{r}}. \end{aligned}$$

**Corollary 3.** In Theorem 11, if  $p$  is a positive integer,  $p > 1$ , then

$$(1 - qt)^p \leq (1 - qt)_q^p,$$

and (19) reduces to

$$\left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x)_a d_q x \right| \leq \frac{q^2(b - a)^2}{1 + q} (\beta_q(p + 1, p + 1))^{\frac{1}{p}} \left( \sup \{ |{}_a D_q^2 f(a)|^r, |{}_a D_q^2 f(b)|^r \} \right)^{\frac{1}{r}}.$$

**Theorem 12.** Let  $f : I = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$  be a twice  $q$ -differentiable function on  $I^0$  with  ${}_a D_q^2 f$  be continuous and integrable on  $I$  where  $0 < q < 1$ . If  $|{}_a D_q^2 f|^r$  is quasi-convex on  $[a, b]$  where  $p, r > 1, \frac{1}{p} + \frac{1}{r} = 1$ , then the following inequality holds:

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x)_a d_q x \right| \\ & \leq \frac{q^2(b - a)^2}{1 + q} \left( \frac{1}{[p + 1]} \right)^{\frac{1}{p}} \left( m_1 \sup \{ |{}_a D_q^2 f(a)|^r, |{}_a D_q^2 f(b)|^r \} \right)^{\frac{1}{r}}, \end{aligned} \tag{20}$$

where

$$m_1 = (1 - q) \sum_{n=0}^{\infty} q^n (1 - q^{n+1})^r$$

and  $[p + 1]$  is the  $q$ -analogue of  $p + 1$ .

**Proof.** Using Lemma 7, Hölder’s inequality and the fact that  $|{}_a D_q^2 f|^r$  is a quasi-convex function, we have

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x)_a d_q x \right| \\ & = \left| \frac{q^2(b - a)^2}{1 + q} \int_0^1 t(1 - qt)_a D_q^2 f((1 - t)a + tb)_0 d_q t \right| \\ & \leq \frac{q^2(b - a)^2}{1 + q} \int_0^1 t(1 - qt) |{}_a D_q^2 f((1 - t)a + tb)|_0 d_q t \\ & \leq \frac{q^2(b - a)^2}{1 + q} \left( \int_0^1 t^p {}_0 d_q t \right)^{\frac{1}{p}} \left( \int_0^1 (1 - qt)^r |{}_a D_q^2 f((1 - t)a + tb)|^r {}_0 d_q t \right)^{\frac{1}{r}} \\ & \leq \frac{q^2(b - a)^2}{1 + q} \left( \int_0^1 t^p {}_0 d_q t \right)^{\frac{1}{p}} \left( \sup \{ |{}_a D_q^2 f(a)|^r, |{}_a D_q^2 f(b)|^r \} \int_0^1 (1 - qt)^r {}_0 d_q t \right)^{\frac{1}{r}} \end{aligned}$$

Applying (14) in Definition 6, we have

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x)_a d_q x \right| \\ & \leq \frac{q^2(b - a)^2}{1 + q} \left( \frac{1}{[p + 1]} \right)^{\frac{1}{p}} \left( \sup \{ |{}_a D_q^2 f(a)|^r, |{}_a D_q^2 f(b)|^r \} \int_0^1 (1 - qt)^r {}_0 d_q t \right)^{\frac{1}{r}} \\ & = \frac{q^2(b - a)^2}{1 + q} \left( \frac{1}{[p + 1]} \right)^{\frac{1}{p}} \left( m_1 \sup \{ |{}_a D_q^2 f(a)|^r, |{}_a D_q^2 f(b)|^r \} \right)^{\frac{1}{r}}. \end{aligned}$$

It is easy to check that

$$m_1 = \int_0^1 (1 - qt)^r {}_0 d_q t = (1 - q) \sum_{n=0}^{\infty} q^n (1 - q^{n+1})^r,$$

thus, we get (20).  $\square$

**Remark 5.** If  $q \rightarrow 1$ , then

$$m_1 = \int_0^1 (1-t)^r dt = \frac{1}{r+1},$$

and (20) reduces to

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{2} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left( \frac{\sup\{|f''(a)|^r, |f''(b)|^r\}}{r+1} \right)^{\frac{1}{r}}. \tag{21}$$

