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SOME RAMSEY-TYPE NUMBERS AND THE INDEPENDENCE RATIO

BY

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ABSTRACT. If each of k, m, and n is a positive integer, there is a smallest positive integer $r = r_k(m, n)$ with the property that each graph G with at least r vertices, and with maximum degree not exceeding k, has either a complete subgraph with m vertices, or an independent subgraph with n vertices. In this paper we determine $r_3(3, n) = r(n)$, for all n. As a corollary we obtain the largest possible lower bound for the independence ratio of graphs with maximum degree three containing no triangles.

From the work of Brooks [2] it follows that if G is a graph with maximum degree k containing no complete graph on $k + 1$ vertices, then the independence ratio of G is at least $1/k$. In case G has no complete graph on k vertices, Albertson, Bollobas, and Tucker [1] proved this ratio is larger than $1/k$, with only two exceptions. And they conjectured that for $k = 3$, with the additional assumption of planarity, this ratio is bounded away from $1/3$. Fajtlowicz [3] verified their conjecture, even without assuming planarity, showing that each cubic graph without triangles has independence ratio at least $12/35$. In addition, he displayed a graph in which the independence ratio is exactly $5/14$. It follows from our main theorem that $5/14$ is a lower bound for the independence ratio in the case $k = 3$, and in light of Fajtlowicz' graph, $5/14$ is the best possible lower bound.

In what follows, all graphs will be finite symmetric graphs with no loops and no multiple edges. If G is a graph, then $v(G)$ and $e(G)$ will be the numbers of vertices and edges of G. If M is a set of vertices of G, no two of which are joined by an edge, then M is called independent. The number of vertices in a largest independent vertex set in G will be denoted $i(G)$. A cycle with n vertices will be denoted C_n.

PROPOSITION 1. *If G is a graph in which each vertex has degree two or degree three, and if $v(G)$ is odd, then either there is a vertex of degree two both of whose neighbors are of degree two, or else there is a vertex of degree two both of whose neighbors are of degree three.*

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Proof. The number of vertices of odd degree is even, so there is an odd number of vertices of degree two. If each vertex of degree two had exactly one neighbor of degree two, these would occur in pairs, which is not the case since there is an odd number of them. So some vertex of degree two has either two neighbors of degree two or no neighbors of degree two.

Until further notice, all graphs will be of maximum degree not exceeding three and will contain no triangles. Several types of deletions of vertices and edges from such graphs will be considered.

(α) If G has a vertex of degree two with two neighbors of degree three, then a deletion of three vertices and six edges may be performed, lowering the independence by at least one. That is, if vertices and edges as in Figure 1(a) are deleted from G and if H is the remaining subgraph, then

$$v(H) = v(G) - 3, \quad e(H) = e(G) - 6, \quad i(H) < i(G) - 1.$$

(β) If G has a vertex of degree two with one neighbor of degree two and one neighbor of degree three, then a deletion of vertices and edges as in Figure 1(b) leaves a subgraph H with

$$v(H) = v(G) - 3, \quad e(H) = e(G) - 5, \quad i(H) < i(G) - 1.$$

a b

Figure 1

(γ) If G has no vertices of degree less than two, and if there is a vertex of degree two with both its neighbors of degree two, then a deletion may be made which leaves a subgraph H with

$$v(H) = v(G) - 5, \quad e(H) < e(G) - 5, \quad i(H) < i(G) - 2.$$

In order to see that this is true, observe that there are several possibilities, illustrated in Figure 2. In cases (a), (b), (c), (d), (e), and (f), deletion of the indicated vertices and edges achieves the desired result. In cases (g) and (h), the indicated vertices and edges must be deleted along with any other vertex.

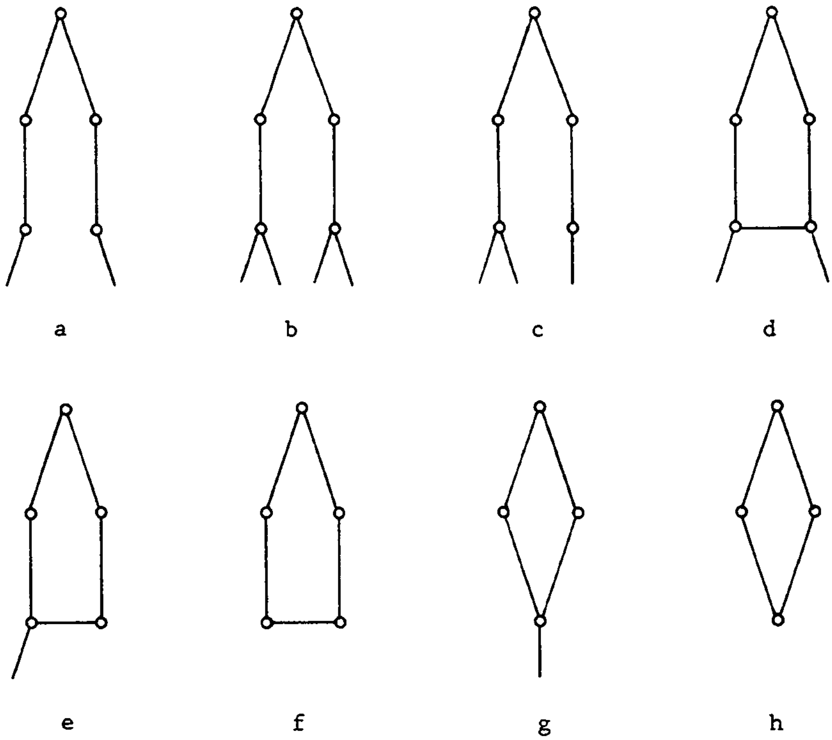

FIGURE 2

(λ) If G has a vertex of degree less than two, and if G is not totally disconnected, then a deletion can be made which leaves a graph H with

$$v(H) = v(G) - 2, \quad e(H) \leq e(G) - 1, \quad i(H) \leq i(G) - 1.$$

In case G has a vertex of degree one, deletion of that vertex and its neighbor suffices. If G has an isolated vertex, deletion of that vertex and any nonisolated vertex suffices.

LEMMA 2. *If G is a graph, $v(G) = 5n$, $e(G) \leq 5n$, and $i(G) < 2n + 1$, then G is the disjoint union of n pentagons.*

PROOF. If $n = 1$, then $v(G) = 5$. If G is a tree, the independence ratio of G is at least $1/2$, so $i(G) \geq 3 = 2n + 1$, so this is not the case. So G has a cycle. Since G has no triangle (recall that "graphs", for the time being, have no triangles), the shortest cycle is C_4 or C_5. If it is C_5, we are finished, so suppose G has a C_4. Since G has no triangle, the C_4 has no diagonal, and the vertex which is not on the C_4 is incident with at most two vertices of the C_4, and those two must be nonadjacent. Hence there is clearly an independent set of 3 vertices. This verifies the case $n = 1$.

