TAMKANG JOURNAL OF MATHEMATICS
Volume 28, Number 3, Autumn 1997

ON REFINEMENTS OF HADAMARD'S INEQUALITIES

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Abstract. Some refinements of Hadamard's inequalities are established.

1. Introduction

The inequalities

$$f\Big(\frac{a+b}{2}\Big)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{2} \tag{1.1}$$

which holds for all convex functions $f:[a,b]\to R$ are known in the literature as Hadamard's inequalities. In [2] and [3], S. S. Dragomir established some refinements of the first inequality of (1.1). In [4], G. S. Yang and M. C. Hong established a refinement of the second inequality of (1.1).

The main purpose of this note is to establish further generalization of the results in [2], [3] and [4].

As in [1] and [2], let E be a nonempty set and let L be a linear class of real-valued functions from E to R having the properties:

$$L_1: f,g\in L\Rightarrow (af+bg)\in L \text{ for all } a,b\in R;$$
$$L_2: 1\in L, \text{ that is, if } f(t)=1(t\in E), \text{ then } f\in L.$$

A linear functional $A:L\to R$ is isotonic if

$$A_1: A(af+bg)=aA(f)+bA(g) \text{ for } f,g\in L \text{ and } a,b\in R;$$
$$A_2: f\in L,\quad f(t)\ge 0 \text{ on } E\Rightarrow A(f)\ge 0\quad (A \text{ is isotonic}).$$

We need the following Jensen's inequality (see [1] or [2]).

Jensen's inequality. Let L satisfy the above properties on E, and suppose Φ is a convex function on an interval $I\subseteq R$. If A is any isotonic linear functional with $A(1)=1$, then, for all $g\in L$ such that $\Phi(g)\in L$, we have $A(g)\in I$ and $\Phi\big(A(g)\big)\le A\big(\Phi(g)\big)$.

Received April 19, 1996.
1991 *Mathematics Subject Classification.* Primary 26D15.
Key words and phrases. Hadamard's inequality, Jensen's inequality, convex function, isotonic linear functional.

2. Preliminary Lemmas

In order to establish the main theorems, we start with the following lemmas.

Lemma 1. *Let C be a convex subset of a real linear space X, and $f : C \to R$, the real numbers, be a convex function. Let $a_i > 0$ $(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n} a_i = 1$ and $a = \min_{1 \le i \le n}\{a_i\}$. Given a sequence $x = \{x_i, x_2, \ldots, x_n\}$ in C, let $\Phi_x : [0, na] \to R$ be defined by*

$$\Phi_x(t) = \sum_{i=1}^{n} a_i f\Big(\Big[1 - \frac{g(t)}{na_i}\Big]x_i + \frac{g(t)}{na_i}x_{i+1}\Big),$$

where g is a linear function on $[0, na]$ such that $0 \le g(t) \le na$ and $x_{n+1} = x_1$. Then

(1) *Φ_x is convex on $[0, na]$,*

$$(2)\ f\Big(\sum_{i=1}^{n} a_i x_i\Big) \le \Phi_x(t) \le \sum_{i=1}^{n} a_i f(x_i) \text{ for all } t \in [0, na]. \tag{2.1}$$

Proof. Let $t_1, t_2 \in [0, na]$ and $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$. Since f is convex on C and g is linear in $[0, na]$, we have

$$\begin{aligned}
\Phi_x(\alpha t_1 + \beta t_2) &= \sum_{i=1}^{n} a_i f\Big(\Big[1 - \frac{g(\alpha t_1 + \beta t_2)}{na_i}\Big]x_i + \frac{g(\alpha t_1 + \beta t_2)}{na_i}x_{i+1}\Big) \\
&= \sum_{i=1}^{n} a_i f\Big(\alpha\Big[\Big(1 - \frac{g(t_1)}{na_i}\Big)x_i + \frac{g(t_1)}{na_i}x_{i+1}\Big] \\
&\qquad + \beta\Big[\Big(1 - \frac{g(t_2)}{na_i}\Big)x_i + \frac{g(t_2)}{na_i}x_{i+1}\Big]\Big) \\
&\le \alpha \sum_{i=1}^{n} a_i f\Big(\Big[1 - \frac{g(t_1)}{na_i}\Big]x_i + \frac{g(t_1)}{na_i}x_{i+1}\Big) \\
&\qquad + \beta \sum_{i=1}^{n} a_i f\Big(\Big[1 - \frac{g(t_2)}{na_i}\Big]x_i + \frac{g(t_2)}{na_i}x_{i+1}\Big) \\
&= \alpha\Phi_x(t_1) + \beta\Phi_x(t_2).
\end{aligned}$$

This completes the proof of (1).