**Corollary 4.** In Theorem 12, if  $r$  is a positive integer,  $r > 1$ , then

$$(1-qt)^r \leq (1-qt)_q^r,$$

and (20) reduces to

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x)_a d_q x \right| \\ & \leq \frac{q^2(b-a)^2}{1+q} \left( \frac{1}{[p+1]} \right)^{\frac{1}{p}} \left( \beta_q(1, r+1) \sup\{|{}_a D_q^2 f(a)|^r, |{}_a D_q^2 f(b)|^r\} \right)^{\frac{1}{r}}. \end{aligned}$$

**Theorem 13.** Let  $f : I = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$  be a twice  $q$ -differentiable function on  $I^0$  with  ${}_a D_q^2 f$  be continuous and integrable on  $I$  where  $0 < q < 1$ . If  $|{}_a D_q^2 f|^r$  is quasi-convex on  $[a, b]$  where  $p, r > 1, \frac{1}{p} + \frac{1}{r} = 1$ , then the following inequality holds:

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x)_a d_q x \right| \\ & \leq \frac{q^2(b-a)^2}{1+q} (n_1)^{\frac{1}{p}} \left( \frac{\sup\{|{}_a D_q^2 f(a)|^r, |{}_a D_q^2 f(b)|^r\}}{[r+1]} \right)^{\frac{1}{r}}, \end{aligned} \tag{22}$$

where

$$n_1 = (1-q) \sum_{n=0}^{\infty} q^n (1-q^{n+1})^p$$

and  $[r+1]$  is the  $q$ -analogue of  $r+1$ .

**Proof.** Using Lemma 7, Hölder’s inequality and the fact that  $|{}_a D_q^2 f|^r$  is a quasi-convex function, we have

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x)_a d_q x \right| \\ & = \left| \frac{q^2(b-a)^2}{1+q} \int_0^1 t(1-qt) {}_a D_q^2 f((1-t)a + tb)_0 d_q t \right| \\ & \leq \frac{q^2(b-a)^2}{1+q} \int_0^1 t(1-qt) |{}_a D_q^2 f((1-t)a + tb)|_0 d_q t \\ & \leq \frac{q^2(b-a)^2}{1+q} \left( \int_0^1 (1-qt)^p {}_0 d_q t \right)^{\frac{1}{p}} \left( \int_0^1 t^r |{}_a D_q^2 f((1-t)a + tb)|_0^r d_q t \right)^{\frac{1}{r}} \\ & \leq \frac{q^2(b-a)^2}{1+q} \left( \int_0^1 (1-qt)^p {}_0 d_q t \right)^{\frac{1}{p}} \left( \sup\{|{}_a D_q^2 f(a)|^r, |{}_a D_q^2 f(b)|^r\} \int_0^1 t^r {}_0 d_q t \right)^{\frac{1}{r}} \end{aligned}$$

Applying (14) in Definition 6, we have

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x)_a d_q x \right| \\ & \leq \frac{q^2(b - a)^2}{1 + q} \left( \int_0^1 (1 - qt)^p {}_0d_q t \right)^{\frac{1}{p}} \left( \frac{\sup\{|{}_aD_q^2 f(a)|^r, |{}_aD_q^2 f(b)|^r\}}{[r + 1]} \right)^{\frac{1}{r}} \\ & = \frac{q^2(b - a)^2}{1 + q} (n_1)^{\frac{1}{p}} \left( \frac{\sup\{|{}_aD_q^2 f(a)|^r, |{}_aD_q^2 f(b)|^r\}}{[r + 1]} \right)^{\frac{1}{r}}. \end{aligned}$$

It is easy to check that

$$n_1 = \int_0^1 (1 - qt)^p {}_0d_q t = (1 - q) \sum_{n=0}^{\infty} q^n (1 - q^{n+1})^p,$$

thus, we get (22).  $\square$

**Remark 6.** If  $q \rightarrow 1$ , then

$$n_1 = \int_0^1 (1 - t)^p dt = \frac{1}{p + 1},$$

and (22) reduces to (21) in Remark 5.