Now suppose that $v(G) = 5k + 5$, $e(G) \leqslant 5k + 5$, and $i(G) < 2k + 3$. Then G is not totally disconnected, so, if G has a vertex of degree less than two, a deletion of type λ may be performed, leaving a subgraph H with $v(H) = 5k + 3$, $e(H) \leqslant 5k + 4$, and $i(H) < 2k + 2$. If H has a vertex of degree less than two, another deletion of type λ would yield a graph J with $v(J) = 5k + 1$, $e(J) \leqslant 5k + 3$, and $i(J) < 2k + 1$. From J, the deletion of any vertex of degree three would yield a subgraph M with $v(M) = 5k$, $e(M) \leqslant 5k$, and $i(M) < 2k + 1$. If the number of edges of J actually exceeds the number of vertices of J, we are assured a vertex of degree three. If not, deletion of any nonisolated vertex would accomplish the same thing. Now, by induction, M is a disjoint union of k pentagons, and so it is easy to see that before the last deletion, we had independence $2k + 1$, which is a contradiction. Thus we may assume that H has no vertex of degree less than two. If H is regular of degree two, it is a disjoint union of cycles, none of which is a triangle. So $i(H) \geqslant \frac{2}{5}(5k + 3) = 2k + 6/5$, so $i(H) \geqslant 2k + 2$, which is not the case. So H has a vertex of degree three, and we may make a deletion of type α or type β. This leaves a subgraph N with $v(N) = 5k$, $e(N) \leqslant 5k - 1$, $i(N) < 2k + 1$. This is impossible by induction, since such an N must be the disjoint union of k pentagons and thus have $5k$ edges. This rules out the possibility that G has vertices of degree less than two. Since $v(G) = e(G)$, G is a disjoint union of cycles. Since the independence ratio of G is not more than $(2k + 2)/(5k + 5) = 2/5$, this may happen only if G is a disjoint union of pentagons.

Proposition 3. *If G is a graph in which each vertex is of degree two or degree three, then the number of vertices of degree three is* $2[e(G) - v(G)]$. *In any graph,* $e(G) \leqslant \frac{3}{2}v(G)$ *with equality only in case G is regular of degree three.*

Lemma 4. *If G is a graph with* $v(G) = 5n + 3$, $e(G) \leqslant 5n + 6$, *and* $i(G) < 2n + 2$, *then G contains a rectangle,* C_4.

Proof. In case $n = 1$, we have $v(G) = 8$, $e(G) \leqslant 11$, and $i(G) < 4$. If there were a vertex of degree less than two, a deletion of type λ would leave a subgraph H with 6 vertices and independence 2. This is not possible because each graph with 6 vertices has either a triangle or an independent set of 3 vertices. Thus, a deletion of type α or β is possible, leaving a subgraph J with $v(J) = 5$, $e(J) \leqslant 6$, and $i(J) < 3$. But the only such graph is a pentagon. So there are two possibilities. Either $e(G) = 11$ and a deletion of type α was made, leaving a pentagon, or else $e(G) = 10$ and a deletion of type β was made leaving a pentagon. There are only three possibilities for G, indicated in Figure 3. All three contain C_4, which completes the proof for $n = 1$. Note that we have also shown that if $v(G) = 8$ and $i(G) < 4$, then $e(G) \geqslant 10$.

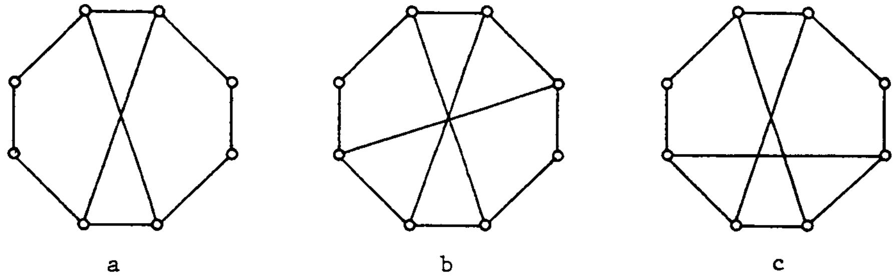

FIGURE 3

Now suppose the statement of the lemma holds for $n = k$ and consider a graph G with $v(G) = 5k + 8$, $e(G) \leqslant 5k + 11$ and $i(G) < 2k + 4$. If G has a vertex of degree less than two, a deletion of type λ leaves a graph H with $v(H) = 5k + 6$, $e(H) \leqslant 5k + 10$, and $i(H) < 2k + 3$. If H has a vertex of degree less than two, another deletion of type λ would leave a graph J with $v(J) = 5k + 4$, $e(J) \leqslant 5k + 9$, and $i(J) < 2k + 2$. The surplus of edges over vertices insures that J has a vertex of degree three. Deletion of such a vertex leaves a graph M with $v(M) = 5k + 3$, $e(M) \leqslant 5k + 6$ and $i(M) < 2k + 2$. By induction M contains C_4, so G does. This takes care of the case where H has a vertex of degree less than two. If every vertex of H has degree two or three, then there must be at least one vertex of degree two since there are not enough edges for H to be cubic. Deletion of a vertex of degree two and its two neighbors yields a graph Q with $v(Q) = 5k + 3$, $e(Q) \leqslant 5k + 6$, and $i(Q) < 2k + 2$. By induction, Q contains C_4. So we have taken care of the case where G has a vertex of degree less than 2. If each vertex of G has degree two or three, then by Proposition 3, the number of vertices of degree three is no more than $2(5k + 11 - 5k - 8) = 6$. Since k is at least one, $v(G)$ is at least 13, so there are at least 7 vertices of degree two. Hence some two vertices of degree two must have a common neighbor. If these two vertices have two common neighbors, there is a C_4 and we are through. If not, the situation must be as in Figure 4. The vertices v and w have the common neighbor x, which may be of degree two or three, as may a and b. By deleting these five vertices, and the edges (at least five) incident with them, we are left with a graph R with $v(R) = 5k + 3$, $e(R) \leqslant 5k + 6$ and $i(R) < 2k + 2$. By induction, R must contain a C_4, and so G does.

From the Ramsey Theorem it follows that if n is a positive integer, there is a smallest positive integer $r(n)$ such that any graph G (with maximum degree three and without triangles!) with $v(g) \geqslant r(n)$ has $i(G) \geqslant n$. If H is a graph with $v(H) = -1 + r(n)$ and $i(H) < n$, then H will be called *n-critical*. A graph G will be called *separated* if each component of G is either regular of degree three or a pentagon, C_5. We now proceed to find $r(n)$ for small values of n, and collect some facts about their n-critical graphs.