Next, using the convexity of f and note that $x_{n+1} = x_1$, we have

$$\begin{aligned}
\Phi_x(t) &\le \sum_{i=1}^{n} a_i \Big[\Big(1 - \frac{g(t)}{na_i}\Big)f(x_i) + \frac{g(t)}{na_i}f(x_{i+1})\Big] \\
&= \sum_{i=1}^{n} a_i f(x_i) + \frac{g(t)}{n}\sum_{i=1}^{n}\big[f(x_{i+1}) - f(x_i)\big] = \sum_{i=1}^{n} a_i f(x_i)
\end{aligned}$$

and

$$\Phi_x(t) \geq f\Big(\sum_{i=1}^{n} a_i\Big[\Big(1-\frac{g(t)}{na_i}\Big)x_i + \frac{g(t)}{na_i}x_{i+1}\Big]\Big)$$
$$= f\Big(\sum_{i=1}^{n} a_i x_i + \frac{g(t)}{n}\sum_{i=1}^{n}[x_{i+1}-x_i]\Big) = f\Big(\sum_{i=1}^{n} a_i x_i\Big),$$

for all $t \in [0, na]$. This proves (2).

Remark 1. Lemma 2.1 in [2] is the special case of our lemma 1 when $n = 2$, $g(t) = t$ and $a_1 = a_2 = \frac{1}{2}$.

In [4], G. S. Yang and M. C. Hong proved:

Lemma 2. *If $f : [a, b] \to R$ is a convex function and $F : [0, 1] \to R$ is defined by*

$$F(t) = \frac{1}{2(b-a)}\int_a^b \Big\{f\Big(\Big[\frac{1+t}{2}\Big]a + \Big[\frac{1-t}{2}\Big]x\Big) + f\Big(\Big[\frac{1+t}{2}\Big]b + \Big[\frac{1-t}{2}\Big]x\Big)\Big\}dx,$$

then F is convex, increasing on $[0, 1]$ and

$$\frac{1}{b-a}\int_a^b f(x)dx = F(0) \leq F(t) \leq F(1) = \frac{f(a)+f(b)}{2}.$$

They used the differentiability of f on $(0, 1)$ to prove F is increasing on $[0, 1]$. Here, we give a proof without using the differentiability of f on $(0, 1)$ as follows:

Proof. That F is convex on $[0, 1]$ is easy to verify. Now, if $0 \leq t < 1$, then

$$F(t) = \frac{1}{2(b-a)}\int_a^b \Big\{f\Big(\frac{[1+t]a + [1-t]x}{2}\Big) + f\Big(\frac{[1+t]b + [1-t]x}{2}\Big)\Big\}dx$$
$$= \frac{1}{(1-t)(b-a)}\Big\{\int_a^{\frac{a+b}{2}-t(\frac{b-a}{2})} f(x)dx + \int_{\frac{a+b}{2}+t(\frac{b-a}{2})}^{b} f(x)dx\Big\}.$$

Since f is convex, we have

$$F'(t) = \frac{1}{(1-t)^2(b-a)}\Big\{\int_a^{\frac{a+b}{2}-t(\frac{b-a}{2})} f(x)dx + \int_{\frac{a+b}{2}+t(\frac{b-a}{2})}^{b} f(x)dx\Big\}$$
$$+\frac{1}{(1-t)(b-a)}\Big\{f\Big(\frac{a+b}{2} - t\Big[\frac{b-a}{2}\Big]\Big)\Big[-\frac{b-a}{2}\Big]$$
$$-f\Big(\frac{a+b}{2} + t\Big[\frac{b-a}{2}\Big]\Big)\Big[\frac{b-a}{2}\Big]\Big\}$$
$$= \frac{1}{(1-t)^2(b-a)}\Big\{\int_a^{\frac{a+b}{2}-t(\frac{b-a}{2})} f(x)dx + \int_{\frac{a+b}{2}+t(\frac{b-a}{2})}^{b} f(x)dx\Big\}$$