**Corollary 5.** In Theorem 13, if  $p$  is a positive integer,  $p > 1$ , then

$$(1 - qt)^p \leq (1 - qt)_q^p,$$

and (22) reduces to

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x)_a d_q x \right| \\ & \leq \frac{q^2(b - a)^2}{1 + q} (\beta_q(1, p + 1))^{\frac{1}{p}} \left( \frac{\sup\{|{}_aD_q^2 f(a)|^r, |{}_aD_q^2 f(b)|^r\}}{[r + 1]} \right)^{\frac{1}{r}}. \end{aligned}$$

**Theorem 14.** Let  $f : I = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$  be a twice  $q$ -differentiable function on  $I^0$  with  ${}_aD_q^2 f$  be continuous and integrable on  $I$  where  $0 < q < 1$ . If  $|{}_aD_q^2 f|^r$  is quasi-convex on  $[a, b]$  for  $r \geq 1$ , then

$$\left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x)_a d_q x \right| \leq \frac{q^2(b - a)^2}{1 + q} (\mu_1)^{\frac{1}{r}} \left( \sup\{|{}_aD_q^2 f(a)|^r, |{}_aD_q^2 f(b)|^r\} \right)^{\frac{1}{r}}, \quad (23)$$

where

$$\mu_1 = (1 - q) \sum_{n=0}^{\infty} (q^n)^{r+1} (1 - q^{n+1})^r.$$

**Proof.** Using Lemma 7, Hölder’s inequality and the fact that  $|{}_aD_q^2f|^r$  is a quasi-convex function, we have

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x)_a d_q x \right| \\ &= \left| \frac{q^2(b - a)^2}{1 + q} \int_0^1 t(1 - qt)_a D_q^2 f((1 - t)a + tb)_0 d_q t \right| \\ &\leq \frac{q^2(b - a)^2}{1 + q} \left( \int_0^1 {}_0d_q t \right)^{1 - \frac{1}{r}} \left( \int_0^1 |t(1 - qt)|^r |{}_aD_q^2 f((1 - t)a + tb)|^r {}_0d_q t \right)^{\frac{1}{r}} \\ &\leq \frac{q^2(b - a)^2}{1 + q} \left( \int_0^1 {}_0d_q t \right)^{1 - \frac{1}{r}} \left( \sup\{|{}_aD_q^2 f(a)|^r, |{}_aD_q^2 f(b)|^r\} \right)^{\frac{1}{r}} \left( \int_0^1 |t(1 - qt)|^r {}_0d_q t \right)^{\frac{1}{r}} \end{aligned}$$

Applying Lemma 3, we have

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x)_a d_q x \right| \\ &\leq \frac{q^2(b - a)^2}{1 + q} \left( \sup\{|{}_aD_q^2 f(a)|^r, |{}_aD_q^2 f(b)|^r\} \right)^{\frac{1}{r}} \left( \int_0^1 |t(1 - qt)|^r {}_0d_q t \right)^{\frac{1}{r}} \\ &= \frac{q^2(b - a)^2}{1 + q} \left( \sup\{|{}_aD_q^2 f(a)|^r, |{}_aD_q^2 f(b)|^r\} \right)^{\frac{1}{r}} (\mu_1)^{\frac{1}{r}}. \end{aligned}$$

It is easy to check that

$$\mu_1 = \int_0^1 t^r(1 - qt)^r {}_0d_q t = (1 - q) \sum_{n=0}^{\infty} (q^n)^{r+1} (1 - q^{n+1})^r,$$

thus, we get (23). □

**Remark 7.** If  $q \rightarrow 1$ , then

$$\mu_1 = \int_0^1 t^r(1 - t)^r dx = \beta(r + 1, r + 1),$$

and (23) reduces to

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) dx \right| \leq \frac{(b - a)^2}{2} (\beta(r + 1, r + 1))^{\frac{1}{r}} \left( \sup\{|f''(a)|^r, |f''(b)|^r\} \right)^{\frac{1}{r}}.$$

**Corollary 6.** In Theorem 14, if  $r$  is a positive integer, then

$$(1 - qt)^r \leq (1 - qt)_q^r,$$

and (23) reduces to

$$\left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x)_a d_q x \right| \leq \frac{q^2(b - a)^2}{1 + q} (\beta_q(r + 1, r + 1))^{\frac{1}{r}} \left( \sup\{|{}_aD_q^2 f(a)|^r, |{}_aD_q^2 f(b)|^r\} \right)^{\frac{1}{r}}.$$