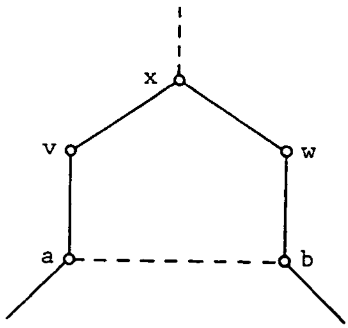

FIGURE 4

(i) $r(3) = 6$ and if G is 3-critical, then G is a pentagon, so $e(G) = 5$. This is the well-known first nontrivial Ramsey number, so it is true even without our maximum degree condition.

(ii) $r(4) = 9$ and if G is 4-critical, then $e(G) \geqslant 10$. Again, this corresponds to the classical Ramsey number. Some 4-critical graphs were displayed in Figure 3 and it was shown that $e(G) \geqslant 10$ in the proof of Lemma 4.

(iii) $r(5) = 12$ and if G is 5-critical, then $e(G) = 16$. This is the first deviation from the classical Ramsey numbers. To see that $r(5) = 12$, suppose $v(G) = 12$ and $i(G) < 5$. If G has a vertex of degree less than three, then a deletion of type α, β, or λ would leave a subgraph H with $v(H) \geqslant 9$ and $i(H) < 4$, which is not possible since $r(4) = 9$. Thus we may suppose G is regular of degree three. Deletion from G of one vertex and its neighbors would leave a graph J with $v(J) = 8$, $e(J) = 9$, and $i(J) < 4$. This is not possible, since a 4-critical graph must have at least 10 edges. This shows that $r(5) \leqslant 12$. In Figure 5 there is a 5-critical graph with 11 vertices, which forces $r(5) = 12$. Note that there are 16 edges, which is the case for any 5-critical graph. For suppose G is 5-critical. Since $e(G) \leqslant \frac{3}{2}v(G)$, it follows that G may have no more than 16 edges. Suppose $e(G) \leqslant 15$. First note that G has no vertex of degree less than two, since a deletion of type λ would leave 9 vertices and independence less than four, which is impossible. By Proposition 1, a deletion of type α or γ is possible. A deletion of type α would leave a graph H with $v(H) = 8$, $e(H) \leqslant 9$ and $i(H) < 4$. This is impossible. A deletion of type γ would leave a graph J with $v(J) = 6$ and $i(J) < 3$, which is impossible. This completes the proof that a 5-critical graph must have 16 edges. Furthermore, a 5-critical graph must contain a C_4. For, by Proposition 3 there must be $2(16 - 11) = 10$ vertices of degree three and hence exactly one vertex of degree 2. Thus an α deletion is possible, leaving a graph H with $v(H) = 8$, $e(H) = 10$ and $i(H) < 4$. There is only one such graph, as seen in the proof of Lemma 4, and it contains a C_4.

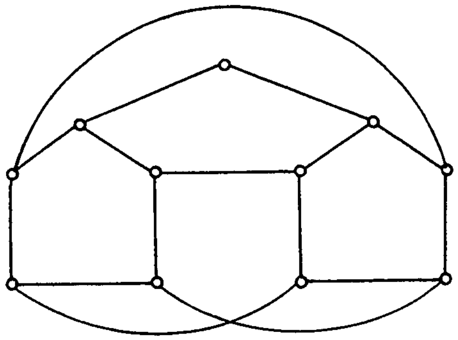

FIGURE 5

(iv) $r(6) = 15$ and if G is 6-critical, then $e(G) = 21$. To see that $r(6) = 15$, suppose that $v(G) = 15$. Since $v(G)$ is odd, G cannot be regular of degree three, so there is a vertex of degree less than three. Deletion of that vertex and its neighbors leaves a graph H with at least 12 vertices, and hence with independent 5 vertices. Thus G has independent 6 vertices, so $r(6) \leq 15$. Fajtlowicz [3] displayed a 6-critical graph, which we show in Figure 6. So $r(6) = 15$.

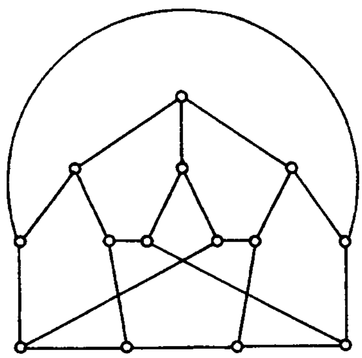

FIGURE 6

To see that each 6-critical graph has 21 edges, note that $\frac{3}{2}(14) = 21$, so no more than 21 edges are possible. And if G is 6-critical with $e(G) \leq 20$, then G may have no vertex of degree less than two or else a λ deletion would leave a graph J with $v(J) = 12$ and $i(J) < 5$, which is not possible. So we may assume G has only vertices of degree two and three. Thus, a deletion of type α, β, or γ is possible. But γ would leave a graph M with $v(M) = 9$ and $i(M) < 4$ which is not possible since $r(4) = 9$, and a deletion of type α or β would leave a graph Q with $v(Q) = 11$, $e(Q) \leq 15$, and $i(Q) < 5$. This is not possible since each 5-critical graph must have 16 edges. So G must have 21 edges.

Now we are ready to state our main theorem.

THEOREM 5. *For each positive integer* n, *the following statements are true*:

(1) *For* $0 \leq k \leq n$, *if* $v(G) = 5n + 3k$ *and* $i(G) < 2n + k + 1$, *then* $e(G) \geq 5n + 5k$.

(2) *For* $0 \leq k \leq 2n$, *if* $v(G) = 8n + 3k$ *and* $i(G) < 3n + k + 1$, *then*

(a) *if* k *is even*, $e(G) \geq 10n + 11k/2$,

(b) *if* k *is odd*, $e(G) \geq 10n + (11k + 1)/2$.

(3) *For* $1 \leq k \leq n$, *if* $v(G) = 5n + 3k$, $i(G) < 2n + k + 1$, *and* $e(G) \leq 5n + 5k + 1$, *then* G *contains a* C_4 *or is separated. And if* $e(G) = 5n + 5k$, *then* G *contains a* C_4.