$$
\begin{aligned}
&-\frac{1}{2(1-t)}\Big\{f\Big(\frac{a+b}{2}-t\Big[\frac{b-a}{2}\Big]\Big)+f\Big(\frac{a+b}{2}+t\Big[\frac{b-a}{2}\Big]\Big)\Big\}\\
\geq&\ \frac{1}{2(1-t)}\Big\{f\Big(\Big[\frac{3+t}{4}\Big]a+\Big[\frac{1-t}{4}\Big]b\Big)+f\Big(\Big[\frac{1-t}{4}\Big]a+\Big[\frac{3+t}{4}\Big]b\Big)\Big\}\\
&-\frac{1}{2(1-t)}\Big\{f\Big(\Big[\frac{1+t}{2}\Big]a+\Big[\frac{1-t}{2}\Big]b\Big)+f\Big(\Big[\frac{1-t}{2}\Big]a+\Big[\frac{1+t}{2}\Big]b\Big)\Big\}\\
=&\ \frac{1}{2(1-t)}\Big\{f\Big(\Big[\frac{1-t}{4}\Big]a+\Big[\frac{3+t}{4}\Big]b\Big)-f\Big(\Big[\frac{1-t}{2}\Big]a+\Big[\frac{1+t}{2}\Big]b\Big)\Big\}\\
&-\frac{1}{2(1-t)}\Big\{f\Big(\Big[\frac{1+t}{2}\Big]a+\Big[\frac{1-t}{2}\Big]b\Big)-f\Big(\Big[\frac{3+t}{4}\Big]a+\Big[\frac{1-t}{4}\Big]b\Big)\Big\}\\
\geq&\ 0.
\end{aligned}
$$

This shows that F is increasing on $[0,1]$. Hence

$$\frac{1}{b-a}\int_a^b f(x)dx = F(0) \leq F(t) \leq F(1) = \frac{f(a)+f(b)}{2}.$$

This completes the proof.

3. Main Results

Now, we give our main results as the following theorems.

Theorem 1. *Under the conditions of Lemma 1, let L, A satisfy the conditions L_1, L_2, A_1 and A_2, and let $h : E \to [0, na]$ be a function such that $h \in L$ and*

$$f\Big(\Big[1-\frac{g(h)}{na_i}\Big]x_i+\frac{g(h)}{na_i}x_{i+1}\Big) \in L \qquad \text{for } i=1,2,\ldots,n.$$

If $A(1) = 1$, then

$$f\Big(\sum_{i=1}^n a_i x_i\Big) \leq \Phi_x\big(A(h)\big) \leq A\big(\Phi_x(h)\big) \leq \sum_{i=1}^n a_i f(x_i). \tag{3.1}$$

Proof. Using Jensen's inequality, we have

$$\Phi_x\big(A(h)\big) \leq A\big(\Phi_x(h)\big).$$

This is the second inequality in (3.1).

Since f is convex on C and A is an istonic linear functional on L, we have

$$
\begin{aligned}
\Phi_x\big(A(h)\big) &= \sum_{i=1}^n a_i f\Big(\Big[1-\frac{g(A(h))}{na_i}\Big]x_i+\frac{g(A(h))}{na_i}x_{i+1}\Big)\\
&\geq f\left(\sum_{i=1}^n a_i\Big[\Big(1-\frac{g(A(h))}{na_i}\Big)x_i+\frac{g(A(h))}{na_i}x_{i+1}\Big]\right) = f\left(\sum_{i=1}^n a_i x_i\right).
\end{aligned}
$$

This is the first inequality of (3.1).

Finally,

$$\Phi_x(h) = \sum_{i=1}^{n} a_i f\Big(\Big[1 - \frac{g(h)}{na_i}\Big]x_i + \frac{g(h)}{na_i}x_{i+1}\Big)$$
$$\le \sum_{i=1}^{n} a_i \Big[\Big(1 - \frac{g(h)}{na_i}\Big)f(x_i) + \frac{g(h)}{na_i}f(x_{i+1})\Big] = \sum_{i=1}^{n} a_i f(x_i).$$

Using A_1, A_2 and $A(1) = 1$, we have $A\big(\Phi_x(h)\big) \le A\left(\sum_{i=1}^{n} a_i f(x_i)\right) = \sum_{i=1}^{n} a_i f(x_i)$. This proves the last inequality of (3.1).