**Theorem 15.** Let  $f : I = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$  be a twice  $q$ -differentiable function on  $I^0$  with  ${}_aD_q^2 f$  be continuous and integrable on  $I$  where  $0 < q < 1$ . If  $\left|{}_aD_q^2 f\right|^r$  is quasi-convex on  $[a, b]$  for  $r \geq 1$ , then

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x) {}_a d_q x \right| \\ & \leq \frac{q^2(b - a)^2}{1 + q} \left( \frac{1}{1 + q} \right)^{1 - \frac{1}{r}} \left( \beta_q(r + 1, 2) \sup\{ \left|{}_aD_q^2 f(a)\right|^r, \left|{}_aD_q^2 f(b)\right|^r \} \right)^{\frac{1}{r}}. \end{aligned} \tag{24}$$

**Proof.** Using Lemma 7, Hölder’s inequality and the fact that  $\left|{}_aD_q^2 f\right|^r$  is a quasi-convex function, we have

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x) {}_a d_q x \right| \\ & = \left| \frac{q^2(b - a)^2}{1 + q} \int_0^1 t(1 - qt) {}_aD_q^2 f((1 - t)a + tb) {}_0 d_q t \right| \\ & \leq \frac{q^2(b - a)^2}{1 + q} \int_0^1 t(1 - qt) \left| {}_aD_q^2 f((1 - t)a + tb) \right| {}_0 d_q t \\ & \leq \frac{q^2(b - a)^2}{1 + q} \left( \int_0^1 (1 - qt) {}_0 d_q t \right)^{1 - \frac{1}{r}} \left( \int_0^1 (1 - qt) t^r \left| {}_aD_q^2 f((1 - t)a + tb) \right|^r \right)^{\frac{1}{r}} \\ & \leq \frac{q^2(b - a)^2}{1 + q} \left( \int_0^1 (1 - qt) {}_0 d_q t \right)^{1 - \frac{1}{r}} \left( \sup\{ \left|{}_aD_q^2 f(a)\right|^r, \left|{}_aD_q^2 f(b)\right|^r \} \int_0^1 (1 - qt) t^r {}_0 d_q t \right)^{\frac{1}{r}} \end{aligned}$$

Applying Lemma 5 and the fact that  $(1 - qt) = (1 - qt)_q^1$ , we have

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x) {}_a d_q x \right| \\ & \leq \frac{q^2(b - a)^2}{1 + q} \left( \frac{1}{1 + q} \right)^{1 - \frac{1}{r}} \left( \sup\{ \left|{}_aD_q^2 f(a)\right|^r, \left|{}_aD_q^2 f(b)\right|^r \} \int_0^1 t^r (1 - qt)_q^1 {}_0 d_q t \right)^{\frac{1}{r}} \\ & = \frac{q^2(b - a)^2}{1 + q} \left( \frac{1}{1 + q} \right)^{1 - \frac{1}{r}} \left( \beta_q(r + 1, 2) \sup\{ \left|{}_aD_q^2 f(a)\right|^r, \left|{}_aD_q^2 f(b)\right|^r \} \right)^{\frac{1}{r}}, \end{aligned}$$

thus, we get (24).  $\square$

**Remark 8.** If  $q \rightarrow 1$ , then

$$\beta(r + 1, 2) = \int_0^1 t^r (1 - t) {}_0 d_q t = \frac{1}{(r + 1)(r + 2)},$$

and (24) reduces to inequality (2) in Theorem 1.

**Theorem 16.** Let  $f : I = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$  be a twice  $q$ -differentiable function on  $I^0$  with  ${}_aD_q^2 f$  be continuous and integrable on  $I$  where  $0 < q < 1$ . If  $\left|{}_aD_q^2 f\right|^r$  is quasi-convex on  $[a, b]$  where  $p, r > 1, \frac{1}{p} + \frac{1}{r} = 1$ , then

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x) {}_a d_q x \right| \\ & \leq \frac{q^2(b - a)^2}{1 + q} (\beta_q(p + 1, 2))^{\frac{1}{p}} \left( \frac{\sup\{ \left|{}_aD_q^2 f(a)\right|^r, \left|{}_aD_q^2 f(b)\right|^r \}}{1 + q} \right)^{\frac{1}{r}}. \end{aligned} \tag{25}$$