(4) *For* $0 \leq k \leq 2n$, *if* $v(G) = 8n + 3k$, $i(G) < 3n + k + 1$ *and*

$$\begin{cases} e(G) \leq 10n + \dfrac{11k}{2} + 1 \ (k \text{ even}) \text{ or} \\ e(G) \leq 10n + \dfrac{11k+1}{2} + 1 \ (k \text{ odd}) \end{cases}$$

then G *contains a* C_4 *or is separated. In addition, if* k *is even*, $k \neq 2n$ *and* $e(G)$ *is exactly* $10n + 11k/2$, *then* G *contains a* C_4.

(5) (a) $r(5n - 3) = 14n - 11$.

(b) *If* G *is* $5n - 3$*-critical, then* $e(G) \geq 21n - 22$.

(6) (a) $r(5n - 2) = 14n - 8$.

(b) *If* G *is* $5n - 2$*-critical, then* $e(G) \geq 21n - 16$.

(7) (a) $r(5n - 1) = 14n - 5$.

(b) *If* G *is* $5n - 1$*-critical, then* $e(G) \geq 21n - 11$.

(8) (a) $r(5n) = 14n - 2$.

(b) *If* G *is* $5n$*-critical, then* $e(G) = 21n - 5$.

(9) (a) $r(5n + 1) = 14n + 1$.

(b) *If* G *is* $5n + 1$*-critical, then* $e(G) = 21n$.

PROOF OF THEOREM. For $n = 1$, the statements have already been verified, in Lemma 2, Lemma 4, and by our determination of small values of $r(n)$. We assume the truth of all nine statements for n, and proceed to state and prove them one at a time for $n + 1$.

(1) For $0 \leq k \leq n + 1$, if $v(G) = 5n + 3k + 5$ and $i(G) < 2n + k + 3$, then $e(G) \geq 5n + 5k + 5$.

PROOF. If $k = 0$, this is true by Lemma 2. So we may induct on k. Suppose $1 \leq k \leq n + 1$, and suppose $e(G) \leq 5n + 5k + 4$. If G has a vertex of degree less than two, then a deletion of type λ would leave a graph H with $v(H) = 5n + 3(k + 1)$, $e(H) \leq 5n + 5(k + 1) - 2$, and $i(H) < 2n + (k + 1) + 1$. For $k + 1 \leq n$, that is for $k \leq n - 1$, this is not possible by part (1) of the induction hypothesis on n. If $k = n$, the situation is that $v(H) = 8n + 3$, $e(H) \leqslant 10n + 3$, and $i(H) < 3n + 2$. This is not possible by part (2) of the induction on n. And if $k = n + 1$, the situation is that $v(H) = 8n + 6$, $e(H) \leqslant 10n + 8$, and $i(H) < 3n + 3$, which is also impossible by part (2) of the induction on n. This rules out the possibility that G has a vertex of degree less than two. A simple computation using Proposition 3 shows that G is not regular of degree three. So G has vertices of degree two, and so a deletion of type α, β, or γ may be made. A deletion of type α or β from G would leave a graph J with $v(J) = 5(n + 1) + 3(k - 1)$, $e(J) \leqslant 5(n + 1) + 5(k - 1) - 1$ and $i(J) < 2(n + 1) + (k - 1) + 1$. This is impossible by induction on k, since $k \geqslant 1$. A deletion of type γ from G would leave a graph M with $v(M) = 5n + 3k$, $i(M) < 2n + k + 1$, and $e(M) \leqslant 5n + 5k - 1$. If $k \leqslant n$, this is impossible by part (1) of the induction on n. If $k = n + 1$, the situation is that $v(M) = 8n + 3$, $e(M) \leqslant 10n + 4$, and $i(M) < 3n + 2$, which is impossible by part (2) of the induction on n. So the assumption $e(G) \leqslant 5n + 5k + 4$ leads to contradictions in every direction. This concludes the derivation of part (1) for $n + 1$.

(2) For $0 \leqslant k \leqslant 2n + 2$, if $v(G) = 8n + 3k + 8$ and $i(G) < 3n + k + 4$, then

$$e(G) \geqslant \begin{cases} 10n + \dfrac{11k}{2} + 10 & \text{if } k \text{ is even,} \\[2ex] 10n + \dfrac{11k + 1}{2} + 10 & \text{if } k \text{ is odd.} \end{cases}$$

PROOF FOR $0 \leqslant k \leqslant 2n - 2$. For $k = 0$, this is a restatement of part (1), proven above, with $k = n + 1$. So, we may induct on k. So suppose that

$$e(G) \leqslant \begin{cases} 10n + \dfrac{11k}{2} + 9 & \text{if } k \text{ is even,} \\[2ex] 10n + \dfrac{11k + 1}{2} + 9 & \text{if } k \text{ is odd.} \end{cases}$$

If G has a vertex of degree less than two, a deletion of type γ from G would leave a graph H with $v(H) = 8n + 3(k + 2)$, $i(H) < 3n + (k + 2) + 1$ and

$$e(H) \leqslant \begin{cases} 10n + \dfrac{11(k + 2)}{2} - 3 & \text{if } k \text{ is even,} \\[2ex] 10n + \dfrac{11(k + 2) + 1}{2} - 3 & \text{if } k \text{ is odd.} \end{cases}$$

Since we are for the moment considering only the case $k \leqslant 2n - 2$, we have $k + 2 \leqslant 2n$, and so the situation described is impossible by part (2) of the induction on n. This rules out vertices of degree less than two. By Proposition 3, G is not regular of degree three, so there are vertices of degree two, and a deletion of type α, β, or γ is possible. A deletion of type γ would leave a graph J with $v(J) = 8n + 3(k + 1)$, $i(J) < 3n + (k + 1) + 1$, and

$$e(J) \leqslant \begin{cases} 10n + \dfrac{11(k+1)}{2} - 1 & \text{if } k \text{ is odd,} \\ 10n + \dfrac{11(k+1)+1}{2} - 2 & \text{if } k \text{ is even.} \end{cases}$$

This is not possible, by part (2) of the induction on n. Since no deletion of type γ is possible, a deletion of type α or β is possible, and if k is odd, $v(G)$ is odd, and by Proposition 1, a deletion of type α is possible. If k is odd, a deletion of type α would leave a graph M with $v(M) = 8(n + 1) + 3(k - 1)$, $i(M) < 3(n + 1) + (k - 1) + 1$, and $e(M) \leqslant 10(n + 1) + 11(k - 1)/2 - 1$. This is not possible by induction on k, since $k - 1$ is even. If k is even, a deletion of type α or β would leave a graph Q with $v(Q) = 8(n + 1) + 3(k - 1)$, $i(Q) < 3(n + 1) + (k - 1) + 1$, and $e(Q) \leqslant 10(n + 1) + (11(k - 1) + 1)/2 - 1$. This is impossible by induction on k, since $k - 1$ is odd. This concludes the proof of (2) for $0 \leqslant k \leqslant 2n - 2$.