Remark 2. We note that Theorem 2.3 in [2] is the special case of Theorem 1 as $n = 2$, $a_1 = a_2 = \frac{1}{2}$, $g(t) = t$ and $h(t) = t$.

Theorem 2. *Under the conditions of Lemma 1, if $x = \{x_1, x_2, \ldots, x_n\}$ is a sequence in C such that $x_i \neq x_{i+1}$, $i = 1, 2, \ldots, n$, and $x_{n+1} = x_1$, then*

$$f\Big(\sum_{i=1}^{n} a_i x_i\Big) \le \sum_{i=1}^{n} a_i f\Big(\Big[1 - \frac{a}{2a_i}\Big]x_i + \frac{a}{2a_i}x_{i+1}\Big)$$
$$\le \sum_{i=1}^{n} \frac{a_i^2}{a(x_{i+1} - x_i)} \int_{x_i}^{x_i + \frac{a}{a_i}(x_{i+1} - x_i)} f(t)dt$$
$$\le \sum_{i=1}^{n} a_i f(x_i). \tag{3.2}$$

Proof. Let $A = \frac{1}{na}\int_0^{na}$, $E = [0, na]$, $g(t) = t$ and $h(t) = t$. Then

$$A\big(\Phi_x(h)\big) = \frac{1}{na}\int_0^{na} \sum_{i=1}^{n} a_i f\Big(\Big[1 - \frac{t}{na_i}\Big]x_i + \frac{t}{na_i}x_{i+1}\Big)dt$$
$$= \sum_{i=1}^{n} \frac{a_i^2}{a(x_{i+1} - x_i)} \int_{x_i}^{x_i + \frac{a}{a_i}(x_{i+1} - x_i)} f(t)dt,$$

and

$$\Phi_x\big(A(h)\big) = \sum_{i=1}^{n} a_i f\left(\left[1 - \frac{\frac{1}{na}\int_0^{na} tdt}{na_i}\right]x_i + \frac{\frac{1}{na}\int_0^{na} tdt}{na_i}x_{i+1}\right)$$
$$= \sum_{i=1}^{n} a_i f\Big(\Big[1 - \frac{a}{2a_i}\Big]x_i + \frac{a}{2a_i}x_{i+1}\Big).$$

Using (3.1), we obtain

$$f\Big(\sum_{i=1}^{n} a_i x_i\Big) \le \sum_{i=1}^{n} a_i f\Big(\Big[1 - \frac{a}{2a_i}\Big]x_i + \frac{a}{2a_i}x_{i+1}\Big)$$

$$\le \sum_{i=1}^{n} \frac{a_i^2}{a(x_{i+1}-x_i)} \int_{x_i}^{x_i+\frac{a}{a_i}(x_{i+1}-x_i)} f(t)dt$$
$$\le \sum_{i=1}^{n} a_i f(x_i).$$

Remark 3. We note that the Hadamard's inequalities (1.1) is the special case of Theorem 2 when $n=2$, $x_1=a$, $x_2=b$, and $a_1=a_2=\frac{1}{2}$.

Tehorem 3. *Under the conditions of Theorem 2, let $H:[0,1]\to R$ be a function defined by*

$$H(t)=\sum_{i=1}^{n} \frac{a_i^2}{a(x_{i+1}-x_i)} \int_{x_i}^{x_i+\frac{a}{a_i}(x_{i+1}-x_i)} f\left(tx+(1-t)\sum_{j=1}^{n} a_j x_j\right) dx.$$

Then (1) *H is convex on* $[0,1]$,

$$(2)\ f\left(\sum_{i=1}^{n} a_i x_i\right) = H(0) = \min_{t\in[0,1]} H(t) \le H(t)$$
$$\le \max_{t\in[0,1]} H(t) = H(1) = \sum_{i=1}^{n} \frac{a_i^2}{a(x_{i+1}-x_i)} \int_{x_i}^{x_i+\frac{a}{a_i}(x_{i+1}-x_i)} f(x)dx$$
$$\le \sum_{i=1}^{n} a_i f(x_i), \tag{3.3}$$

for all $t\in[0,1]$,
(3) *H is increasing on* $[0,1]$.

