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Some Refinements of Geršgorin Discs

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Abstract

This paper refines the work on Geršgorin Discs in the article "Geometric Multiplicities and Geršgorin Discs", *The American Mathematical Monthly*, 120(2013), 452-455 by R. Marsli and F. Hall. Some consequences of this refinement and examples are provided.

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1 Introduction

One of the most attractive and useful results to locate the eigenvalues of a matrix is Geršgorin's theorem, which goes back to 1931. The main part of this theorem is the following.

Geršgorin Theorem. *Let $A = [a_{ij}]$ be an $n \times n$ real or complex matrix and let*

$$R'_i = \sum_{j=1,\, j\neq i}^{n} |a_{ij}|, \quad 1 \leq i \leq n$$

denote the deleted absolute row sums of A. Then every eigenvalue of A is located in the union of its n Geršgorin discs (G-discs)

$$\cup_{i=1}^{n} D_i$$

where

$$D_i = \{z \in C : |z - a_{ii}| \leq R'_i\}.$$

The usual proof of the theorem involves a clever idea. Let λ be an eigenvalue A, and suppose that

$$Ax = \lambda x,\ x = [x_i] \neq 0.$$

Some entry of x has largest modulus, say $|x_p| \geq |x_i|$ for all $i = 1, 2, \ldots, n$, and $x_p \neq 0$. Then

$$x_p(\lambda - a_{pp}) = \sum_{j=1,\, j\neq p}^{n} a_{pj}x_j$$

and hence

$$|x_p||\lambda - a_{pp}| \leq |x_p| \sum_{j=1,\, j\neq p}^{n} |a_{pj}| = |x_p|R'_p$$

so that $|\lambda - a_{pp}| \leq R'_p$; that is, λ lies in the pth Geršgorin disc.

An $n \times n$ complex matrix A has n Geršgorin discs D_i, some of which may degenerate into points and some of which may be duplicates, as in the trivial example of an identity matrix. Recently, the authors extended the Geršgorin theory in the articles [1], [3], and [4]. In particular, the following result was proved in [3].

Theorem 1.1. *Let λ be an eigenvalue of A with geometric multiplicity k. Then λ is in at least k of the Geršgorin discs D_i of A.*

In the proof, a key preliminary result was employed: Each subspace S of C^n has a basis whose vectors have largest modulus entries in *different* positions. The argument for this uses a deflation process that has the same flavor as the proof in [2] of Schur's triangularization theorem. The preliminary result then insures that there is a basis $\{x_1, x_2, \ldots, x_k\}$ of the eigenspace S of λ and distinct integers $p_1, \ldots, p_k \in \{1, \ldots, n\}$ such that each each vector x_i has a largest modulus entry in position p_i. This construction in the proof of the Geršgorin theorem shows that λ lies in Geršgorin discs $D_{p_1}, \ldots, D_{p_k}$.

From Theorem 1.1, it follows [3] that an eigenvalue with geometric multiplicity at least $k \geq 1$ is contained in *any* union of $n-k+1$ different Geršgorin discs of A. That is a generalization of Geršgorin's general theorem, which is the case $k = 1$.

Corollary 1.2. *Let λ be an eigenvalue of A with geometric multiplicity at least $k \geq 1$. Then*

$$\lambda \in \cup_{j=1}^{n-k+1} \{z \in C : |z - a_{i_j i_j}| \leq R'_{i_j}\}$$

for any choices of indices $1 \leq i_1 < \ldots < i_{n-k+1} \leq n$. There are $\binom{n}{k-1}$ possibilities for such a union, so that λ is contained in their intersection.

The purpose of this paper is to give a refinement of Theorem 1.1. Some consequences and examples are also provided.

2 A Refinement of Theorem 1.1

We will employ the following key result, which is contained in Theorem 1.4.10 in [2].

Lemma 2.1. *Let λ be an eigenvalue of the $n \times n$ matrix A with geometric multiplicity at least k. If $\hat{A}$ is an $m \times m$ principal submatrix of A and if $m > n-k$, then λ is an eigenvalue of $\hat{A}$.*

By taking m to be $n-k+1$ and applying Lemma 2.1 we obtain the following improvement of Theorem 1.1.