(3) For $1 \leqslant k \leqslant n + 1$, if $v(G) = 5n + 3k + 5$, $e(G) \leqslant 5n + 5k + 6$, and $i(G) < 2n + k + 3$, then G contains a C_4 or is separated. And if $e(G) = 5n + 5k + 5$, then G contains a C_4.

Proof. If $k = 1$, the statement is true by Lemma 4, so we may induct on k. If G has a vertex of degree less than two, then a deletion of type λ would leave a graph H with $v(H) = 5n + 3(k + 1)$, $e(H) \leqslant 5n + 5(k + 1)$, and $i(H) < 2n + (k + 1) + 1$. If $k \leqslant n - 1$, this implies that H contains a C_4 by the induction on n, part (3). If $k = n$, the situation is that $v(H) = 8n + 3$, $e(H) \leqslant 10n + 5$, and $i(H) < 3n + 2$. But this is not possible by part (2) of the induction on n. If $k = n + 1$, the situation is that $v(H) = 8n + 6$, $e(H) \leqslant 10n + 10$, and $i(H) < 3n + 3$, which is also impossible by part (2) of the induction on n. Thus, in case G has a vertex of degree less than two, we have shown that G contains a C_4. Now suppose G has only vertices of degree two and degree three. By Proposition 3, G is not regular of degree three, so a deletion of type α, β, or γ is possible. A deletion of type α or β would leave a graph J with $v(J) = 5(n + 1) + 3(k - 1)$, $e(J) \leqslant 5(n + 1) + 5(k - 1) + 1$ and $i(J) < 2(n + 1) + (k - 1) + 1$. This implies, by the induction on k, that J is separated or contains a C_4. If J contains a C_4, then G does. If J is separated, then $J = R \cup S$ where R is regular of degree three and S is a disjoint union of pentagons. The deleted edges do not go into R, since vertices in R already have degree three. Thus, Lemma 4 applied to $G - R$ assures us that $G - R$ contains a C_4. Thus, G contains a C_4.

Now, if a deletion of type γ is made from G, the result is a graph M with $v(M) = 5n + 3k$, $e(M) \leqslant 5n + 5k + 1$, and $i(M) < 2n + k + 1$. If $k \leqslant n$, part (3) of the induction on n implies that M contains a C_4 or is separated. If M contains C_4, then G does. If M is separated, we must look back at the several different types of γ deletions. Types (g) and (h) actually contain a C_4, so they need not be considered. Types (a), (b), (c), (d) and (e) result in the deletion of at least 6 edges, so in these cases we would have the slightly stronger edge inequality $e(M) \leqslant 5n + 5k$. By induction on n, part (3), it would follow that M contains C_4. Finally, if a deletion of type (f) occurs, and if M is separated, then G is separated, since the components of G would be those of M along with the deleted pentagon. But so far we have considered only $k \leqslant n$. If $k = n + 1$, the γ deletion leaves a graph Q with $v(Q) = 8n + 3$, $e(Q) \leqslant 10n + 6$, and $i(Q) < 3n + 2$. By part (4) of the induction on n, this implies that Q contains C_4 or is separated. If Q contains C_4, then G does. If Q is separated, an exact repetition of the argument above involving the various types of γ deletions shows that the deletion is a pentagon and so G is separated.

To conclude the derivation of (3), we must show that if $e(G) = 5n + 5k + 5$, then G contains a C_4. To do this, we need only show that G is not separated. Suppose G is separated. By Proposition 3, the number of vertices of degree three in G is

$$2[e(G) - v(G)] = 2[5n + 5k + 5 - (5n + 3k + 5)] = 4k.$$

Hence there are $5n - k + 5$ vertices of degree two. The vertices of degree two make up pentagons, and so they contribute exactly $\frac{2}{5}(5n - k + 5)$ to $i(G)$. The $4k$ vertices of degree 3 contribute at least $\frac{5}{14}(4k)$ to $i(G)$, by parts (5)–(9) of the induction on n. Hence $i(G) \geqslant 2n - \frac{2}{5}k + 2 + \frac{10}{7}k = 2n + 2 + \frac{36}{35}k > 2n + k + 2$. So $i(G) \geqslant 2n + k + 3$, which is contrary to assumption. Hence G is not separated, so G contains a C_4, and the derivation of (3) for $n + 1$ is complete.

(4) For $0 \leqslant k \leqslant 2n + 2$, if $v(G) = 8n + 3k + 8$, $i(G) < 3n + k + 4$, and

$$e(G) \leqslant \begin{cases} 10n + \dfrac{11k}{2} + 11 & \text{if } k \text{ is even,} \\[2ex] 10n + \dfrac{11k + 1}{2} + 11 & \text{if } k \text{ is odd,} \end{cases}$$

then G contains a C_4 or is separated. In addition, if k is even, $k \neq 2n + 2$, and $e(G) = 10n + 11k/2 + 10$, then G contains a C_4.

Proof for $0 \leqslant k \leqslant 2n - 2$. For $k = 0$, this is a restatement of part (3) with $k = n + 1$, just proven above. So again we may induct on k. If G has a vertex of degree less than two, a deletion of type λ would leave a graph H with $v(H) = 8n + 3(k + 2)$, $i(H) < 3n + (k + 2) + 1$, and

$$e(H) \leqslant \begin{cases} 10n + \dfrac{11(k + 2)}{2} - 1, & k \text{ even,} \\[2ex] 10n + \dfrac{11(k + 2) + 1}{2} - 1, & k \text{ odd.} \end{cases}$$

Since $k \leq 2n - 2$, $k + 2 \leq 2n$, and so this situation is impossible by part (2) of the induction on n. So G has only vertices of degree two and three. Proposition 3 insures that G is not regular of degree three, so we may make a deletion of type α, β, or γ, and by Proposition 1, if k is odd we may make a deletion of type α or γ. If k is even and a deletion of type α or β is made from G, the result is a graph J with $v(J) = 8(n + 1) + 3(k - 1)$, $i(J) < 3(n + 1) + (k - 1) + 1$ and $e(J) \leq 10(n + 1) + (11(k - 1) + 1)/2 + 1$. By induction on k, this implies that J contains a C_4 or is separated. If J contains a C_4, then G does. If J is separated, then J is the union of a regular graph R of degree three and some pentagons. Lemma 4 applied to $G - R$ guarantees that $G - R$ contains a C_4. Thus G contains a C_4. If k is odd and a deletion of type α is made, the result is a graph M with $v(M) = 8(n + 1) + 3(k - 1)$, $i(M) < 3(n + 1) + (k - 1) + 1$, and $e(M) \leq 10(n + 1) + 11(k - 1)/2 + 1$. An argument identical to the one just presented for k even shows that G must contain a C_4.