Proof. Let $t_1,t_2\in[0,1]$ and $\alpha,\beta\ge 0$ with $\alpha+\beta=1$. Since f is convex on C, we have

$$H(\alpha t_1+\beta t_2)$$
$$=\sum_{i=1}^{n} \frac{a_i^2}{a(x_{i+1}-x_i)} \int_{x_i}^{x_i+\frac{a}{a_i}(x_{i+1}-x_i)} f\Big((\alpha t_1+\beta t_2)x+(1-\alpha t_1-\beta t_2)\sum_{j=1}^{n} a_j x_j\Big)dx$$
$$=\sum_{i=1}^{n} \frac{a_i^2}{a(x_{i+1}-x_i)}$$
$$\times \int_{x_i}^{x_i+\frac{a}{a_i}(x_{i+1}-x_i)} f\Big(\alpha\Big[t_1x+(1-t_1)\sum_{j=1}^{n} a_j x_j\Big]+\beta\Big[t_2x+(1-t_2)\sum_{j=1}^{n} a_j x_j\Big]\Big)dx$$
$$\le \alpha \sum_{i=1}^{n} \frac{a_i^2}{a(x_{i+1}-x_i)} \int_{x_i}^{x_i+\frac{a}{a_i}(x_{i+1}-x_i)} f\Big(t_1x+(1-t_1)\sum_{j=1}^{n} a_j x_j\Big)dx$$

$$+\beta\sum_{i=1}^{n}\frac{a_i^2}{a(x_{i+1}-x_i)}\int_{x_i}^{x_i+\frac{a}{a_i}(x_{i+1}-x_i)} f\Big(t_2x+(1-t_2)\sum_{j=1}^{n}a_jx_j\Big)dx$$
$$=\alpha H(t_1)+\beta H(t_2).$$

This completes the proof of (1).

Now, observe that $H(0)=f\Big(\sum_{i=1}^{n}a_ix_i\Big)$ and $H(1)=\sum_{i=1}^{n}\frac{a_i^2}{a(x_{i+1}-x_i)}$ $\int_{x_i}^{x_i+\frac{a}{a_i}(x_{i+1}-x_i)}f(x)dx$. Using the convexity of f and the inequality (3.2), we have

$$\begin{aligned}H(t)&=\sum_{i=1}^{n}\frac{a_i^2}{a(x_{i+1}-x_i)}\int_{x_i}^{x_i+\frac{a}{a_i}(x_{i+1}-x_i)} f\Big(tx+(1-t)\sum_{j=1}^{n}a_jx_j\Big)dx\\
&\le t\sum_{i=1}^{n}\frac{a_i^2}{a(x_{i+1}-x_i)}\int_{x_i}^{x_i+\frac{a}{a_i}(x_{i+1}-x_i)} f(x)dx\\
&\quad+(1-t)\sum_{i=1}^{n}\frac{a_i^2}{a(x_{i+1}-x_i)}\int_{x_i}^{x_i+\frac{a}{a_i}(x_{i+1}-x_i)} f\Big(\sum_{j=1}^{n}a_jx_j\Big)dx\\
&=t\sum_{i=1}^{n}\frac{a_i^2}{a(x_{i+1}-x_i)}\int_{x_i}^{x_i+\frac{a}{a_i}(x_{i+1}-x_i)} f(x)dx+(1-t)f\Big(\sum_{j=1}^{n}a_jx_j\Big)\\
&\le\sum_{i=1}^{n}\frac{a_i^2}{a(x_{i+1}-x_i)}\int_{x_i}^{x_i+\frac{a}{a_i}(x_{i+1}-x_i)} f(x)dx\le\sum_{i=1}^{n}a_if(x_i),\end{aligned}$$

for all $t\in[0,1]$.

On the other hand, let $y_i=tx_i+(1-t)\sum_{j=1}^{n}a_jx_j$, $1\le i\le n$, and $y_{n+1}=y_1$, then

$$\begin{aligned}H(t)&=\sum_{i=1}^{n}\frac{a_i^2}{a(y_{i+1}-y_i)}\int_{y_i}^{y_i+\frac{a}{a_i}(y_{i+1}-y_i)} f(y)dy\\
&\ge f\Big(\sum_{j=1}^{n}a_iy_i\Big)=f\Big(\sum_{i=1}^{n}a_ix_i\Big),\end{aligned}$$

for all $t\in[0,1]$.

This completes the proof of (2).