**Proof.** Using Lemma 7, Hölder’s inequality and the fact that  $|{}_aD_q^2f|^r$  is a quasi-convex function, we have

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x)_a d_q x \right| \\ &= \left| \frac{q^2(b - a)^2}{1 + q} \int_0^1 t(1 - qt)_a D_q^2 f((1 - t)a + tb)_0 d_q t \right| \\ &\leq \frac{q^2(b - a)^2}{1 + q} \int_0^1 t(1 - qt) \left| {}_aD_q^2 f((1 - t)a + tb) \right|_0 d_q t \\ &\leq \frac{q^2(b - a)^2}{1 + q} \left( \int_0^1 t^p(1 - qt)_0 d_q t \right)^{\frac{1}{p}} \left( \int_0^1 (1 - qt) \left| {}_aD_q^2 f((1 - t)a + tb) \right|^r_0 d_q t \right)^{\frac{1}{r}} \\ &\leq \frac{q^2(b - a)^2}{1 + q} \left( \int_0^1 t^p(1 - qt)_0 d_q t \right)^{\frac{1}{p}} \left( \sup\{ \left| {}_aD_q^2 f(a) \right|^r, \left| {}_aD_q^2 f(b) \right|^r \} \int_0^1 (1 - qt)_0 d_q t \right)^{\frac{1}{r}} \end{aligned}$$

Applying Lemma 5 and the fact that  $(1 - qt) = (1 - qt)_q^1$ , we have

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x)_a d_q x \right| \\ &\leq \frac{q^2(b - a)^2}{1 + q} \left( \int_0^1 t^p(1 - qt)_q^1 d_q t \right)^{\frac{1}{p}} \left( \frac{\sup\{ \left| {}_aD_q^2 f(a) \right|^r, \left| {}_aD_q^2 f(b) \right|^r \}}{1 + q} \right)^{\frac{1}{r}} \\ &= \frac{q^2(b - a)^2}{1 + q} (\beta_q(p + 1, 2))^{\frac{1}{p}} \left( \frac{\sup\{ \left| {}_aD_q^2 f(a) \right|^r, \left| {}_aD_q^2 f(b) \right|^r \}}{1 + q} \right)^{\frac{1}{r}}, \end{aligned}$$

thus, we get (25). □

**Remark 9.** If  $q \rightarrow 1$ , then

$$\beta(p + 1, 2) = \int_0^1 t^p(1 - t)dt = \int_0^1 s(1 - s)^p ds = \beta(2, p + 1).$$

Inequality (25) reduces to inequality (3) in Theorem 2.

**Theorem 17.** Let  $f : I = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$  be a twice  $q$ -differentiable function on  $I^0$  with  ${}_aD_q^2f$  be continuous and integrable on  $I$  where  $0 < q < 1$ . If  $|{}_aD_q^2f|$  is quasi-convex on  $[a, b]$ , then

$$\left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x)_a d_q x \right| \leq \frac{q^2(b - a)^2 \sup\{ \left| {}_aD_q^2 f(a) \right|, \left| {}_aD_q^2 f(b) \right| \}}{(1 + q)^2(1 + q + q^2)}. \tag{26}$$

**Proof.** Using Lemma 7, Hölder’s inequality and the fact that  $|{}_aD_q^2f|$  is a quasi-convex function, we have

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x) {}_a d_q x \right| \\ &= \left| \frac{q^2(b - a)^2}{1 + q} \int_0^1 t(1 - qt) {}_a D_q^2 f((1 - t)a + tb) {}_0 d_q t \right| \\ &\leq \frac{q^2(b - a)^2}{1 + q} \int_0^1 t(1 - qt) \left| {}_a D_q^2 f((1 - t)a + tb) \right| {}_0 d_q t \\ &\leq \frac{q^2(b - a)^2}{1 + q} \sup\{ |{}_a D_q^2 f(a)|, |{}_a D_q^2 f(b)| \} \int_0^1 t(1 - qt) {}_0 d_q t \end{aligned}$$