Now, if a deletion of type γ is made from G, we are left with a graph Q with $v(Q) = 8n + 3(k + 1)$ and $i(Q) < 3n + (k + 1) + 1$. We consider the several types of deletions of type γ. Since the object is to find a C_4, types (g) and (h) need not be considered. Types (b), (c), and (d) would lower the number of edges by at least seven, leaving

$$e(Q) \leq \begin{cases} 10n + \dfrac{11(k+1)+1}{2} - 2, & k \text{ even}, \\[2mm] 10n + \dfrac{11(k+1)}{2} - 1, & k \text{ odd}. \end{cases}$$

Since we are considering $k \leq 2n - 2$, $k + 1 \leq 2n$, and so the induction on n, part (2) makes this situation impossible. Deletions of types (a) and (e) would lower the number of edges by six, leaving

$$e(Q) < \begin{cases} 10n + \dfrac{11(k+1)+1}{2} - 1, & k \text{ even}, \\[2mm] 10n + \dfrac{11(k+1)}{2}, & k \text{ odd}. \end{cases}$$

If k is even, this is impossible by part (2) of the induction on n. If k is odd, part (4) of the induction on n shows that Q contains a C_4, since $k + 1$ is even. All that remains is to consider a deletion of type γ (f), in which the deleted vertices form a pentagon. This deletion yields

$$e(Q) < \begin{cases} 10n + \dfrac{11(k+1)+1}{2}, & k \text{ even}, \\[2mm] 10n + \dfrac{11(k+1)}{2} + 1, & k \text{ odd}. \end{cases}$$

By part (4) of the induction on n, this implies that Q contains a C_4 or is separated. Hence G contains a C_4 or is separated since it is the union of Q and a pentagon.

All that remains is to show that if k is even and $e(G) = 10n + 11k/2 + 10$, then G contains a C_4. It suffices to show that G is not separated. So suppose that G is separated. By Proposition 3, the number of vertices of degree three is

$$2\left[\left(10n + \frac{11k}{2} + 10\right) - (8n + 3k + 8)\right] = 4n + 5k + 4.$$

And so the number of vertices of degree two is $4n - 2k + 4$. Since $k \leqslant 2n - 2$, it follows that $4n + 5k + 4 \leqslant 14n - 6$. By parts (5)–(9) of the induction on n, there must be at least $\frac{5}{14}(4n + 5k + 4)$ independent vertices among the vertices of degree three. And there are exactly $\frac{2}{5}(4n - 2k + 4)$ independent vertices among the vertices of degree two. Hence $i(G)$ is at least $\frac{5}{14}(4n + 5k + 4) + \frac{2}{5}(4n - 2k + 4) = 3n + (\frac{69}{70}k + \frac{1}{35}n) + 3\frac{1}{35} \geqslant 3n + k + 3\frac{1}{35}$, since $k \leqslant 2n - 2 < 2n$. Thus $i(G) \geqslant 3n + k + 4$, which is contrary to assumption. So G is not separated, and must therefore contain a C_4. The derivation of (4) for $n + 1$ is concluded for $0 \leqslant k \leqslant 2n - 2$.

(5) (a) $r(5n + 2) = 14n + 3$.

(b) If G is $5n + 2$-critical, then $e(G) \geqslant 21n - 1$.

PROOF. (a) Suppose G is a graph with $v(G) = 14n + 3$ and $i(G) < 5n + 2$. If G had a vertex of degree less than two, a λ deletion would leave a graph H with $v(H) = 14n + 1$ and $i(H) < 5n + 1$. This is not possible by part (9) of the induction on n, since $r(5n + 1) = 14n + 1$. Thus, each vertex has degree two or three, and since $v(G)$ is odd, a deletion of type α or γ is possible. A deletion of type γ would leave a graph J with $v(J) = 14n - 2$ and $i(J) < 5n$. This is not possible since $r(5n) = 14n - 2$ by part (8) of the induction on n. A deletion of type α would leave a graph M with $v(M) = 14n$ and $i(M) < 5n + 1$, that is, a $5n + 1$-critical graph. But by part (9) of the induction on n $e(M) = 21n$, so M is regular of degree three. So the three deleted vertices form a separate component and must therefore contribute two to $i(G)$. Hence $i(G) = 5n + 2$. Thus $r(5n + 2) \leqslant 14n + 3$. To show $r(5n + 2) = 14n + 3$ we must show a $5n + 2$-critical graph with $14n + 2$ vertices. Recall that we have already displayed a 6-critical graph, a 5-critical graph, a 4-critical graph, and a 3-critical graph, the pentagon. In addition we mention that a pair of vertices joined by an edge is 2-critical. Now, a $5n + 2$-critical graph may be formed by taking the disjoint union of n copies of a 6-critical graph and one copy of a 2-critical graph.

(b) To show that if G is $5n + 2$ critical, $e(G) \geqslant 21n - 1$, we appeal to part (2) of the theorem, proven above for $n + 1$, and we let $k = 2n - 2$. Then we have that if G is a graph with $v(G) = 8n + 3(2n - 2) + 8$ and $i(G) < 3n + (2n - 2) + 4$, it follows that $e(G) \geqslant 10n + 11(2n - 2)/2 + 10$. That is, if

$v(G) = 14n + 2$ and $i(G) < 5n + 2$, then $e(G) \geq 21n - 1$.

(6) (a) $r(5n + 3) = 14n + 6$.

(b) If G is $5n + 3$-critical, then $e(G) \geq 21n + 5$.

PROOF. (a) Suppose $v(G) = 14n + 6$ and $i(G) < 5n + 3$. If G had a vertex of degree less than three, a deletion of that vertex and its neighbors would leave a graph H with $v(H) \geq 14n + 3$ and $i(H) < 5n + 2$. This is impossible since $r(5n + 2) = 14n + 3$. So G is regular of degree three, and $e(G) = \frac{3}{2}v(G) = 21n + 9$. Deletion from G of any vertex and its neighbors leaves a graph J with $v(J) = 14n + 2$, $e(J) = 21n$, and $i(J) < 5n + 2$. But this is precisely the situation described in part (4) of the theorem, for $n + 1$, with $k = 2n - 2$. So, the conclusion is that J contains a C_4 or is separated. But if J were separated, the number of vertices of degree two would be a multiple of five, which is not the case, since by Proposition 3 there are $2[21n - (14n + 2)] = 14n - 4$ vertices of degree three, and hence exactly six vertices of degree two. Thus we have established that J, and hence G, contains a C_4. We may delete from G a C_4 along with the vertices adjacent to two nonadjacent vertices of the C_4, as in Figure 7. This leaves a graph M with $v(M) = 14n$ and $i(M) < 5n + 1$. Thus M is $5n + 1$-critical, so $e(M) = 21n$, which means that M is regular of degree three. So no edges from the deletion go into M. So the independence of G is the sum of the independence of M and the independence of the 6 deleted vertices. So $i(G) \geq 5n + 3$, and we have shown that $r(5n + 3) \leq 14n + 6$. The disjoint union of a pentagon and n copies of a 6-critical graph is $5n + 3$-critical, so we have shown $r(5n + 3) = 14n + 6$.