Theorem 2.2. *Let λ be an eigenvalue of the $n \times n$ matrix A with geometric multiplicity at least k. Construct the $n \times n$ matrix C_k in the following way: in every row of A, replace the smallest $k-1$ off-diagonal entries in absolute value by zeros. Then λ is in at least k of the Geršgorin discs of C_k.*

Proof. We choose an arbitrary principal submatrix B_1 of A of order $n-k+1$. By Lemma 2.1, λ is an eigenvalue of B_1, and hence in one of it's G-discs. In the corresponding row (say row r) of the matrix A, we can replace the smallest $k-1$ off-diagonal entries in absolute value by zeros, so that λ is in the associated G-disc of the new matrix.

Next, delete row and column r from A to obtain a principal submatrix A_2 of A (which can be considered as A_1) of order $n-1$. Choose an arbitrary principal submatrix B_2 of A_2 of order $n-k+1$ and continue as above. We continue this process until we reach a principal submatrix A_k of order $n-(k-1)$, ie $n-k+1$, and repeat the procedure on A_k. This completes k steps in which we have replaced in k rows of A the smallest $k-1$ off-diagonal entries in absolute value by zeros, and λ is in each of the corresponding G-discs.

Finally, in each of the remaining $n-k$ rows, also replace the smallest $k-1$ off-diagonal entries in absolute value by zeros. □

Example 2.3. Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 2 & 0 \\ -1 & 1 & -1 & 1 & 1 \end{bmatrix}.$$

Now, 1 is an eigenvalue of A with algebraic multiplicity 5 and geometric multiplicity $k=3$. Notice that in this case the eigenvalue is in all five G-discs. (We recall that for an $n \times n$ matrix A, an eigenvalue λ could be in any number t of G-discs where (geom mult λ) $\leq t \leq n$, see [1], [4].) We can go through the process in the proof of Theorem 2.2, or just see that the eigenvalue 1 is in all five (the last three of which are smaller) G-discs of the matrix

$$C_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Of course, other choices for the matrix C_3 are possible, such as

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 2 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Remark 2.4. Clearly, the G-discs of the matrix C_k have radii less than or equal to the the radii of the G-discs of the original matrix A, which in general gives a better inclusion region for the eigenvalues of A with geometric multiplicity $\geq k$. Furthermore, as in Example 2.3, it should be clear that the matrix C_k is not unique, since many off-diagonal entries in the same row may have the same absolute value. We also point out that the eigenvalues of A and C_k are in general not the same.

For the purposes in the sequel, we denote matrices C_i constructed from the $n \times n$ matrix A in the same way as in Theorem 2.2: in every row of A, replace the smallest $i-1$ off-diagonal entries in absolute value by zeros.

We can now give a refinement of Corollary 1.2.

Corollary 2.5. *Let λ be an eigenvalue of A with geometric multiplicity at least $k \geq 1$. Then*

$$\lambda \in \cup_{j=1}^{n-k+1} \{z \in C : |z - a_{i_j i_j}| \leq R'_{i_j}(C_k)\}$$

for any choices of indices $1 \leq i_1 < \ldots < i_{n-k+1} \leq n$. There are $\binom{n}{k-1}$ possibilities for such a union, so that λ is contained in their intersection.

The following is another interesting, immediate corollary of Theorem 1.1, based again on the matrix C_k.

Corollary 2.6. *Let A be an $n \times n$ matrix. If each collection of G-discs of the matrix C_k that is separated from the remaining G-discs of C_k consists of at most $k-1$ discs, then each eigenvalue of A has geometric multiplicity less than k. In particular, if this is the case for $k = 2$, then each eigenvalue has geometric multiplicity 1.*

Example 2.7. Only observing that the three G-discs of the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

form a connected region, we cannot determine the multiplicities of its eigenvalues. However, consider the matrix

$$C_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

which has three mutually disjoint G-discs. By Corollary 2.6, each eigenvalue of A has geometric multiplicity 1.

3 Applications

The following result is a direct consequence of Theorem 2.2.

Theorem 3.1. *Let λ be an eigenvalue of A with geometric multiplicity k and suppose that $|a_{ij}| = \alpha$ for all $i \neq j$. Then the inequality*

$$|\lambda - a_{ii}| \leq (n-k)\alpha$$

holds for at least k values of i.

Corollary 3.2. *Let λ be an eigenvalue of A with geometric multiplicity k. Suppose that $|a_{ij}| = \alpha$ for all $i \neq j$ and that $|a_{ii}| = \beta$ for all i. Then*

$$|\lambda| \leq \beta + (n-k)\alpha.$$

Example 3.3. The above corollary is illustrated by the adjacency matrix A of the complete graph of order n, which has zeros on the diagonal and one in each off-diagonal position. The eigenvalues of A are $n-1$ and -1 with (algebraic and geometric) multiplicities 1 and $n-1$, respectively. For the eigenvalue $n-1$, Corollary 3.2 says that $n-1 \leq n-1$ and for the eigenvalue -1 that $1 \leq 1$, both of which are trivially true.