Applying Lemma 6, we have

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x) {}_a d_q x \right| \\ &\leq \frac{q^2(b - a)^2 \sup\{ |{}_a D_q^2 f(a)|, |{}_a D_q^2 f(b)| \}}{(1 + q)^2(1 + q + q^2)}, \end{aligned}$$

thus, we get (26).  $\square$

**Remark 10.** If  $q \rightarrow 1$ , then inequality (26) reduces to inequality (4) in Theorem 3.

**Theorem 18.** Let  $f : I = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$  be a twice  $q$ -differentiable function on  $I^0$  with  ${}_aD_q^2f$  be continuous and integrable on  $I$  where  $0 < q < 1$ . If  $|{}_aD_q^2f|^r$  is quasi-convex on  $[a, b]$  for  $r \geq 1$ , then the following inequality holds:

$$\left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x) {}_a d_q x \right| \leq \frac{q^2(b - a)^2}{(1 + q)^2(1 + q + q^2)} \left( \sup\{ |{}_aD_q^2 f(a)|^r, |{}_aD_q^2 f(b)|^r \} \right)^{\frac{1}{r}}. \tag{27}$$

**Proof.** Using Lemma 7, Hölder’s inequality and the fact that  $|{}_aD_q^2f|^r$  is a quasi-convex function, we have

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x) {}_a d_q x \right| \\ &= \left| \frac{q^2(b - a)^2}{1 + q} \int_0^1 t(1 - qt) {}_a D_q^2 f((1 - t)a + tb) {}_0 d_q t \right| \\ &\leq \frac{q^2(b - a)^2}{1 + q} \int_0^1 t(1 - qt) \left| {}_a D_q^2 f((1 - t)a + tb) \right| {}_0 d_q t \\ &\leq \frac{q^2(b - a)^2}{1 + q} \left( \int_0^1 t(1 - qt) {}_0 d_q t \right)^{1 - \frac{1}{r}} \left( \int_0^1 t(1 - qt) \left| {}_a D_q^2 f((1 - t)a + tb) \right|^r {}_0 d_q t \right)^{\frac{1}{r}} \\ &\leq \frac{q^2(b - a)^2}{1 + q} \left( \int_0^1 t(1 - qt) {}_0 d_q t \right)^{1 - \frac{1}{r}} \left( \sup\{ |{}_aD_q^2 f(a)|^r, |{}_aD_q^2 f(b)|^r \} \int_0^1 t(1 - qt) {}_0 d_q t \right)^{\frac{1}{r}} \end{aligned}$$



Applying Lemma 6, we have

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x) {}_a d_q x \right| \\ & \leq \frac{q^2(b - a)^2}{1 + q} \left( \frac{1}{(1 + q)(1 + q + q^2)} \right)^{1 - \frac{1}{r}} \left( \frac{\sup\{|{}_a D_q^2 f(a)|^r, |{}_a D_q^2 f(b)|^r\}}{(1 + q)(1 + q + q^2)} \right)^{\frac{1}{r}} \\ & = \frac{q^2(b - a)^2}{(1 + q)^2(1 + q + q^2)} \left( \sup\{|{}_a D_q^2 f(a)|^r, |{}_a D_q^2 f(b)|^r\} \right)^{\frac{1}{r}}, \end{aligned}$$

thus, we get (27). □

**Remark 11.** If  $q \rightarrow 1$ , then inequality (27) reduces to inequality (6) in Theorem 5.

#### 4. Discussion of New Perspectives

Currently, the Hermite-Hadamard inequality plays a significant role in the development of all fields of Mathematics. It has significant applications in a variety of applied Mathematics, such as integral inequalities, approximation theory, special means theory, optimization theory, information theory and numerical analysis. In recent years, a number of authors have discovered new Hermite-Hadamard-type inequalities for convex,  $s$ -convex functions, logarithmic convex functions,  $h$ -convex functions, quasi-convex functions,  $m$ -convex functions,  $(K, m)$ -convex functions, co-ordinated convex functions, and the Godunova-Levin function,  $P$ -function, and so on. In this paper, we use a new quantum integral identity established in [11] (Lemma 4.1) to develop some quantum estimates for Hermite-Hadamard type inequalities in which some quasi-convex functions are involved.

Since quantum calculus has large applications in many mathematical areas such as number theory, special functions, quantum mechanics and mathematical inequalities, we hope interested readers will continue to explore more quantum estimates of Hermite-Hadamard type inequalities for other kinds of convex functions, and, furthermore, to find applications in the above-mentioned mathematical areas.

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