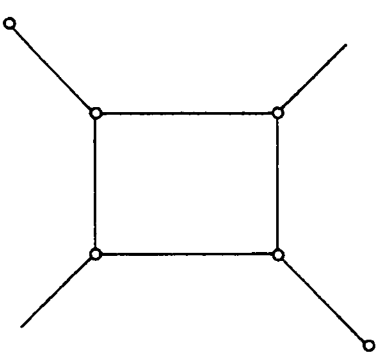

FIGURE 7

(b) If G is $5n + 3$-critical, then $v(G) = 14n + 5$ and $i(G) < 5n + 3$. Note that G may have no vertex of degree less than two, for a deletion of type λ would leave a graph H with $v(H) = 14n + 3$ and $i(H) < 5n + 2$, which is not possible since $r(5n + 2) = 14n + 3$. Suppose $e(G) \leq 21n + 4$. Since $v(G)$ is odd and G has only vertices of degree two and three, we can make a deletion of type α or γ. A deletion of type α would leave a graph J with

$v(J) = 14n + 2$, $e(J) \leq 21n - 2$ and $i(J) < 5n + 2$, which is impossible since a $5n + 2$-critical graph must have at least $21n - 1$ edges. A deletion of type γ would leave a graph M with $v(M) = 14n$, $i(M) < 5n + 1$, and $e(M) \leq 21n - 1$, which is impossible since a $5n + 1$-critical graph must have $21n$ edges. Thus $e(G) \geq 21n + 5$.

(7) (a) $r(5n + 4) = 14n + 9$.

(b) If G is $5n + 4$-critical, then $e(G) \geq 21n + 10$.

PROOF. (a) Suppose $v(G) = 14n + 9$. Since $v(G)$ is odd, G is not regular of degree three. Delete from G any vertex of degree less than three and its neighbors, leaving a graph H with $v(H) \geq 14n + 6$. Since $r(5n + 3) = 14n + 6$, $i(H) \geq 5n + 3$, so $i(G) \geq 5n + 4$. Thus $r(5n + 4) \leq 14n + 9$. And the union of a 4-critical graph with n copies of a 6-critical graph is $5n + 4$-critical. Thus $r(5n + 4) = 14n + 9$.

(b) If G is $5n + 4$-critical, then $v(G) = 14n + 8$ and $i(G) < 5n + 4$. G may not have a vertex of degree less than two, for a deletion of type λ would leave a graph H with $v(H) = 14n + 6$ and $i(H) < 5n + 3$. This is impossible since $r(5n + 3) = 14n + 6$. Suppose that $e(G) \leq 21n + 9$. Then G is not regular of degree three, so a deletion of type α, β, or γ is possible. A deletion of type α or β would leave a graph J with $v(J) = 14n + 5$, $i(J) < 5n + 3$ and $e(J) \leq 21n + 4$. This is not possible since each $5n + 3$-critical graph has at least $21n + 5$ edges. A deletion of type γ would leave a graph M with $v(M) = 14n + 3$ and $i(M) < 5n + 2$. This is impossible since $r(5n + 2) = 14n + 3$. Hence $e(G) \geq 21n + 10$.

(8) (a) $r(5n + 5) = 14n + 12$.

(b) If G is $5n + 5$-critical, then $e(G) = 21n + 16$.

PROOF. (a) Suppose $v(G) = 14n + 12$. If G is not regular of degree three, deletion of any vertex of degree less than three, along with its neighbors, leaves a graph H with $v(H) \geq 14n + 9$. Since $r(5n + 4) = 14n + 9$, this would imply that $i(H) \geq 5n + 4$ and hence $i(G) \geq 5n + 5$. Thus we may assume that G is regular of degree three. Deletion of any vertex along with its neighbors leaves a graph J with $v(J) = 14n + 8$ and $e(J) = \frac{3}{2}(14n + 12) - 9 = 21n + 9$. Since a $5n + 4$-critical graph must have at least $21n + 10$ edges, we have $i(J) \geq 5n + 4$, and so $i(G) \geq 5n + 5$. Thus $r(5n + 5) \leq 14n + 12$. A $5n + 5$-critical graph may be constructed by taking n copies of a 6-critical graph and one copy of a 5-critical graph. Thus $r(5n + 5) = 14n + 12$.

(b) If G is $5n + 5$-critical, then $v(G) = 14n + 11$ and $i(G) < 5n + 5$. If G has a vertex of degree less than two, a deletion of type λ leaves a graph H, $v(H) = 14n + 9$ and $i(H) < 5n + 4$, which is impossible since $r(5n + 4) = 14n + 9$. So G has only vertices of degree two and three. Suppose $e(G) \leq 21n + 15$. Since $v(G)$ is odd a deletion of type α or γ may be made. A deletion of type α would leave a graph J with $v(J) = 14n + 8$, $i(J) < 5n + 4$, and $e(J) \leq 21n + 9$, which is impossible since each $5n + 4$-critical graph has at least $21n + 10$ edges. A deletion of type γ would leave a graph M with $v(M) = 14n + 6$ and $i(M) < 5n + 3$. This is not possible since $r(5n + 3) = 14n + 6$. Thus, $e(G) \geq 21n + 16$. And $\frac{3}{2}(14n + 11) = 21n + 16\frac{1}{2}$, so $e(G) = 21n + 16$.

(9) (a) $r(5n + 6) = 14n + 15$.

(b) If G is $5n + 6$-critical, then $e(G) = 21n + 21$.

PROOF. (a) If G is a graph with $v(G) = 14n + 15$, then, since $v(G)$ is odd, G is not regular of degree three, so there is a vertex of degree less than three. Deletion of such a vertex and its neighbors leaves a graph H with $v(H) = 14n + 12$. Thus $i(H) \geq 5n + 5$, since $r(5n + 5) = 14n + 12$. Thus $i(G) \geq 5n + 6$. It follows that $r(5n + 6) \leq 14n + 15$. A $5n + 6$-critical graph can be gotten by taking $n + 1$ copies of a 6-critical graph. Thus $r(5n + 6) = 14n + 15$.