Finally, let $0<t<u\le 1$. Since H is convex on $[0,1]$ and $H(t)\ge H(0)$, we have

$$\frac{H(u)-H(t)}{u-t}\ge\frac{H(t)-H(0)}{t}\ge 0,$$

that is $H(t)\le H(u)$.

This completes the proof of (3).

Remark 4. We note that Theorem 1 in [3] is the special case of Theorem 3 when $n=2$, $a_1=a_2=\frac{1}{2}$.

Theorem 4. *Under the conditions of Theorem 2, let $K : [0,1] \to R$ be a function defined by*

$$K(t) = \sum_{i=1}^{n} \frac{a_i^2}{2a(x_{i+1}-x_i)} \int_{x_i}^{x_i+\frac{a}{a_i}(x_{i+1}-x_i)} \Big\{ f\Big(\Big[\frac{1-t}{2}\Big]x + \Big[\frac{1+t}{2}\Big]x_i\Big) + f\Big(\Big[\frac{1-t}{2}\Big]x + \Big[\frac{1+t}{2}\Big]\Big[x_i + \frac{a}{a_i}(x_{i+1}-x_i)\Big]\Big)\Big\} dx$$

Then K is convex, increasing on $[0,1]$ and

$$\sum_{i=1}^{n} \frac{a_i^2}{a(x_{i+1}-x_i)} \int_{x_i}^{x_i+\frac{a}{a_i}(x_{i+1}-x_i)} f(x)dx = K(0) \le K(t) \le K(1) \le \sum_{i=1}^{n} a_i f(x_i)$$

for all $t \in [0,1]$.

Proof. Using Lemma 2, it is easy to see that K is convex and increasing on [0,1]. Now,

$$\begin{aligned} K(0) &= \sum_{i=1}^{n} \frac{a_i^2}{a(x_{i+1}-x_i)} \int_{x_i}^{x_i+\frac{a}{a_i}(x_{i+1}-x_i)} f(x)dx \\ K(1) &= \sum_{i=1}^{n} \frac{a_i^2}{a(x_{i+1}-x_i)} \\ &\quad \int_{x_i}^{x_i+\frac{a}{a_i}(x_{i+1}-x_i)} \frac{1}{2}\Big[f(x_i) + f\Big(x_i + \frac{a}{a_i}(x_{i+1}-x_i)\Big)\Big]dx \\ &= \sum_{i=1}^{n} \frac{a_i\Big[f(x_i) + f\Big(x_i + \frac{a}{a_i}(x_{i+1}-x_i)\Big)\Big]}{2} \\ &= \frac{1}{2}\sum_{i=1}^{n} a_i f(x_i) + \frac{1}{2}\sum_{i=1}^{n} a_i f\Big(x_i + \frac{a}{a_i}(x_{i+1}-x_i)\Big) \\ &\le \sum_{i=1}^{n} a_i f(x_i). \end{aligned}$$

This completes the proof.

Remark 5. Let $a_i = \frac{1}{n}(i = 1,2,\ldots,n)$ and $x_{n+1} = x_1 < x_2 < \cdots < x_n$. Then, from Theorem 3 and Theorem 4, we have

$$H(t) = \sum_{i=1}^{n} \frac{1}{n(x_{i+1}-x_i)} \int_{x_i}^{x_{i+1}} f\Big(tx + \frac{1-t}{n}\sum_{j=1}^{n} x_j\Big)dx,$$

and

$$K(t) = \sum_{i=1}^{n} \frac{1}{2n(x_{i+1} - x_i)} \int_{x_i}^{x_{i+1}} \Big\{ f\Big(\Big[\frac{1-t}{2}\Big] x + \Big[\frac{1+t}{2}\Big] x_i \Big) + f\Big(\Big[\frac{1-t}{2}\Big] x + \Big[\frac{1+t}{2}\Big] x_{i+1} \Big) \Big\} dx$$

such that H and K are convex increasing on $[0,1]$ and

$$\begin{aligned} f\Big(\frac{1}{n}\sum_{i=1}^{n} x_i\Big) &= H(0) \le H(t) \\ &\le H(1) = \sum_{i=1}^{n} \frac{1}{n(x_{i+1} - x_i)} \int_{x_i}^{x_{i+1}} f(x)dx = K(0) \\ &\le K(t) \le K(1) = \frac{1}{n}\sum_{i=1}^{n} f(x_i). \end{aligned}$$

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