The corollary is also illustrated by the all ones matrix J_n of order n, which has eigenvalues n and 0 with (algebraic and geometric) multiplicities 1 and $n-1$, respectively.

Another application of Theorem 2.2 is the following.

Theorem 3.4. *Let A be an $n \times n$ matrix with all off-diagonal entries nonzero and let λ be an eigenvalue of A such that every G-circle of A passes through λ. Then the geometric multiplicity of λ is 1.*

Proof. We are given that all off-diagonal entries of A are nonzero and that λ is an eigenvalue of A such that every G-circle of A passes through λ. Suppose that the geometric multiplicity k of λ is greater than 1. By Theorem 2.2, λ is in at least k G-discs of C_k. However, by the construction of C_k, λ then cannot be on any of those k G-circles of A, since the G-discs of C_k are respectively properly contained in the G-discs of the original matrix A. We have a contradiction. Thus, $k = 1$. □

Remark 3.5. If the $n \times n$ matrix B is irreducible and the eigenvalue λ of B satisfies

$$|\lambda - b_{ii}| \geq R'_i(B),$$

for all $i = 1, \cdots, n$, then according to Theorem 6.2.8 in [2], every G-circle of B passes through λ. By what we have proved, if also all the off-diagonal entries of B are nonzero, then in fact the geometric multiplicity of λ is 1.

Example 3.6. We use the all ones matrix J_n. Now, the eigenvalue n is on every G-circle of J_n and the geometric multiplicity of n is 1.

We can relax the condition on the off-diagonal entries in Theorem 3.4 in the following way. The proof is similar to the proof of Theorem 3.4.

Corollary 3.7. *Let A be an $n \times n$ matrix and let λ be an eigenvalue of A such that every G-circle of A passes through λ. Suppose $k > 1$ and that there are $n-k+1$ or more rows of A with the following property: in each of these rows, the number of zero off-diagonal entries is less than $k-1$. Then the geometric multiplicity of λ is less than k.*

When $k = 2$ we have the following.

Corollary 3.8. *Let A be an $n \times n$ matrix with all off-diagonal entries nonzero in at least $n-1$ rows of A and let λ be an eigenvalue of A such that every G-circle of A passes through λ. Then the geometric multiplicity of λ is 1.*

This allows the matrix A to be reducible.

We present a last result which makes use of Theorem 2.2. Recall that $\|A\|_\infty$ is the maximum of the absolute row sums of A and that $\|A\|_1$ is the maximum of the absolute column sums of A.

Theorem 3.9. *Let A be an $n \times n$ matrix with all off-diagonal entries nonzero in at least $n-1$ rows ($n-1$ columns) and let λ be an eigenvalue of A such that $|\lambda| = \|A\|_\infty$ ($|\lambda| = \|A\|_1$). Then the geometric multiplicity of λ is 1.*

Proof. Without loss of generality, suppose that all the off-diagonal entries in the first $n-1$ rows are nonzero, and assume that $|\lambda| = \|A\|_\infty$. Suppose that the geometric multiplicity k of λ is greater than 1. By Theorem 2.2, λ is in at least k G-discs of C_k. Letting one of these k G-discs be disc i, $i \neq n$, we then have

$$|\lambda - c_{ii}| \leq R_i'(C_k)$$

so that

$$|\lambda| \leq |c_{ii}| + R_i'(C_k) \; < |a_{ii}| + R_i'(A) \leq \|A\|_\infty$$

and hence

$$|\lambda| < \|A\|_\infty \, ,$$

which is a contradiction. Thus, the geometric multiplicity of λ is 1.

For the proof involving $|\lambda| = \|A\|_1$, we can use the transpose of A. □

Remark 3.10. We can relax the condition on the off-diagonal entries in Theorem 3.9 in a similar way as in Corollary 3.7.

Example 3.11. Theorem 3.9 is illustrated by any doubly stochastic matrix with all off-diagonal entries nonzero and its eigenvalue of 1. Such a matrix is actually a nonnegative irreducible matrix.

Remark 3.12. There has been so much written on Geršgorin, his work, and follow-up results. We simply refer to Chapter 6 in the very valuable book [2] and also to the important book [5].

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