(b) Suppose that G is $5n + 6$-critical. Then $v(G) = 14n + 14$ and $i(G) < 5n + 6$. If G has a vertex of degree less than two, a deletion of type λ would leave a graph H with $v(H) = 14n + 12$ and $i(H) < 5n + 5$. This is impossible since $r(5n + 5) = 14n + 12$, so G has only vertices of degree two and three. Suppose $e(G) \leq 21n + 20$. Then G is not regular of degree three, and so a deletion of type α, β, or γ is possible. A deletion of type α or β would leave a graph J with $v(J) = 14n + 11$, $i(J) < 5n + 5$, and $e(J) \leq 21n + 15$. But this is not possible since a $5n + 5$-critical graph must have $21n + 16$ edges. Finally, a deletion of type γ would leave a graph M with $v(M) = 14n + 9$, and $i(M) < 5n + 4$, which is not possible since $r(5n + 4) = 14n + 9$. Thus $e(G) = 21n + 21$.

All that remains in the proof of the theorem is to consider the cases $k = 2n - 1, 2n, 2n + 1$, and $2n + 2$ of part (2) and part (4). But these four cases of part (2) are exactly the statements (6)(b), (7)(b), (8)(b), and (9)(b), which we have now shown. So we consider the four cases of part (4).

If $k = 2n - 1$, we must show that if $v(G) = 14n + 5$, $i(G) < 5n + 3$, and $e(G) \leq 21n + 6$, then G contains C_4 or is separated. If G had a vertex of degree less than two, a deletion of type λ would leave a graph H with $v(H) = 14n + 3$ and $i(H) < 5n + 2$, which is not possible since $r(5n + 2) = 14n + 3$. Since $v(G)$ is odd, we can make a deletion of type α or γ. A deletion of type α would leave a graph J with $v(J) = 14n + 2$, $i(J) < 5n + 2$, and $e(J) \leq 21n$. By the case $k = 2n - 2$ of part (4), J must contain a C_4. If we make a γ deletion from G, what remains is a graph M with $v(M) = 14n$, and $i(M) < 5n + 1$. So M is $5n + 1$-critical and thus must have exactly $21n$ edges, which means M is regular of degree three, and $i(M) = 5n$. So the deletions must have removed no more than six edges, and must have no three independent vertices. Looking at the several types of deletions of type λ, we see that types (a)–(d) have too many edges, types (g) and (h) contain a C_4, which is acceptable, and type (e) has an edge going into M, which is impossible since M is already regular of degree three. The only other case is type (f). If this happens, G is separated since G is the disjoint union of M and a pentagon. This concludes the proof of the case $k = 2n - 1$.

If $k = 2n$, we will show that if $v(G) = 14n + 8$, $i(G) < 5n + 4$, and $e(G) \leqslant 21n + 11$, then G contains a C_4. If G had a vertex of degree less than two, a deletion of type λ would leave a graph H with $v(H) = 14n + 6$ and $i(H) < 5n + 3$, which is not possible since $r(5n + 3) = 14n + 6$. Thus, G has only vertices of degree two and three, and G is not regular of degree three. A deletion of type α or β would leave a graph J with $v(J) = 14n + 5$, $i(J) < 5n + 3$, and $e(J) \leqslant 21n + 6$. By the previous case, $k = 2n - 1$, J must have a C_4 or be separated. If J has a C_4, then G does. If J is separated, then Lemma 4 applies to the part of G consisting of the pentagons in J and the deleted vertices and edges. The result is that G contains a C_4. Finally, a deletion of type γ from G would leave a graph M with $v(M) = 14n + 3$ and $i(M) < 5n + 2$, which is impossible since $r(5n + 2) = 14n + 3$. This concludes the proof of the case $k = 2n$.

If $k = 2n + 1$, we will show that if $v(G) = 14n + 11$, $i(G) < 5n + 5$, and $e(G) \leqslant 21n + 17$, then G contains a C_4. Actually the only possibility for $e(G)$ is $e(G) = 21n + 16$, by (8)(b) and Proposition 3. The number of vertices of degree three is $2[(21n + 16) - (14n + 11)] = 14n + 10$, so there is exactly one vertex of degree two, and all other vertices are of degree three. So a deletion of type α is possible, leaving a graph H with $v(H) = 14n + 8$, $i(H) < 5n + 4$, and $e(H) = 21n + 10$. But by the case $k = 2n$, just proven, this implies that H contains a C_4. Thus G contains a C_4.

Finally, if $k = 2n + 2$, we will show that if $v(G) = 14n + 14$, $i(G) < 5n + 6$, and $e(G) \leqslant 21n + 22$, then G is separated. By (9)(b), $e(G) = 21n + 21$, which means that G is regular of degree three, and hence trivially separated.

This concludes the proof of the theorem. Henceforth the word "graph" will again mean graph, with no restriction about degree or containment of triangles.

THEOREM 6. *If G is a graph with maximum degree $k \geqslant 3$, and if G contains no triangles, then the independence ratio $i(G)/v(G)$ is at least $5/(5k - 1)$.*

PROOF. For $k = 3$, this follows immediately from Theorem 5, since the smallest possible independence ratio would necessarily occur in some n-critical graph. So we induct on k. Suppose $k \geqslant 4$ and we have shown the statement of the theorem true for each r with $3 \leqslant r < k$. Let G be a graph with maximum degree k containing no triangles. Let M be a maximum cardinality independent vertex set in G, so that $i(G) = |M|$. Suppose $|M|/v(G) < 5/(5k - 1)$, in contradiction to the theorem. Now $G - M$ is a graph with maximum degree not exceeding $k - 1$, and containing no triangles, and so the independence ratio of $G - M$ is at least $5/(5(k - 1) - 1)$. So there is an independent subset N of $G - M$ with at least $5/(5k - 6)(|G - M|)$ vertices. But $|M| < 5/(5k - 1)|G|$, so $|G - M| > (1 - 5/(5k - 1))|G| = (5k - 6)|G|/(5k - 1)$. Thus,

$$|N| > \frac{5}{5k-6}|G - M| > \left(\frac{5}{5k-6}\right)\left(\frac{5k-6}{5k-1}\right)|G| = \left(\frac{5}{5k-1}\right)|G|.$$

Since N is independent in $G - M$, it is independent in G, and so the independence ratio of G is at least $5/(5k - 1)$, which was to be proven.

In the case $k = 3$, the independence ratio we have just shown is best possible since $5/14$ is achieved in the 6-critical graph due to Fajtlowicz. For larger k, $5/(5k - 1)$ is evidently quite weak.

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